Proposition:

a U-valued Q-Wiener, iff
$$W_t = \sum_{k \in \mathbb{N}} \sqrt{\lambda_k}\, \beta_k(t) e_k,$$

(β_k) a real-valued family of Brownian motions
Q, non-negative symmetric, $\operatorname{tr} Q < \infty$,
$\{e_k\}$ orthonormal basis of U, given by the eigenvectors of Q, such that that the sequence $\{\lambda_k\}$ of eigenvalues is decreasing.

Definition:

For a real valued separable Banach space E, let $(\mathcal{F}_t)_{t \geq 0}$ be a filtration on $(\Omega, \mathcal{F}, P)$, an M_t E-valued stochastic process is an $(\mathcal{F}_t)$-martingale if for all t

i $\mathbb{E}(\|M_t\|_E) < \infty$

ii. M_t is $\mathcal{F}_t$-measurable

iii. $\mathbb{E}_s(M_t) = M_s$ P-a.e. , $0 \leq s \leq t$.

Let $M^2_T(E)$ be the space of square integrable E-valued martingales on $[0, T]$, with norm
$$\|M_t\|_{M^2_T} = \sup_{0 \leq s \leq T} \left(\mathbb{E}\left(\|M_t\|_E^2\right)\right)^{1/2}.$$

Example: Every Q-Wiener W_t,
$$W \in M^2_T(U).$$

Definition:

An $L(U, H)$-valued $\Phi(t)$ over $(\Omega, \mathcal{F}, P)$, is elementary if for
$$0 = t_0 < t_1 < \dots < t_k = T$$
$$\Phi(t) = \sum_{m=0}^{k-1} \Phi^m \chi_{(t_m, t_{m+1}]}(t),$$

for $\Phi^m : \Omega \to L(\mathcal{U}, \mathcal{H})$, $\mathcal{F}_{t_m}$-measurable, wrt to the strong Borel σ-algebra on $L(\mathcal{U}, \mathcal{H})$, and such that Φ^m takes finitely many values in $L(\mathcal{U}, \mathcal{H})$.

Define:

$$\mathrm{Int}(\Phi)_t = \int_0^t \Phi_s \, dW_s = \sum_{m=0}^{k-1} \Phi^m (W_{t_{m+1} \wedge t} - W_{t_m \wedge t})$$

Proposition:

$$\mathrm{Int} : \mathcal{E} \to \mathcal{M}_T^2(\mathcal{H}),$$

where $\mathcal{E}$ is the set of elementary processes.

Proposition: (Itô's Isometry)

The norm:

$$\|\Phi\|_T^2 = \mathbb{E}\left(\int_0^T \|\Phi_s \circ Q^{1/2}\|_2^2 \, ds \right)$$

is equal to

$$\left\| \int_0^{\cdot} \Phi_s \, dW_s \right\|_{\mathcal{M}_T^2}^2$$

such that $\mathrm{Int} : (\mathcal{E}, \|\cdot\|_T) \to \mathcal{M}_T^2$ is an isometry.

Definition:

Let $\bar{\mathcal{E}}$ be the closure of $\mathcal{E}$ under $\|\cdot\|_T$.

$\mathcal{U}_0 = Q^{1/2}(\mathcal{U})$

$L_2^0 = L^2(\mathcal{U}_0, \mathcal{H})$ (Hilbert - Schmidt)

$\|L\|_{L_2^0} = \|L \circ Q^{1/2}\|$

Proposition:

$$\bar{\mathcal{E}} = \left\{ \Phi : [0, T] \times \Omega \to L_2^0 \;\middle|\; \begin{array}{l} \Phi \text{ is predictable} \\ \|\Phi\|_T < \infty \\ \forall t \in [0,T],\ \Phi_t \text{ is} \\ \mathcal{F}_t\text{-measurable} \end{array} \right\}$$

Consider Q non-negative, symmetric.

Recall $U_0 = Q^{1/2}(U)$

Choose Hilbert space U_1, such ~~that~~

$J: U_0 \longrightarrow U_1$

$J: u \longmapsto \sum_{k=1}^{\infty} \alpha_k \langle u, e_k \rangle e_k$, such that $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$.

Then J is Hilbert - Schmidt.

Proposition:

Let (e_k) be an orthonormal basis for U_0, β_k an independent family of $\mathbb{R}$-valued Brownian motion

Define $Q_1 = JJ^* : U_1 \longrightarrow U_1$

Q_1 is non-negative definite, symmetric, $\operatorname{tr} Q_1 < \infty$.

$$W_t = \sum_{k=1}^{\infty} \beta_k J(e_k) \in \mathcal{M}_T^2(U)$$

is a Q_1-Wiener process.

Define W_t is a <u>cylindrical Q-Wiener process</u>.

Definition:

Given a cylindrical Q-Wiener process W_t, a process Φ_t is integrable with respect to W_t if it is predictable, $L^2(Q_1^{1/2}(U_1), H)$-valued, and

$$P\left(\int_0^T \|\Phi_s\|_2^2 \, ds < \infty \right) = 1,$$

taken as a function over Ω.

Define $\int_0^t \Phi_s \, dW_s := \int_0^t \Phi_s \circ J^{-1} \, dW_s$

Gelfand triples

Separable Hilbert space $\mathcal{H}$,

Reflexive Banach space V, V dense in $\mathcal{H}$

$$V \hookrightarrow \mathcal{H} \hookrightarrow V^*$$

$(V, \mathcal{H}, V^*)$ a Gelfand triple.

Let W_t be a cylindrical Q-Wiener process, over Hilbert space $\mathcal{U}$, for $Q = 1$.

We consider SDEs

$$dX_t = A(t, X_t)\,dt + B(t, X_t)\,dW_t.$$

The solution is an $\mathcal{H}$-valued process X_t

$$A : [0, T] \times V \times \Omega \to V^*$$

$$B : [0, T] \times V \times \Omega \to L^2(\mathcal{U}, \mathcal{H}).$$

Require that A, B are progressively measurable; for any $t \in [0, T]$, the restriction to $[0, t] \times V \times \Omega$, is $\mathcal{B}([0, t]) \times \mathcal{B}(V) \times \mathcal{F}_t$ measurable.

We impose four conditions on A, B.

H1 Hemicontinuity

$\forall u, v, w \in V, \omega \in \Omega, t \in [0, T]$

the map $\lambda \in \mathbb{R} \longmapsto \langle A(t, u + \lambda v, \omega), w\rangle$

is continuous.

H2 Weak monotonicity

$\exists c \in \mathbb{R}\ \forall u, v \in V$

$$2\langle A(\cdot, u) - A(\cdot, v), u - v\rangle$$

$$+ \| B(\cdot, u) - B(\cdot, v)\|^2_{L^2(\mathcal{U}, V)}$$

$$\leq c\| u - v\|^2_{\mathcal{H}}$$

over $[0, T] \times \Omega$.

H3: Coercivity

$\exists\, \alpha \in (1,\infty), c_1 \in \mathbb{R}, c_2 \in (0,\infty)$ and an $\mathcal{F}_t$-adapted process

$f \in L^1([0,T]\times\Omega, dt\otimes P)$,

such that for all $v \in V$, $t \in [0,T]$,

$$2\langle A(t,v),v\rangle + \| B(t,v)\|_2^2 \leq$$

$$c_1 \|v\|_{\mathcal{H}}^2 - c_2\|v\|_V^\alpha + f(t)$$

over Ω

H4 Boundedness

$\exists\, c_3 \in (0,\infty)$ $\mathcal{F}_t$-adapted process $g \in L^{\alpha/\alpha-1}([0,T]\times\Omega)$

$\forall v \in V$, $t \in [0,T]$

$$\|A(t,v)\|_{V^*} \leq g(t) + c_3\|v\|_V^{\alpha-1},$$

on Ω, α the same as H_3.

Theorem:

$$dX_t = A(t,X_t)dt + B(t,X_t)\,dW_t$$

For A, B satisfying H1-H4, and if

$X_0 \in L^2(\Omega, \mathcal{F}_0, P; \mathcal{H})$,

then the SDE admits a unique solution.

Proof idea:

Let $\{e_n\}$ an orthonormal basis for $\mathcal{H}$.

$\mathcal{H}_n = \mathrm{span}\{e_1,\dots,e_n\}$

$P_n : V^* \to \mathcal{H}_n \qquad y \mapsto \sum_{j=1}^n \langle y, e_j\rangle e_j$

Let $\{g_n\}$ be an orthonormal basis for $\mathcal{U}$

$$W^{(n)}(t) = \sum_{j=1}^n \langle W_t, g_j\rangle g_j$$

For $g \in \mathcal{U}$, let $\langle W_t, g\rangle = \int_0^t \langle g, \cdot\rangle_{\mathcal{U}}\, dW_s$.

For $n \in \mathbb{N}$, we can solve the following in $\mathcal{H}_n$

$$dX_t^{(n)} = P_n A(t, X_t^{(n)})dt + P_n B(t, X_t^{(n)}) dW_t^{(n)}$$

$$X_0^{(n)} = P_n X_0 .$$

$$K = L^\alpha([0,T] \times \Omega; dt \otimes P; V)$$

There exists a subsequence $(n_k)_{k \geq 0}$

i. $X^{(n_k)} \longrightarrow X$ weakly in K and in $L^2([0,T] \times \Omega; dt \otimes P; \mathcal{H})$

ii. $Y^{(n_k)} = A(\cdot, X^{(n_k)}) \longrightarrow Y$ weakly in K

iii. $Z^{(n_k)} = B(\cdot, X^{(n_k)}) \longrightarrow Z$ weakly in $\mathcal{J} = L^2([0,T] \times \Omega, dt \otimes P, L^2(\mathcal{U}, \mathcal{H}))$

Our solution is

$$X_t = X_0 + \int_0^t Y_s ds + \int_0^t Z_s dW_s .$$