Vector-Valued and Noncommutative Analysis

Why UMD Banach spaces?

Definition:

A Banach space E is UMD (unconditional martingale differences) if for all $p \in (1,\infty)$, $\exists \beta_p \geq 0$ such that for any σ-finite measure space $(\Omega, \mathcal{F}, \mu)$, a filtration $(\mathcal{F}_n)_n$, a finite martingale $(f_n)_{n=1}^N$ in $L^p(\Omega; E)$, any scalars $|\varepsilon_n| = 1$,

$$\left\| \sum_{n=1}^N \varepsilon d f_n \right\|_{L^p(\Omega;E)} \leq \beta_p \left\| \sum_{n=1}^N d f_n \right\|_{L^p(\Omega;E)}$$

$$d_n = f_n - f_{n-1}, \quad n \quad 0$$
$$d_0 = f_0$$

Examples. L^p, $p \in (1,\infty)$, Noncommutative L^p, Orlicz, Sobolev spaces.

$L^1, L^\infty, C[0,1], c'$ are not UMD.

Definition:

For an E-valued function f, the Hilbert transform Hf is defined by

$$Hf = \lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \frac{1}{\pi} \int_{\varepsilon < |x-y| < R} \frac{f(y)}{x-y}\, dy.$$

For a predictable sequence $(v_n)_{n\geq 1} \in L(X)$, $s \longmapsto v_n(s)x$ is $\mathcal{F}_{n-1}$-measurable,

$$\|v_n\|_{L^\infty(\Omega;L(X))} = \sup_{\|x\|\leq 1} \|v_n x\|_{L^\infty(\Omega;X)} < \infty$$

If $p \in [1,\infty)$, $f \in L^p(\Omega; X)$, finitely many non-zero difference terms $(\mathbb{E}_n(f) - \mathbb{E}_{n-1}(f) = df_n)$

The martingale transform

$$T_v f = \sum_{n\geq 1} v_n df_n$$

$L^{p,\infty} = \{ f \in L^0 \mid \|f\|_{p,\infty} < \infty \}$ (Weak L^p)

$$\|f\|_{p,\infty} = \sup_{t>0} t^{-1/p} f^*(t)$$

$T: L^p \to L^{p,\infty}$ weak type (p,p)

$T: L^p \to L^p$ strong type (p,p)

Theorem: (Bourgain, Burkholder)

The following are equivalent:

i. E is UMD

ii. E-valued Hilbert transform is weak type $(1,1)$

iii. E-valued martingale transform is weak type $(1,1)$.

Noncommutative Integration

Definition:

A closed $*$-subalgebra of $B(\mathcal{H})$ is a von Neumann algebra if it is closed in SOT.

Theorem: (Bicommutant theorem)

$\mathcal{A} \subseteq B(\mathcal{H})$

$\mathcal{A}' = \{ x \in B(\mathcal{H}) \mid [x,y] = 0, \ \forall y \in \mathcal{A} \}$

$\mathcal{A} \subseteq B(\mathcal{H})$ is a von Neumann algebra if and only if $\mathcal{A} = \mathcal{A}''$

Examples:

$L^\infty(\Omega)$, Ω a σ-finite measure space

Group G, $\mathcal{L}(G)$ the group vNa.

Quantum groups.
Foliations (Connes)
Poisson manifolds
Open quantum systems
$M_n(\mathbb{C})$, $\mathcal{B}(\mathcal{H})$
$L^\infty(\Omega) \bar{\otimes} \mathcal{B}(\mathcal{H})$

Not a Theorem. (Gelfand-Naimark)
The category of commutative C^*-algebras is anti-equivalent to the category of LC Hausdorff spaces.
Not a Corollary:
The category of commutative von Neumann algebras is anti-equivalent to the category of measure spaces.

Theorem: (Segal)
Fix a measure space (Ω, Σ, μ). TFAE
i. Ω is localizable
ii. The Radon-Nikodym theorem holds for Ω *
iii. The Riesz representation theorem holds for functionals on $L^1(\Omega)$
iv. the space $L^\infty(\Omega)$ is a maximal abelian von Neumann subalgebra (MASA) of $\mathcal{B}(L^2(\Omega))$.

Definition:
A projection $p \in \mathcal{B}(\mathcal{H})$ $p = p^* = p^2$.

Factor is a vNa with trivial centre ($Z(M) \cong \mathbb{C}$)

$M = \int^\oplus M_t \, dt$

Classification:

The trace of projections in a gives a linearly ordered set, of three kinds.

type I: $[0, 1, 2, \dots, n]$ $\quad n \in \mathbb{N} \cup \{\infty\}$

type II: $[0, T]$, $\quad T \in \mathbb{R}_+ \cup \{\infty\}$

type III: $\{0, \infty\}$

Substitute trace for integrals on type I / II algebras

$L^\infty(\Omega)$, $f \longmapsto \int f \, d\mu$ is an extended real-valued functional.

$\{x \eta M\}$ $\quad \eta$ = affiliated to M

"unbounded bicommutant" of vNa $M \subseteq B(H)$

Choose a trace τ as an integral

$S(M, \tau) \subseteq \{x \eta M\}$ space of τ-measurable operators

Definition:

$$L^p(M, \tau) = \{x \in S(M, \tau) \mid \|x\|_p < \infty\}$$

$$\|x\|_p^p = \tau\left(|x|^p\right)$$

Example:

$$R = \left(\bigotimes_{\mathbb{N}} (M_2(\mathbb{C}), \mathrm{tr}_2)\right)''$$

$$\mathrm{tr}_2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = 1$$

limit of $\bigotimes_{i=1}^{N} M_2(\mathbb{C})$

Exercise $L^\infty(0,1) \hookrightarrow R$

Theorem: (Calderón-Zygmund)

For $f \in L^1(\mathbb{R}^n)$, $\alpha > 0$, $\exists\, g, b$

i. $f = g + b$

ii. $\|g\|_1 \leq \|f\|_1$, $\|g\|_\infty \leq 2^n \alpha$

iii. $b = \sum_j b_j$, b_j supported on disjoint dyadic blocks

iv. $\displaystyle\int_{Q_j} b_j \, d\lambda = 0$

v. $\|b_j\|_1 \leq 2^{n+1} \alpha |Q_j|$ vi. $\sum_j |Q_j| \leq \alpha^{-1} \|f\|_1$

Theorem. (Cuculescu)

For a finite vNa $\mathcal{M}$, filtration $(\mathcal{M}_n)$, $x = (x_n)_n$ a self-adjoint L^1-bounded martingale, $\forall \lambda > 0$ $\exists\, (q_n^{(\lambda)}) \subseteq \mathrm{Proj}(\mathcal{M})$,

i. $q_n^{(\lambda)} \in \mathcal{M}_n$

ii. $q_n^{(\lambda)}$ commutes with $q_{n-1}^{(\lambda)} x_n q_{n-1}^{(\lambda)}$

iii. $|q_{n-1}^{(\lambda)} x_n q_{n-1}^{(\lambda)}| \leq \lambda q_n^{(\lambda)}$

iv. $q^{(\lambda)} = \bigwedge q_n^{(\lambda)}$

$$\tau(1 - q^{(\lambda)}) \leq \lambda^{-1} \|x\|_1$$

Theorem:

For a finite von Neumann algebra, the NC martingale transform is weak type $(1,1)$, and strong type (p,p), for $p \in (1,\infty)$.