On the Calculus of Variations and Its Major Influences on the Mathematics of the First Half of Our Century.* Part I.

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1. JOHANN BERNOULLI'S BRACHYSTOCHRONE. CARATHÉODORY'S METHOD. The calculus of variations evolved from the differential and integral calculus, "the calculus," for short. An initial motivation of the latter was the determination of extrema of *functions*, as shown by the title of the earliest relevant *published* paper *A new method for the determination of maxima and minima*... (Leibniz, 1684). However, the calculus had to be extended to the calculus of variations in order to take care of more general problems involving the determination of stationary values of *functionals*,† given in the simplest case by a definite integral involving an unknown function and boundary conditions. The earliest (not very simple) *solved* problem of this kind was Newton's determination of the shape of a gun shell of least air resistance (letter to Gregory of July 14, 1694).

The earliest problem that received general publicity [due to a rather bombastic advertisement in Acta Eruditorum by Johann Bernoulli (1667–1748)] in 1696 was the problem of determining the *brachystochrone*, the curve along which a particle will fall from one given point to another in the shortest time. This problem was solved by Newton, Leibniz, and Johann Bernoulli as well as by his brother Jacob (1654–1705), the solution being a *cycloid*. Thus 1696 can be called the birthyear of the calculus of variations. Johann Bernoulli not only posed that problem but also gave a solution capable of extensive generalization worked out in 1908 by Carathéodory. The resulting general method was later named after Carathéodory.

2. SIMPLEST GENERAL PROBLEMS. Although they arose from different geometric and physical applications, many of the early problems led to functionals that depended on real functions defined on an interval and satisfying boundary condi-

*Abbreviated version of a paper with the same title.

†A functional is a function defined on a set of functions. A stationary value of a functional is its value at a "point" (= function) that satisfies the necessary conditions for an extremum (see Section 3 below).

tions, and all functionals were of the *same form* (as needed to create a general theory).

$$J[y] = \int_{x_0}^{x_1} L(x, y, y')\, dx, \quad y(x_0) = y_0, \quad y(x_1) = y_1, \quad x_0 < x_1. \quad (2.1)$$

The task was to determine a function $y(x)$ that satisfied the boundary conditions in (2.1) and rendered $J[y]$ stationary, possibly yielding a minimum or a maximum of $J[y]$. At this early stage, the existence and uniqueness of solutions was "obvious" for physical reasons because a solution could be verified experimentally if desired. Also, with the concept of function not yet sharply defined, nobody made an attempt to characterize the set of functions in which such a $y(x)$ was to be found. This was accomplished well over one hundred years later in the works of Jacobi (1804–51) around 1835 and, especially, of Weierstrass (1815–97) around 1880.

3. EULER AND LAGRANGE. As the birthyear of the *theory* of the calculus of variations one usually considers 1744, the year in which Euler published his famous book *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutio problematis isoperimetrici latissimo sensu accepti* (A method for discovering curved lines that enjoy a maximum or minimum property, or the solution of the isoperimetric problem taken in its widest sense). Thus Euler replaced "art of invention" (*ars inveniendi*), a very popular term in the works of Tschirnhaus and in other works of Leibniz's time, by "method of invention," a remarkable turn toward systematization. This book, a landmark in the development of the subject, contained the *Euler equation*

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0, \quad (3.1)$$

(first published by Euler in 1736) as a necessary condition for $y(x)$ satisfying (2.1) to yield a minimum of $J[y]$. In more explicit form it is the equation

$$L_{y'y'}y'' + L_{y'y}y' + L_{y'x} - L_y = 0. \quad (3.2)$$

This equation suggests calling (2.1) a *regular problem* when $L_{y'y'}$ is never zero, and then assuming that $L_{y'y'} > 0$.

Euler's book also contains a fascinating collection of 66 problems. Carathéodory, the editor of the book as a volume of Euler's *Works*, said that it

> "is one of the most beautiful mathematical works ever written. We cannot emphasize enough the extent to which that *Lehrbuch* over and over again served later generations as a prototype in the endeavor of presenting special mathematical material in its [logical, intrinsic] connection."

Euler's inspiration came from geometry and even more from the *principle of least action*, according to which nature realizes all motions in the most economical manner; more precisely, among all possible ways of reaching a given goal, nature chooses the one which minimizes the *action integral* $\int mv\, ds$ over the path (m = mass, v = speed, s = arc length). The beginning of the principle is often dated back to Leibniz because of a (lost) letter he is supposed to have written on the principle in 1707, but the question is still an open one. The principle is usually named after de Maupertuis (1698–1759), president of the Berlin Academy under Frederick the Great. Actually, Euler most likely discovered it earlier, formulated it mathematically more rigorously, and applied it to a nontrivial problem (involving central forces). In contrast to this, Maupertuis published the principle (in 1744 and 1746) in a vague and almost theological form. He defended vigorously his (questionable) priority, but failed to realize that a rigorization of the principle would call for specification of conditions to be satisfied by the motions with which the actual motion was to be compared. Accordingly, his main merit seems to be that he was *searching for* a minimum principle.

The great significance of the calculus of variations in mathematical physics is due to the transparent and coordinate-free form that the laws of nature take in this calculus. That fact became apparent in the work of Euler, and even more impressively in that of Lagrange. In his path-breaking memoir *Essai d'une nouvelle méthode pour déterminer les maxima et les minima des formules intégrales indéfinies* (1760–61) Lagrange substantially overtook Euler (as Euler was well aware). His work was a milestone in the development of the field and in its application to geometry and analytical mechanics. In it he invented the "method of variations" together with the symbol δ. His new idea was to use "*comparison functions*,"

$$\bar{y} = y + \varepsilon\eta, \qquad \eta \in C^2([x_0, x_1]), \qquad \eta(x_0) = \eta(x_1) = 0, \tag{3.3}$$

in (2.1) and to conclude from the vanishing of the first variation of (2.1),

$$\delta J = \varepsilon \frac{\partial J[\bar{y}]}{\partial \varepsilon}\bigg|_{\varepsilon=0} = \varepsilon \cdot \int_{x_0}^{x_1} (L_y \eta + L_{y'}\eta')\, dx = 0, \tag{3.4}$$

(and an integration by parts) that Euler's equation (3.1) gave a necessary condition for $y(x)$ to render $J[y]$ stationary. (For a detailed and transparent explanation of this vital point see pp. 505–508 of G. F. Simmons, *Differential Equations*, second ed., McGraw-Hill, 1991.) In that paper Lagrange also started working on problems with variable endpoints, with application to brachystochrone and other problems. As another important step forward, he explicitly formulated his *multiplier rule* (without proof). This rule became a basic tool in his *Méchanique analitique*, in which he also included his theory of the calculus of variations and derived from the principle of least action his *equations of motion*, equivalent to *Newton's second law* and constituting analogues of Euler's equation. They are:

$$\frac{\partial V}{\partial x_i} + \frac{d}{dt}\left(\frac{\partial T}{\partial x_i}\right) = 0, \qquad i = 1, 2, 3, \tag{3.5}$$

where V and T are the potential and kinetic energy, respectively.

Whereas Euler's interest in the calculus of variations centered around applications, Lagrange's emphasis was on algorithmic aspects of analysis. Lagrange's entire work excels by its wealth of original discoveries as well as by the outstanding assimilation of the historical material.

> "By generalizing Euler's method he arrived at his remarkable formulas which in one line contain the solution of all problems of analytical mechanics.
>
> [In his Memoir of 1760–61] he created the whole calculus of variations with one stroke. This is one of the most beautiful articles that have ever been written. The ideas follow one another like lightning with the greatest rapidity..."

These enthusiastic lines are taken from lecture notes by C. G. J. Jacobi (1804–51).

4. MINIMAL SURFACES. Euler's *Methodus inveniendi* of 1744 marked not only the beginning of the *theory* of the calculus of variations but also of one of its most fascinating geometric applications related to the creation of a remarkable class of surfaces called *minimal surfaces*. These were originally obtained from the calculus of variations as (portions of) surfaces of least area among all surfaces bounded by a given space curve. Nowadays we define them as surfaces with vanishing mean curvature H,

$$H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2}\left(\frac{GL - 2FM + EN}{EG - F^2}\right) = 0, \tag{4.1}$$

using the discovery of Meusnier (1756–93) in 1776 that (4.1) is a necessary condition for least area. Here κ_1 and κ_2 are the principal curvatures, and E, F, G and L, M, N are the respective coefficients of the first and second fundamental forms of the surface.

What is most important to us here is Euler's discovery of the first non-trivial minimal surface, the *catenoid*. Euler obtained it by minimizing area, as the surface generated by rotating a *catenary* [a cosh curve, the curve of a hanging chain (*catena*) or cable], say,

$$r = A \cosh x, \tag{4.2}$$

where r is the distance, in 3-dimensional space, from the x-axis.

Euler's discovery of the catenoid was a major accomplishment in his geometric work and marked the beginning of the study of minimal surfaces. It was followed by Lagrange's systematic theory developed in his Memoir of 1760–61. In this paper and in a subsequent one he extended his method to *double integrals* for functions of two variables.

$$J[z] = \int_{\Omega}\!\int L(x, y, z, p, q)\, dx\, dy \qquad (p = z_x,\ q = z_y) \tag{4.3}$$

over a domain Ω in the xy-plane subject to given boundary conditions; the corresponding *Euler-Lagrange equation* [taking the place of (3.1)] is

$$L_z - \frac{\partial}{\partial x}L_p - \frac{\partial}{\partial y}L_q = 0. \tag{4.4}$$

5. LEGENDRE, JACOBI, WEIERSTRASS. In the calculus, $y' = 0$ is only a necessary condition for a minimum of a function $y(x)$, and for a decision one must also consider y''. Similarly, in the calculus of variations Euler's equation is only a necessary condition for a minimum, and for a decision one must also consider the *second variation* of (2.1),

$$\delta^2 J = \frac{\varepsilon^2}{2}\frac{\partial^2 J[\tilde{y}]}{\partial \varepsilon^2}\bigg|_{\varepsilon=0} = \frac{\varepsilon^2}{2}\int_{x_0}^{x_1}\left(L_{yy}\eta^2 + 2L_{yy'}\eta\eta' + L_{y'y'}\eta'^2\right) dx, \tag{5.1}$$

introduced by Legendre (1752–1833) in 1786. It was formally motivated by Taylor's theorem

$$J[y + \varepsilon\eta] = J[y] + \delta J + \delta^2 \tilde{J}, \tag{5.2}$$

where the tilde means that the arguments are $y + \tilde{\varepsilon}\eta$, $y' + \tilde{\varepsilon}\eta'$ with $\tilde{\varepsilon} \in (0, \varepsilon]$. Legendre obtained the condition $L_{y'y'} \geq 0$ along a minimizing curve and $L_{y'y'} \leq 0$ along a maximizing curve (very similar to the calculus!) but he did not justify his analysis completely.

In fact, it took another fifty years before Jacobi succeeded in rigorously demonstrating that $L_{y'y'} > 0$ and the so-called *Jacobi condition*, which asserts that x_1 should be closer to x_0 than the so-called conjugate point* of x_0, suffice for a local minimum, that is a minimizing $\tilde{y}$ among $y \in C^1[x_0, x_1]$ satisfying the boundary conditions in (2.1) and lying close to $\tilde{y}$ in the C^1 sense, that is, satisfying

$$(a)\ |y - \tilde{y}| < \rho, \qquad (b)\ |y' - \tilde{y}'| < \rho \quad \text{for small positive } \rho. \tag{5.3}$$

Jacobi's discovery of the conjugate point and of its significance closed a substantial gap. However, condition (5.3b) seemed to be too restrictive and in no way suggested by the nature of the problem. Weierstrass emphasized that one should extend the domain of (2.1) and consider *strong minima*, that is, one should drop (5.3b).

For this program of obtaining a sufficient condition for a strong minimum, Weierstrass set up entirely new machinery centering around two ingenious concepts. The first was a *field of extremals* of (2.1), which he defined as a domain Ω in the xy-plane such that through each of its points there passes precisely one extremal of a one-parameter family of extremals of (2.1) (solution curves of Euler's equation) depending continuously on the parameter, and the second was the so-called E-function, a turning point in the history of the calculus of variations.

To define the E-function, Weierstrass started from the *slope function* $p = p(x, y)$, the slope at (x, y) of the extremal of a field of extremals $y = h(x, \alpha)$; thus

$$p(x, y) = h'(x, \alpha)|_{\alpha = \alpha(x, y)}, \tag{5.4}$$

where $'$ refers to differentiation with respect to x. Then he defined the *E-function* by

$$E(x, y, p, y') = L(x, y, y') - L(x, y, p) - (y' - p) L_{y'}(x, y, p), \tag{5.5}$$

where $y = y(x)$ is any C^1-curve in the region covered by the field of extremals. Now he could prove that if, for an extremal $y = \tilde{y}(x)$ of the field, the above sufficient conditions for a local minimum are satisfied, and if $E \geq 0$ at every point in the field and for every y', then $\tilde{y}(x)$ gives a strong minimum of (2.1).

In addition to his path-breaking new method, Weierstrass also revolutionized the calculus of variations by stressing—practically for the first time—the importance of a precise definition of the domain $D(J)$ of the functional $J[y]$ and of *admissible functions*, the functions $y \in D(J)$ satisfying the side conditions.

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*The *conjugate point* of x_0 is the first value $x > x_0$ where a nonzero solution of

$$\frac{d}{dx}\left(L_{y'y'}\frac{dw}{dx}\right) - \left(L_{yy} - \frac{d}{dx}L_{yy'}\right)w = 0, \qquad w(x_0) = 0 \quad (x \geq x_0)$$

vanishes. Here $w(x) = \partial y/\partial\alpha|_{\alpha=0}$ and $\alpha = 0$ corresponds to $\tilde{y}$ in the family of extremals $y = y(x, \alpha)$.