Online Workshop on Stochastic Analysis

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Course outline:

Day 1 – Stochastic processes
Stochastic integrals
Semi-martingales
Itô's lemma
Example SDEs, weak and strong solutions

Day 2 – Stochastic analysis in L^2
Solution of the stochastic heat equation

Day 3 – Vector-valued integration
Characterisation of UMD Banach spaces
Banach space valued stochastic analysis

Day 4 – von Neumann algebras and integration
Semi-commutative analysis
Calderón–Zygmund type decompositions

Motivating examples:

i. Consider a population growth model:

$$\frac{dP}{dt} = a(t)P(t)$$

We expect there to be random fluctuations over time. As such, let

$$a(t) = b(t) + \text{"noise"},$$

where $b(t)$ is deterministic.

This suggests two problems:

a. What kind of random process is the noise?

b. How can we integrate "against a random process"?

ii. Consider a financial market. Typically, "white noise", a continuous process models the random changes, however, unexpected events (such as a pandemic) may cause sudden large shifts in prices.

iii. An electrical circuit satisfies the equation

$$L\cdot Q''(t) + R\cdot Q'(t) + C^{-1} Q(t) = F(t),$$

for inductance L
resistance R
capacitance C
charge Q
potential source F

The potential source may suffer both from random noise, and power spikes / surges.

iv. Long term measurements may drift due to noise. (Allegedly, a nearby emu farm was a substantial source of interference for LIGO in Australia).

Consider tracking a satellite by measuring acceleration. The accumulation of errors will cause the measured position to drift. Can we "closely" approximate the true position?

I. Martingales and Stochastic Processes

Definition:

For a given probability space (Ω, Σ, μ), and a linearly ordered index set T (e.g. $\mathbb{N}$, $\mathbb{R}_+$, $[0,t]$) a filtration on Ω is an increasing family of σ-algebras $(\Sigma_t)_{t\in T}$, $(s \le t \Rightarrow \Sigma_s \subseteq \Sigma_t)$.

We assume (Ω,μ) is complete. Furthermore, the filtration is said to satisfy the "usual hypotheses" if
i. Σ_0 contains all μ-null sets of Σ
ii. The filtration is right continuous,
$$\Sigma_t = \bigcap_{u>t} \Sigma_u .$$

We always assume the usual hypotheses.
For $t \in T$, let $\mathcal{E}_t(X) = \mathcal{E}(X/\Sigma_t)$ denote the conditional expectation onto Σ_t.

A stochastic process is a parameterised collection of random variables (μ-measurable functions) $\{X_t\}_{t\in T}$ over (Ω,μ).

- While for fixed $t \in T$, $X_t(\omega)$ is a random variable over Ω, we may also fix $\omega \in \Omega$, and instead consider the <u>path</u>
$$t \longmapsto X_t(\omega).$$
As such, we take $X_t(\omega)$ to be measurable in $T\times\Omega$.
- Identifying $\omega \in \Omega$ with the path
$$t \longmapsto X_t(\omega) \in \mathbb{R}^T$$
induces a probability measure on $\tilde{\Omega} = \mathbb{R}^T$ through the embedding
$$\Omega \hookrightarrow \tilde{\Omega} = \mathbb{R}^T.$$
$$\omega \longmapsto (X_t(\omega) : T \to \mathbb{R})$$

Given stochastic processes $\{X_t\}_{t\in T}$, $\{Y_t\}_{t\in T}$, over Ω, say that:

- $\{X_t\}_{t\in T}$ is a <u>modification</u> of $\{Y_t\}_{t\in T}$ if for each $t\in T$,
$$\mu\{\omega\in\Omega \,/\, X_t(\omega) = Y_t(\omega)\} = 1 \quad (X_t = Y_t \text{ a.s.})$$

- $\{X_t\}_{t\in T}$ is <u>indistinguishable</u> from $\{Y_t\}_{t\in T}$ if for almost all $t\in T$, $X_t = Y_t$.

A stochastic process $\{X_t\}_{t\in T}$ is said to be <u>adapted</u> to the filtration $(\Sigma_t)_{t\in T}$ if for each $t\in T$, X_t is Σ_t-measurable.

A <u>martingale</u> is an adapted real-valued stochastic process $\{X_t\}_{t\in T}$, such that for any $t\in T$,

i. $\mathbb{E}(|X_t|) < \infty$

ii. $\mathcal{E}_s(X_t) \overset{a.e.}{=} X_s$, for all $s\le t$.

If we replace condition (ii), $\{X_t\}_{t\in T}$ is a

ii'. $\mathcal{E}_s(X_t) \le X_s$ - <u>super-martingale</u>

ii''. $\mathcal{E}_s(X_t) \ge X_s$ - <u>sub-martingale</u>

Example:

i. Given a fair coin, let X_n denote the value of the n^{th} coin flip. $\{X_n\}_{n\in\mathbb{N}}$ is a martingale.

ii. Given $f\in L^2(\Omega,\Sigma,\mu)$, let $X_t = \mathcal{E}_t(f)$. Then $\{X_t\}_{t\in T}$ is a martingale, with limit f. Is the converse true?

Theorem: (Kolmogorov's Extension Theorem)

For all $t_1,\dots,t_k\in T$, $k\in\mathbb{N}$, if there exist probability measures $\nu_{t_1,\dots,t_k}$ on $\mathbb{R}^{nk}$, such that

i. for all permutations $\sigma\in S_k$

$$\nu_{t_{\sigma(1)},\dots,t_{\sigma(k)}}(F_1\times\cdots\times F_k) = \nu_{t_1,\dots,t_k}(F_{\sigma^{-1}(1)}\times\cdots\times F_{\sigma^{-1}(k)}) \qquad (K1)$$

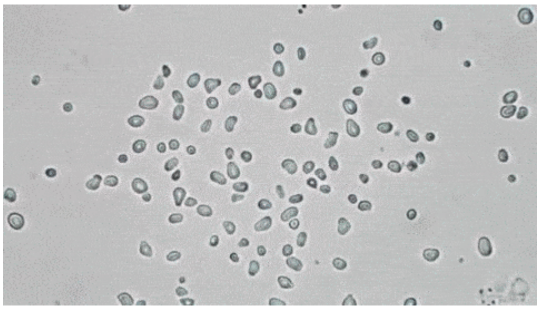
ii. for all $m\in\mathbb{N}$

$$\nu_{t_1,\dots,t_k}(F_1\times\cdots\times F_k) = \nu_{t_1,\dots,t_k,t_{k+1},\dots,t_m}(F_1\times\cdots\times F_k\times\mathbb{R}^n\times\cdots)$$

then there exists a probability space $(\Omega,\Sigma,\mathbb{P})$ and an $\mathbb{R}^n$-valued stochastic process $\{X_t\}_{t\in T}$ on $\mathbb{R}^n$, such that

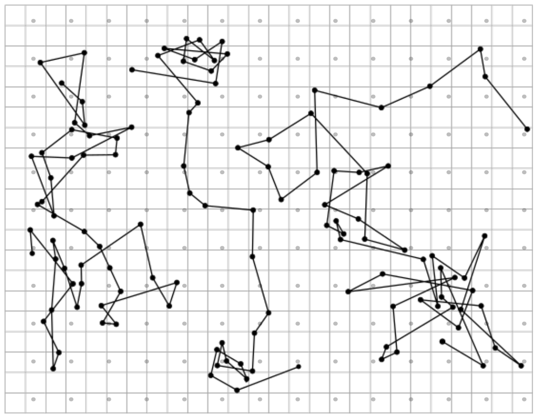
$$\nu_{t_1,\dots,t_k}(F_1 \times \dots \times F_k) = \mathbb{P}(X_{t_1} \in F_1, \dots, X_{t_k} \in F_k)$$

for all $t_1, \dots, t_k \in T$, $k \in \mathbb{N}$, Borel sets $F_1, \dots, F_k \subseteq \mathbb{R}^n$.



In 1828, Robert Brown noticed that pollen grains on the surface of water move irregularly. We can model this motion

If we describe the behaviour of particles as measures satisfying (K1, K2), we may then extend to a stochastic process.



i. Fix $x \in \mathbb{R}^n$, define

$$p(t; x, y) = \frac{1}{\sqrt{(2\pi t)^n}} \cdot \exp\left(-\frac{|x-y|^2}{2t}\right)$$

for all $y \in \mathbb{R}^n$, $t \in (0, \infty)$.

ii. For any $0 \leq t_1 \leq \dots \leq t_k$, define a measure $\nu_{t_1,\dots,t_k}$ on $\mathbb{R}^{n \times k}$ by $\nu_{t_1,\dots,t_k}(F_1 \times \dots \times F_k) =$

$$\int_{F_1 \times \dots \times F_k} p(t_1; x, x_1)\, p(t_2 - t_1; x_1, x_2) \cdots p(t_k - t_1; x_{k-1}, x_k)\, dx_1 \cdots dx_k,$$

with respect to the Lebesgue measure on $\mathbb{R}^k$, where

$$p(0; x, y)\, dy = \delta_x(y),$$

for unit point mass δ.

Definition:

By Kolmogorov's extension theorem, there exists a stochastic process $\{B_t\}_{t\in[0,\infty)}$ on a probability space $(\Omega, \Sigma, \mathbb{P}^x)$, with joint distributions

$$\mathbb{P}^x(B_{t_1}\in F_1, \dots, B_{t_k}\in F_k) =$$
$$\int_{F_1\times\cdots\times F_k} p(t_1; x, x_1)\, p(t_2 - t_1, x_1, x_2)\cdots p(t_k - t_{k-1}, x_{k-1}, x_k)\, dx_1\cdots dx_k.$$

Call $\{B_t\}$ a <u>Brownian motion</u> starting at $x\in\mathbb{R}^n$.

Remark:

i. The joint distribution $\mathbb{P}^x$ does not uniquely define $\{B_t\}$.

ii. We may choose a modification of $\{B_t\}$ such that the sample path $t\mapsto B_t(\omega)$ are continuous for almost all ω.

iii. It follows that Brownian motion is a probability measure $\mathbb{P}^x$ on $C([0,\infty), \mathbb{R}^n)$. Call this the <u>canonical Brownian motion.</u>

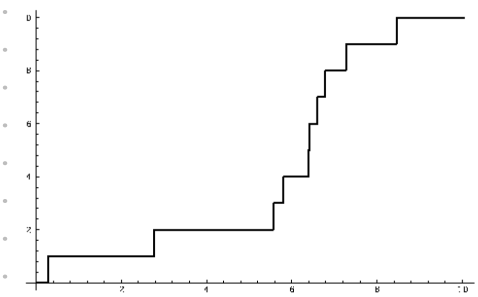
Properties:

i. $\{B_t\}$ is a Gaussian process, for any $t_1,\dots,t_k$, the joint distribution of $B_{t_1},\dots,B_{t_k}$ is a $n\times k$ dimensional multi-normal distribution.

ii. $\{B_t\}$ has independent increments. For $0\le t_1\le\cdots\le t_k$,

$B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_k} - B_{t_{k-1}}$

are independent random variables.



Let us now consider discontinuous processes. For example, a random number of customers arrive at a shop at random times.

Definition:

A <u>stopping time</u> is a random variable
$\tau : \Omega \longrightarrow [0, \infty]$,
such that $\{\omega \in \Omega : \tau(\omega) \leq t\} \in \Sigma_t$ for all t.
For stochastic process $\{X_t\}$, the <u>stopped process</u> $\{X_{\tau \wedge t}\}$ is given by
$X_{(\tau\wedge t)(\omega)}(\omega)$, for all $\omega \in \Omega$.

Let $(T_n)_{n\geq 0}$ be a strictly increasing sequence of positive random variables.
The process $(N_t)_{t\in T} : \Omega \longrightarrow \mathbb{N} \cup \{\infty\}$, given by
$$N_t(\omega) = \sum_{n\geq 0} \chi_{\{t \geq T_n\}}(\omega),$$
where χ_E is the indicator function over the set Ω, is called a <u>counting process</u>.

Theorem:

A counting process is adapted if and only if the associated random variables $\{T_n\}_{n\geq 0}$ are stopping times.

An adapted counting process is said to be a <u>Poisson process</u> if

i. For any $0 \leq s < t$,
$N_t - N_s$ is independent of Σ_s;

ii. For any $0 \leq s < t$, $0 \leq u < v$, $v - u = t - s$ implies that N_{v-u} has the same distribution as N_{t-s}.

Theorem:

For any Poisson process $\{N_t\}$, and any $t \in T$, there exists $\lambda \geq 0$ such that

$$P(N_t = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!},$$

for $n \geq 0$. So, a Poisson process has Poisson distribution at time t, with parameter λt.

We are interested in processes which account for both continuous noise, and random jumps.

Definition:

An adapted process $(X_t)_{t\in T}$ is said to be a <u>Lévy process</u> if $X_0 \overset{a.e.}{=} 0$,

i. $(X_t)_{t\in T}$ has increments independent of the past, in that for any $0 \leq s < t$, $X_t - X_s$ is independent of Σ_s;

ii. $(X_t)_{t\in T}$ has stationary increments; for $0 \leq s < t$, $X_t - X_s$ has the same distribution as X_{t-s};

iii. $(X_t)_{t\in T}$ is continuous in probability;
$\lim_{s\to t} X_s = X_t$ with probability 1.

A stochastic process is said to be <u>càdlàg</u> (or <u>RCLL</u>) if for almost every $\omega \in \Omega$, the path $t \mapsto X_t(\omega)$ is right continuous (RC) and all left limits exist (LL).

This is an abbreviation of

continue à droite, limites à gauche.

Theorem:

Every Lévy process admits a unique modification to a càdlàg Lévy process.

For a càdlàg process, time $t > 0$, let
$$X_{t-} = \lim_{s \to t^-} X_s,$$
and then let $\Delta X_t = X_t - X_{t-}$ denote any jump.

Theorem:

Let (X_t) be a Lévy process, with jumps bounded by a ($\sup_{s \geq 0} |\Delta X_s| \leq a$ a.s.), and set

$$Z_t = X_t - \mathbb{E}(X_t).$$

i. (Z_t) is a martingale.

ii. $Z_t = Z_t^c + Z_t^d$

iii. (Z_t^c) is a martingale with continuous paths.

iv. (Z_t^d) is a martingale defined by
$$Z_t^d = \int_{\{|x| \leq a\}} x \big(N_t(\cdot, dx) - t\nu(dx) \big)$$
for counting process (N_t), "Lévy measure" ν.
(See Protter for details).

v. (Z_t^c) and (Z_t^d) are independent Lévy processes.

Theorem: (Lévy Decomposition Theorem)

Given Lévy process (X_t), there exists a Brownian motion (B_t), a poisson process (N_t) which is independent of (B_t), and Lévy measure ν such that

$$X_t = B_t + \alpha t + \sum_{0 < s \leq t} \Delta X_s \chi_{\{|\Delta X_s| \geq 1\}} + \int_{\{|x| < 1\}} x \big(N_t(\cdot, dx) - t\nu(dx) \big)$$

II. Stochastic Integration and Differential Equations

Definition:

A stochastic process (H_t) is said to be a simple predictable process, if

$$H_t = H_0 \chi_{\{0\}}(t) + \sum_{j=0}^{n} H_j \chi_{(T_j, T_{j+1}]}(t),$$

for random variables $H_j \in L^\infty(\Omega, \Sigma_{T_j})$, and a finite collection of stopping times

$$0 = T_0 \leq \cdots \leq T_{n+1} < \infty.$$

Let $\mathbb{S}$ denote the collection of simple predictable processes. Let $\mathbb{S}_u$ denote $\mathbb{S}$, taken with the topology of uniform convergence over $T \times \Omega$. Also, let $\mathbb{L}^0$ denote the space of "finite random variables" (real valued measurable functions) over Ω, considered with the topology given by convergence in probability.

We would like stochastic integrals

i. to be linear;

ii. to satisfy the bounded convergence theorem.

For a given stochastic process X, define

$$I_X : \mathbb{S} \longrightarrow \mathbb{L}^0$$

by

$$I_X(H) = H_0 X_0 + \sum_{j=0}^{n} H_i (X_{T_{j+1}} - X_{T_j}).$$

Definition:

A stochastic process (X_t) is called a <u>total semimartingale</u> if X is càdlàg, adapted, and $I_X : \mathbb{S} \longrightarrow \mathbb{L}^0$ is continuous.

A process $(X_t)_{t \in T}$ is called a <u>semimartingale</u> if for each $t \in T$, $(X_s)_{s \in [0,t)}$ is a total semimartingale.

Example:

i. Any adapted process with càdlàg paths, and finite variation on compact intervals is a semimartingale.

ii. Every L^2 - martingale with càdlàg paths is a semimartingale. (e.g. Brownian motion).

iii. Definition:

An adapted process X, with càdlàg paths is said to be <u>decomposable</u> if

$$X_t = X_0 + M_t + A_t,$$

where $M_0 = A_0 = 0$, M is a locally square-integrable martingale, and A is càdlàg, adapted, and has paths of finite variation over compact sets.

Theorem:

Every decomposable process is a semimartingale.

Corollary:

Every Lévy process is a semimartingale.

Can we integrate processes which are not simple adapted?

Definition:

Let $\mathbb{D}$ denote the space of adapted processes with càdlàg paths, and $\mathbb{L}$ the space of adapted processes with càglàd (RLLC) paths. Then let $b\mathbb{L}$ be the subspace of processes in $\mathbb{L}$ with bounded paths.

Definition:

A sequence $(H^n)_{n\geq 0}$ of stochastic processes converges to a process H <u>uniformly on compacts in probability</u> (ucp) if for each $t > 0$,

$$\sup_{0\leq s\leq t} |H_s^n - H_s|$$

converges to 0 in probability.

Denote

$$H_t^* = \sup_{0 \le s \le t} |H_s|.$$

Theorem:

The space S is dense in $\mathbb{L}$ under the ucp topology.

Definition:

Given $H \in S$ and a càdlàg process X, define

$$J_X : S \longrightarrow \mathbb{D}$$

by

$$J_X(H) = H_0 X_0 + \sum_{j=0}^{n} H_j (X_{T_{j+1}} - X_{T_j}),$$

for stopping times $0 = T_0 \le T_1 \le \cdots \le T_{n+1} < \infty$.

We then call $J_X(H)$ the <u>stochastic integral</u> of H with respect to X. $J_X(H)$ may also be denoted by

$$J_X(H) = \int H_s \, dX_s = H \cdot X.$$

Theorem:

For any semimartingale, the mapping

$$J_X : S_{ucp} \longrightarrow \mathbb{D}_{ucp}$$

is continuous.

By density under the ucp topology, and since $\mathbb{D}_{ucp}$ is a complete metric space, we may extend

$$J_X : S \longrightarrow \mathbb{D}$$

to

$$J_X : \mathbb{L} \longrightarrow \mathbb{D}$$

by linearity.

Example:

What is $\int_0^t B_s \, dB_s$?

Let B denote a standard Brownian motion, with $B_0 = 0$.
Let $(\pi_n)_{n=1}^{\infty}$ be a refining sequence of partitions of $[0,\infty)$, with $\lim_{n\to\infty} \mathrm{mesh}(\pi_n) = 0$.

- $B^n = \sum_{t_k \in \pi_n} B_{t_k} \chi_{(t_k, t_{k+1}]}$
- $(B^n)_n$ converges to B in the ucp topology.
- If $t \in \pi_n$ for all sufficiently large n,
$$\mathcal{J}_B(B^n)_t = \sum_{\substack{t_k \in \pi_n \\ t_k < t}} B_{t_k}(B_{t_{k+1}} - B_{t_k})$$
- So:
$$\mathcal{J}_B(B)_t = \lim_{n\to\infty} \mathcal{J}_B(B^n)_t$$
$$= \lim_{n\to\infty} \sum_{\substack{t_k \in \pi_n \\ t_k < t}} B_{t_k}(B_{t_{k+1}} - B_{t_k})$$
$$= \lim_{n\to\infty} \sum_{\substack{t_k \in \pi_n \\ t_k < t}} \Big\{ \frac{1}{2}(B_{t_{k+1}} + B_{t_k})(B_{t_{k+1}} - B_{t_k}) - \frac{1}{2}(B_{t_{k+1}} - B_{t_k})(B_{t_{k+1}} - B_{t_k}) \Big\}$$
$$= \frac{1}{2} B_t^2 - \frac{1}{2} \lim_{n\to\infty} \sum_{\substack{t_k \in \pi_n \\ t_k < t}} (B_{t_{k+1}} - B_{t_k})^2 .$$
$$= \frac{1}{2} B_t^2 - \frac{1}{2} t \quad \text{[By variance of Brownian motion]}$$

That is to say that
$$\int_0^t B_s \, dB_s = \frac{1}{2}(B_t^2 - t).$$

Notation

For a stochastic process (Y_t), define (Y_-) by
$$(Y_-)_s = \lim_{u \to s^-} Y_u. \quad [(Y_-)_0 = 0]$$

Definition:

Let X and Y be semimartingales.

The quadratic variation process of X is
$$[X,X] = ([X,X]_t)_{t\in T},$$
defined by
$$[X,X] = X^2 - 2\int X_- dX.$$
The quadratic covariation (or the bracket process) of X and Y is defined by
$$[X,Y] = XY - \int X_- dY - \int Y_- dX.$$
Corollary: (Integration by parts)
$$XY = \int X_- dY + \int Y_- dX + [X,Y].$$

Definition:

Given a semimartingale X, let $[X,X]^c$ denote the path-by-path continuous part of $[X,X]$, such that
$$[X,X]_t = [X,X]_t^c + \sum_{0\le s\le t} (\Delta X_s)^2.$$

Theorem: (Change of Variables)

For a finite variation process X, with right continuous paths, $f\in C^1$ such that f' is continuous, then $(f(X_t))_{t\in T}$ is again a finite variation process, and
$$f(V_t) - f(V_0)$$
$$= \int_{(0,t]} f'(V_{s-})dV_s + \sum_{0<s\le t}\{f(V_s) - f(V_{s-}) - f'(V_s)\Delta V_s\}.$$

Theorem: (Itô's Lemma)

Let X be a semimartingale, and f a real C^2 function. Then, $f(X)$ is a semimartingale, and

$$f(X_t) - f(X_0) = \int_{(0,t]} f'(X_{s-})\,dX_s + \frac{1}{2}\int_{(0,t]} f''(X_{s-})\,d[X,X]^c_s + \sum_{0<s\le t}\left\{ f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s \right\}.$$

Corollary:

If X is a continuous semimartingale, and f is a real C^2 function, then $f(X)$ is a semimartingale and

$$f(X_t) - f(X_0) = \int_{(0,t]} f'(X_s)\,dX_s + \frac{1}{2}\int_{(0,t]} f''(X_s)\,d[X,X]_s.$$

Tomorrow we will discuss the solution of SDEs.

To conclude, let us better understand what a semimartingale is.

Definition:

A stochastic process X is said to be a <u>local martingale</u> if it is càdlàg, and there exists an increasing sequence $(T_n)_{n\ge 0}$ of stopping times, such that $\lim_{n\to\infty} T_n = \infty$ a.e.

and

$(X_{t\wedge T_n})\chi_{\{T_n>0\}}$

is a uniformly integrable martingale for each n.

$$\left[\lim_{n\to\infty} \sup_{t\ge 0} \int_{\{|Y_n|\ge n\}} |Y_n|\,dP = 0\right]$$

Definition:

An adapted càdlàg process A is of <u>finite variation</u> if almost all paths $t \mapsto A_t(\omega)$ are of finite variation on compact intervals.

Theorem: (Bichteler – Dellacherie)

Let X be an adapted càdlàg process. The following are equivalent:

i. X is a semimartingale.

ii. X is decomposable, in that there exists a locally square integrable local martingale M, and a process A of finite variation, such that $M_0 = 0 = A_0$, and

$$X_t = X_0 + M_t + A_t.$$