Online Workshop on Stochastic Analysis

III. Solution of SDEs

Example: (Population Dynamics)

$$dP_t = \alpha(t) P_t \, dt + \beta(t) \underline{P_t} \, dW_t ,$$

for Brownian motion W_t.

Equation is separable:

$$\frac{dP_t}{P_t} = \underline{\alpha(t)\, dt} + \underline{\beta(t)\, dW_t}$$

Proposition: (Itô's Lemma 3)

(X_t), defined

$$dX_t = u\,dt + v\,dB_t .$$

For $g \in C^2$,

$$d(g(t, X_t)) = \frac{\partial}{\partial t} g(t, X_t)\,dt + \frac{\partial}{\partial x} g(t, X_t)\,dt$$

$$+ \frac{1}{2} \frac{\partial^2}{\partial x^2} g(t, X_t)(dX_t)^2$$

$$dt^2 = dt\,dB_t = dB_t\,dt = 0$$

$$(dB_t)^2 = dt$$

$$d(\ln P_t) = \frac{dP_t}{P_t} - \frac{1}{2P_t^2}(dP_t)^2$$

$$(dP_t)^2 = (\alpha(t) P_t\,dt + \beta(t) P_t\,dB_t)^2$$

$$= (\alpha(t) P_t\,dt)^2$$

$$+ \alpha_t \beta_t P_t^2 (dt\,dB_t + dB_t\,dt)$$

$$+ (\beta(t) P_t\,dB_t)^2$$

$$= (\beta(t) P_t)^2\,dt$$

$$d(\ln P_t) = \frac{dP_t}{P_t} - \frac{\beta(t)^2}{2}\,dt$$

$$\frac{dP_t}{P_t} = \alpha(t)\,dt + \beta(t)\,dB_t$$

$$d(\ln P_t) = \alpha(t)dt + \beta(t)dB_t - \frac{\beta(t)^2}{2}dt$$

$$\ln P_t - \ln P_0 = \int \alpha(t)dt + \int \beta(t)\,dB_t - \frac{1}{2}\int \beta(t)^2 dt$$

$$P_t = P_0 \exp\left(\int \alpha(t)dt - \frac{1}{2}\int \beta(t)^2 dt + \int \beta(t)dB_t\right)$$

Theorem:

Assume that

$b : \mathbb{R}^n \times [0,T] \longrightarrow \mathbb{R}^n$

$B : \mathbb{R}^n \times [0,T] \longrightarrow \mathbb{M}_{m \times n}$,

continuous, and for $C, D > 0$,

(C1) $|b(x,t) - b(y,t)| \leq C|x-y|$,

$|B(x,t) - B(y,t)| \leq C|x-y|$,

$\forall t \in [0,T]$, $x, y \in \mathbb{R}^n$,

(C2) $|b(x,t)| \leq D(1+|x|)$,

$|B(x,t)| \leq D(1+|x|)$,

let X_0 be an $\mathbb{R}^n$-valued RV, such that

$\mathbb{E}(|X_0|^2) < \infty$,

with X_0 independent of ΣW_0^+, the σ-algebra generated by $\{W_t - W_s \mid s \geq 0\}$.

There then exists a unique $X \in L^2([0,T] \times \Omega; \mathbb{R}^n)$,

$$dX_t = b(X,t)dt + B(X,t)dW_t.$$

THEOREM. *Suppose now that $n = 1$ but $m \geq 1$ is arbitrary. The solution of*

$$(19) \qquad \begin{cases} dX = (c(t) + d(t)X)dt + \sum_{l=1}^{m}(e^l(t) + f^l(t)X)dW^l \\ X(0) = X_0 \end{cases}$$

is

$$(20) \qquad \begin{aligned} X(t) = \Phi(t)\Bigg(X_0 &+ \int_0^t \Phi(s)^{-1}\left(c(s) - \sum_{l=1}^{m} e^l(s)f^l(s)\right) ds \\ &+ \int_0^t \sum_{l=1}^{m} \Phi(s)^{-1} e^l(s)\, dW^l\Bigg), \end{aligned}$$

where

$$\Phi(t) := \exp\left(\int_0^t d - \sum_{l=1}^{m} \frac{(f^l)^2}{2}\, ds + \int_0^t \sum_{l=1}^{m} f^l\, dW^l\right).$$

Motivation:

Given an insulated 1-D rod, of finite length L.
Initial temperature, at $x \in [0, L]$, given by $f(x)$,
let $u(x,t)$ be temperature at $x \in [0, L]$, time t.

i. Density of heat is proportional to temp.

ii. The rate of flow of heat between points is proportional to their difference in temp.

$$\frac{d}{dt} \int_0^L u(x,t)\,dx = \eta^2 \left[\partial_x u(L,t) - \partial_x u(0,t)\right].$$

$$\int_0^L \partial_t u(x,t)\,dx = \eta^2 \int_0^L \partial_x^2 u(x,t)\,dx$$

By differentiation:

$$\partial_t u = \eta^2 \partial_x^2 u.$$

Assume there is a source of random heat transfer:

$$\partial_t u = \eta^2 \partial_x^2 u + f$$

IV Vector-Valued Analysis

Fix Banach space X, with Borel σ-algebra $\mathcal{B}(X)$, and measure space $(\Omega, \mathcal{F}, \mu)$.

Definition:

A function $f : \Omega \to X$, is simple if

$$f = \sum_{i=1}^{n} x_i \chi_{E_i},$$

$x_i \in X$, $E_i \in \mathcal{F}$.

A function $f : \Omega \to X$, is called μ-measurable if there exists a sequence (f_n) of simple functions,

$\lim_{n \to \infty} \| f_n - f \|_X = 0$ μ-a.e.

We say that a μ-measurable function is Bochner integrable, if for some sequence (f_n) of simple functions

$$\lim_{n\to\infty}\int_\Omega \|f_n - f\|_X d\mu = 0.$$

Then, for $E\in\mathcal{F}$,

$$\int_E f d\mu = \lim_{n\to\infty}\int_E f_n d\mu$$

$$= \lim_{n\to\infty}\sum_{j=1}^{m_n}\mu(E_{j,n})x_{j,n}.$$

For two Hilbert spaces $\mathcal{H}, \mathcal{U}$.

Definition:

For Borel σ-algebra $\mathcal{B}(\mathcal{U})$, a measure μ on $(\mathcal{U},\mathcal{B}(\mathcal{U}))$ is <u>Gaussian</u> if for all $v\in\mathcal{U}$, the functional

$$v' : u \longmapsto \langle u, v\rangle, \quad (u\in\mathcal{U})$$

has Gaussian law. There exist constants $m\in\mathbb{R}$, $\sigma\geq 0$,

i. If $\sigma = 0$, $\mu\circ(v')^{-1} = \delta_m$.

ii. If $\sigma > 0$,

$$(\mu\circ(v')^{-1})\cdot A = \mu(v'\in A)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}\int_A e^{-\frac{(x-m)^2}{2\sigma^2}}dx,$$

for all $A\in\mathcal{B}(\mathbb{R})$.

Theorem:

A measure μ on $(\mathcal{U},\mathcal{B}(\mathcal{U}))$ is Gaussian if and only if

$$\hat{\mu}(u) = \int_U e^{i\langle u,v\rangle}\mu(dv) = e^{i\langle m,u\rangle - \frac{1}{2}\langle Qu,u\rangle},$$

$m\in\mathcal{U}$, $Q\in L(\mathcal{U})$

Q is non-negative, symmetric, $\mathrm{tr}\, Q < \infty$.
Denote Q by $N(m, Q)$
- mean m
- covariance operator Q

Properties:

i. For any $g, h \in U$
$$\int_U \langle x, h \rangle \mu(dx) = \langle m, h \rangle.$$

ii. $$\int_U (\langle x, h \rangle - \langle m, h \rangle)(\langle x, g \rangle - \langle m, g \rangle) \mu(dx) = \langle Q h, g \rangle.$$

iii. $$\int \| x - m \|^2 \mu(dx) = \mathrm{tr}(Q)$$

Proposition:

For $m \in U$, $Q \in L(U)$ non-negative, symmetric, finite trace, let $\{e_k\}$ be an orthonormal basis for U, given by eigenvectors of Q, such λ_k is a decreasing of eigenvalues of Q.

A U-valued RV X over $(\Omega, \mathcal{F}, P)$, is Gaussian with
$$P \circ X^{-1} = N(m, Q)$$
if and only if
$$X = \sum_{k \in \mathbb{N}} \sqrt{\lambda_k}\, \beta_k e_k + m \in U$$
for $P \circ \beta_k^{-1} = N(0, 1)$, for all k, with $\lambda_k > 0$.

Definition:

A $\mathcal{U}$-valued stochastic process (W_t) is a (standard) Q-Wiener process if

i. $W_0 = 0$

ii. W_t has P-a.e. continuous paths

iii. W_t has independent increments

iv. For all $0 \le s \le t$

$$P \circ (W_t - W_s)^{-1} = N(0, (t-s)Q),$$

for Q non-negative, symmetric, and trace class in $L(\mathcal{U})$.