The free stochastic calculus of Biane and Speicher

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What this talk is about

These talks are intended as an introduction to the paper:

P. Biane & R. Speicher.
Stochastic calculus with respect to free Brownian motion and analysis on Wigner space.
Probab. Theory Related Fields **112** (1998), no. 3, 373–409.

Which covers the analogy in free probability of Itô calculus and some aspects of Malliavin calculus. Important earlier papers include Fagnola (1991) [4] and Vincent-Smith (1997) [7].

What these talks are about

My interest in this paper stems from its essential use of the theory of double operator integrals.
A double operator integral is a formal expression

$$\mathcal{T}_{\phi}^{A,B}(X) = \iint_{\mathbb{R}^2} \phi(\lambda,\mu)\, dE^A(\lambda) X dE^B(\mu).$$

Biane and Speicher's version of Itô's lemma can be stated as

$$f(X_t) = f(X_0) + \int_0^t \mathcal{T}_{f^{[1]}}^{X_s,X_s}(dX_s) + \frac{1}{2}\int_0^t \Delta_s f(X_s)\, ds$$

where $(X_t)_{t>0}$ is a free Brownian motion, and $\Delta_s f$ is a second order term that can be defined with a triple operator integral.

Plan for these talks

1. Some background on free probability.
2. Free stochastic processes, in particular the free Brownian motion.
3. Integration of biprocesses and the Itô isometry.
4. Formulation of the free Itô lemma and related details.

Free probability

Free probability theory is an analogy of probability theory for noncommutative algebras, originating with D. V. Voiculescu in the 1980s. Voiculescu wanted to get a better understanding of the von Neumann algebras $L(\mathbb{F}_n)$, the free group factors. His insights led to a large number of advances in that area. Later, it was realized that free probability is very important in the study of large random matrices.

Algebraic probability

Traditional probability theory is based on measurable spaces equipped with probability measures $(\Omega, \Sigma, \mathbb{P})$.
The idea is that one considers σ-algebras of events $E \in \Sigma$, and their probabilities

$$E \mapsto \mathbb{P}(E).$$

A measurable $\mathbb{C}$-valued function on Ω is called a random variable, and the expected value $\mathbb{E}(X)$ is defined as the integral of X with respect to $\mathbb{P}$.
It is also possible to take random variables as being the primitive notion, and consider algebras $\mathcal{A}$ of random variables with a linear map τ taking the role of the expectation.

Algebraic probability

The basic ingredients of algebraic probability theory are:

- An associative unital $*$-algebra $\mathcal{A}$ (an "algebra of random variables")
- A faithful state τ on $\mathcal{A}$ (an "expectation function") That is, a linear functional on $\mathcal{A}$ such that

$$\tau(x^*x) \geq 0, \quad x \in \mathcal{A}$$

  with equality if and only if $x = 0$, and

$$\tau(1) = 1.$$

Algebraic probability

The main commutative example is to take

$$\mathcal{A} = L_\infty(\Omega, \Sigma)$$

and

$$\tau(f) = \mathbb{E}(f) = \int_\Omega f \, d\mathbb{P}.$$

W^*-probability spaces

In the sequel we will assume that all our algebraic probability spaces are actually von Neumann algebras.

Definition

A W^*-probability space is a *finite von Neumann algebra* $(\mathcal{A}, \tau)$. That is, $\mathcal{A}$ is a von Neumann algebra (a unital $*$-subalgebra of bounded linear operators on some Hilbert space H, closed in the weak operator topology). We equip $\mathcal{A}$ with a finite faithful normal trace $\tau \in \mathcal{A}^*$.These adjectives mean:

- Finite: $\tau(1) = 1$.
- Faithful: $\tau(x^*x) = 0$ if and only if $x = 0$.
- Normal: If $\{x_i\}_{i \in I}$ is a directed family of operators in $\mathcal{A}$ such that $\langle \xi, x_i \xi \rangle_H \uparrow \langle \xi, x\xi \rangle_H$ for all $\xi \in H$ we have

$$\tau(x_i) \uparrow \tau(x).$$

Definition

(cont.) Finally we need that τ is a trace. That is, $\tau(u^*xu) = \tau(x)$ for all unitary elements $u \in \mathcal{A}$. Equivalently,

$$\tau(xy) = \tau(yx), \quad x, y \in \mathcal{A}.$$

More generally, if τ is a trace we have

$$\tau(x_1 x_2 \cdots x_n) = \tau(x_n x_1 x_2 \cdots x_{n-1}), \quad x_1, \ldots, x_n \in \mathcal{A}.$$

That is, $\tau(x_1 x_2 \cdots x_n)$ is invariant under all cyclic permutations of $(x_1, \ldots, x_n)$.

The commutative case

Theorem (Segal (1953))

Let $(\mathcal{A}, \tau)$ be a W^*-probability space such that $\mathcal{A}$ is abelian. There exists a probability space $(\Omega, \Sigma, \mathbb{P})$ such that $\mathcal{A} = L_\infty(\Omega)$ and

$$\tau(f) = \mathbb{E}(f) = \int_\Omega f\, d\mathbb{P}.$$

More examples

Let $n \geq 1$, and $\mathcal{A} = M_n(\mathbb{C})$ (the algebra of all complex $n \times n$ matrices). If we take $\tau = \frac{1}{n}\mathrm{Tr}$, then $(\mathcal{A}, \tau)$ is a W^*-probability space.
We can also consider algebras of random matrices, $\mathcal{A} = L_\infty(\Omega) \otimes M_n(\mathbb{C})$, with $\tau = \frac{1}{2}\mathbb{E} \otimes \mathrm{Tr}$.

The GNS construction

The trace τ defines an inner product on $\mathcal{A}$,

$$\langle x, y\rangle_\tau = \tau(x^* y), \quad x, y \in \mathcal{A}.$$

The completion of $\mathcal{A}$ with this inner product is a Hilbert space denoted $L_2(\mathcal{A}, \tau)$ (or just $L_2(\tau)$.)

We can also define L_p-spaces $L_p(\tau)$ with norm

$$\|x\|_p = \tau(|x|^p)^{\frac{1}{p}}, x \in \mathcal{A}$$

where $|x| = (x^*x)^{\frac{1}{2}}$. It turns out that $\|x+y\|_p \leq \|x\|_p + \|y\|_p$ for $1 \leq p < \infty$.

Definition

For $1 \leq p < \infty$, we define $L_p(\tau)$ to be the completion of $\mathcal{A}$ with the norm $\|\cdot\|_p$.

We write $L_\infty(\tau) = \mathcal{A}$.

The noncommutative dictionary

Instead of probability spaces $(\Omega, \Sigma, \mathbb{P})$, we consider von Neumann algebras $\mathcal{A}$ with finite faithful normal traces τ (corresponding to $L_\infty(\Omega)$.) The L_p-spaces $L_p(\mathbb{P})$ are replaced with $L_p(\tau)$. Random variables (measurable functions on Ω) are replaced with elements of $\mathcal{A}$, or more generally operators on $L_2(\tau)$ affiliated with $\mathcal{A}$. Random events $E \in \Sigma$ are replaced with projections $e \in \mathcal{A}$, and $\tau(e)$ is the probability of an event. Sub σ-algebras of Σ are instead replaced with subalgebras of $\mathcal{A}$.

Noncommutative random variables

If $X = X^* \in \mathcal{A}$, we can consider the moments

$$\tau(X^k), \quad k = 0, 1, 2, \dots$$

More generally, if f is a continuous function on the spectrum of X then $f(X) \in \mathcal{A}$. By the Riesz theorem, there exists a unique measure ν_X on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} f \, d\nu_X = \tau(f(X)), \quad f \in C_0(\mathbb{R}).$$

The measure ν_X is equal to

$$d\nu_X = \tau(dE_X)$$

where dE_X is the spectral measure of X.

Independence

Let $(\mathcal{A}, \tau)$ be a W^*-probability space. What does it mean for two random variables in $\mathcal{A}$ to be independent?
In traditional probability theory, we say that $E, F \in \Sigma$ are independent if

$$\mathbb{P}(E \cap F) = \mathbb{P}(E)\mathbb{P}(F).$$

More generally, if Σ_1, Σ_2 are σ-subalgebras of Σ, then we say that Σ_1 and Σ_2 are independent if

$$\mathbb{P}(E_1 \cap E_2) = \mathbb{P}(E_1)\mathbb{P}(E_2), \quad E_1 \in \Sigma_1,\ E_2 \in \Sigma_2.$$

Algebraic Independence

Independence can also be stated in terms of random variables. Two random variables X, Y on (Ω, Σ) are independent if they generate independent sub-σ-algebras of Σ.
Equivalently, real-valued variables X and Y are independent if

$$\mathbb{E}(f(X)g(Y)) = \mathbb{E}(f(X))\mathbb{E}(g(Y))$$

for all Borel functions f and g such that $f(X), g(Y) \in L_2(\mathbb{P})$. Another equivalent perspective is to say that Σ_1 and Σ_2 are independent if and only if

$$\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$$

for all Σ_1-measurable random variables X and Σ_2-measurable random variables Y.
What should this be for algebraic probability theory?

Tensor Independence

Let $(\mathcal{A}, \tau)$ be a W^*-probability space, and let $\mathcal{A}_1, \mathcal{A}_2$ be W^*-subalgebras of $\mathcal{A}$. Mimicking the traditional situation, we might like to say that $\mathcal{A}_1$ and $\mathcal{A}_2$ are independent if

$$\tau(X_1 X_2) = \tau(X_1)\tau(X_2), \quad X_1 \in \mathcal{A}_1, X_2 \in \mathcal{A}_2.$$

But this is not really adequate, since this does not tell us how to compute τ on the subalgebra $\langle \mathcal{A}_1 \cup \mathcal{A}_2 \rangle$ of $\mathcal{A}$ generated by $\mathcal{A}_1$ and $\mathcal{A}_2$.

Free independence

Definition

Say that subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are *freely independent* (or just free) if

$$\tau(X_1 Y_1 \cdots X_n Y_n) = 0$$

whenever $X_j \in \mathcal{A}_1$, $Y_j \in \mathcal{A}_2$ are such that $\tau(X_j) = \tau(Y_j) = 0$. Elements $X, Y \in \mathcal{A}$ are said to be freely independent if they generate freely independent subalgebras of $\mathcal{A}$.

Free independence in the commutative situation

Free independence formally resembles classical independence, but is actually far stronger! Suppose that X and Y are symmetric freely independent random variables in $(\mathcal{A}, \tau)$, and $XY = YX$. Subtracting $\tau(X)1$ from X and $\tau(Y)1$ from Y if necessary, we may assume that X and Y have mean zero. Then freeness implies

$$\tau(XYXY) = 0.$$

Since $XY = YX$ we get

$$\tau(X^2Y^2) = 0.$$

Freeness again implies that

$$\tau((X^2 - \tau(X^2)1)(Y^2 - \tau(Y^2)1)) = 0 \Rightarrow \tau(X^2Y^2) = \tau(X^2)\tau(Y^2) = 0.$$

Therefore $\tau(X^2) = 0$ or $\tau(Y^2) = 0$. It follows that one of X or Y must be constant.

Tensor independence

It turns out that there are several ways of defining independence in algebraic probability theory.

Definition

We say that subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are *tensor independent* if we have

$$\tau(X_1Y_1X_2Y_2\cdots X_nY_n) = \tau(X_1\cdots X_n)\tau(Y_1\cdots Y_n), \quad X_j \in \mathcal{A}_1, Y_j \in \mathcal{A}_2.$$

Elements $X, Y \in \mathcal{A}$ are said to be tensor independent if they generate tensor independent subalgebras.

There are a handful of others depending on what is considered to be valid. Tensor independence is the one that reduces to the traditional notion of independence in the commutative case.

Free independence and free products

If $(\mathcal{A}_1, \tau_1)$ and $(\mathcal{A}_2, \tau_2)$ are W^*-probability spaces, their free product $(\mathcal{A}_1 * \mathcal{A}_2, \tau_1 * \tau_2)$ is a von Neumann algebra generated by all expressions of the form

$$\sum_{j=1}^{n} a_{1,n}^{1} a_{2,n}^{2} \cdots a_{2,n}^{k_n}, \quad a_{1,j}^{k} \in \mathcal{A}_1, a_{2,j}^{k} \in \mathcal{A}_2$$

with no further relations.

Theorem

If $\mathcal{A}_1$ and $\mathcal{A}_2$ are freely independent subalgebras of the W^*-probability space $(\mathcal{A}, \tau)$, then the subalgebra of $\mathcal{A}$ generated by $\mathcal{A}_1$ and $\mathcal{A}_2$ is isomorphic to the free product $\mathcal{A}_1 * \mathcal{A}_2$.
If $\mathcal{A}_1$ and $\mathcal{A}_2$ are tensor independent, then the subalgebra is isomorphic to the von Neumann tensor product $\mathcal{A}_1 \overline{\otimes} \mathcal{A}_2$.

In this sense, if X and Y are freely independent, then there are "no algebraic relations" between X and Y.

Free vs. classical independence

Free independence is not a generalisation of classical independence, instead it is an *analogy*. Free independence has formal similarities to classical independence that turn out to be useful and appropriate in the general noncommutative setting, but it is actually far stronger than independence in the classical sense.

One explanation for this difference comes from category theory; the free product is the coproduct in the category of associative algebras, the tensor product is the coproduct in the category of commutative associative algebras.

Free probability theory is important for a number of reasons, in particular it has deep and interesting combinatorial aspects which I will not touch on.

Noncommutative random variables

Recall that if μ and ν are probability measures on $\mathbb{R}$, there is a measure $\mu * \nu$ (the convolution), which is the probability measure of the sum of two independent random variables with distributions μ and ν respectively. An analogy for freely independent random variables exists, the *free convolution*, but it is much more challenging to compute.

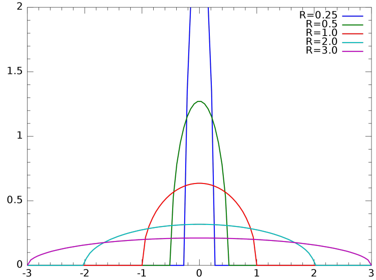
Noncommutative random variables

In classical probability theory, the Gaussian distributions play a distinguished role

$$d\gamma_\sigma(t) = (2\pi\sigma^2)^{-\frac{1}{2}} \exp(-\frac{1}{2\sigma^2}t^2)\,dt.$$

In free probability theory, instead what is important is the semicircular (or Wigner) distribution

$$d\nu_R(t) = \frac{2}{\pi R^2}(R^2 - t^2)^{\frac{1}{2}}\chi_{|t|<R}\,dt$$



The free central limit theorem

Definition

A sequence $\{x_n\}_{n=0}^\infty$ of self-adjoint random variables converges in distribution to the probability measure ν if

$$\lim_{n\to\infty} \tau(f(x_n)) = \int_{\mathbb{R}} f\, d\nu, \quad f \in C_b(\mathbb{R}).$$

This is the same as saying that the sequence of distributions $\{\nu_{x_n}\}_{n=0}^\infty$ converges weakly to ν.

The free central limit theorem

Theorem (Voiculescu)

Let $\{x_n\}_{n=0}^{\infty}$ *be a sequence of freely independent and identically distributed self-adjoint random variables in a* W^**-probability space* $(\mathcal{A},\tau)$*. Put*

$$\tau(x_n) = 0, \quad \tau(x_n^2) = \frac{R^2}{4}$$

and

$$z_n := n^{-\frac{1}{2}} \sum_{k=0}^{n-1} x_k, \quad n \geq 1.$$

Then $\{z_n\}_{n=1}^{\infty}$ *converges in distribution to a semicircular distribution with radius* R*.*

Free stochastic calculus

What about a free stochastic calculus? Some of the basic requirements are as follows:

1. A good notion of stochastic process
2. A free Wiener process (Brownian motion)
3. A theory of free stochastic integration
4. A free Itô lemma
5. A theory of free SDE
6. A free Wiener chaos decomposition & and Malliavin calculus.

I will cover the first four points, Biane and Speicher did much more.

Stochastic processes in free probability

Let $(\mathcal{A}, \tau)$ be a W^*-probability space. An $\mathcal{A}$-valued stochastic process should be a function

$$[0, \infty) \to \mathcal{A}$$

which is measurable in some sense.
A *filtration* of $\mathcal{A}$ is a nested family $(\mathcal{A}_s)_{s>0}$ of von Neumann subalgebras such that $\bigcup_{s>0} \mathcal{A}_s$ is dense in $\mathcal{A}$ in the weak operator topology.

Definition

A stochastic process $(X_t)_{t>0}$ valued in $\mathcal{A}$ is said to be adapted to the filtration $(\mathcal{A}_t)_{t>0}$ if $X_t \in \mathcal{A}_t$ for all $t > 0$.

Free Wiener process

Recall that a Wiener process $\{W_t\}_{t\geq 0}$ on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a family of random variables such that for all $t > s$ we have

1. W_t is $\mathcal{F}_t$-measurable and $W_0 = 0$.
2. $W_t - W_s$ is a Gaussian r.v. with mean zero and variance $t - s$.
3. For $t > s$, the increment $W_t - W_s$ is independent of $\mathcal{F}_s$.

A *free Wiener process* is a family $\{X_t\}_{t>0}$ is a family of square-integrable random variables adapted to $(\mathcal{A}_t)_{t\geq 0}$ such that for all $t > s$,

1. $X_t \in \mathcal{A}_t$ and $X_0 = 0$.
2. $X_t - X_s$ is a semicircular r.v. with mean zero and variance $t - s$.
3. $X_t - X_s$ is freely independent of $\mathcal{A}_s$.

Free Wiener process

Theorem

A free Wiener process exists.

Proof (sketch).

Let $H = L_2[0,\infty)$, and let $\mathcal{H}$ be the free Fock space

$$\mathcal{H} = \mathbb{C}\Omega \oplus H \oplus (H \otimes H) \oplus (H \otimes H \otimes H) \oplus \cdots = \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} H^{\otimes n}.$$

Here, Ω is a fixed unit vector. For $h \in H$, the annihilation operator $a(h)$ is defined by

$$a(h)(\xi_1 \otimes \cdots \xi_n) = \langle h, \xi_1 \rangle \otimes \xi_2 \cdots \xi_n, \quad \xi_1 \otimes \cdots \otimes \xi_n \in H^{\otimes n}$$

and $a(h)\Omega = 0$.

Free Wiener process (cont.)

(cont.)

Define

$$X_t = \frac{1}{2}(a(\chi_{[0,t]}) + a(\chi_{[0,t]})^*).$$

Let $\mathcal{A}_t$ denote the von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $\{X_s\}_{s\leq t}$, and $\mathcal{A} = \overline{\bigcup_{t>0} \mathcal{A}_t}$. Define

$$\tau(x) = \langle \Omega, x\Omega \rangle.$$

Then

$$\{X_t\}_{t\geq 0}$$

is a free Wiener process adapted to the filtered noncommutative probability space

$$(\mathcal{A}, (\mathcal{A}_t)_{t\geq 0}, \tau).$$

□

Free stochastic integrals

We now wish to define integrals with respect to the free Wiener process. Formally these should be expressions of the shape

$$\int_0^T Y_t dX_t = \lim \sum_j Y_{t_j}(X_{t_{j+1}} - X_{t_j}).$$

But this is not enough, because we can also consider integrals of the shape

$$\int_0^T (dX_t) Y_t = \lim \sum_j (X_{t_{j+1}} - X_{t_j}) Y_{t_j}$$

using *right multiplication*, since the multiplication is noncommutative.

Biprocesses

A key insight is that it is best to define *two-sided* integrals

$$\int_0^T Y_t \sharp dX_t$$

where Y_t is a so-called *biprocess*, and the integral a limit of sums of the form

$$\sum_{j,k} A^k_{t_j}(X_{t_{j+1}} - X_{t_j})B^k_{t_j}, \quad A^k_t \in \mathcal{A}_t,\ B^k_t \in \mathcal{A}^{\mathrm{op}}_t.$$

This level of generality is necessary to get a good Itô calculus.

Biprocesses

Why do we need biprocesses?
Consider $d(X_t^2)$. This should be

$$d(X_t^2) = X_t dX_t + (dX_t)X_t + (dX_t)^2 = X_t dX_t + (dX_t)X_t + dt.$$

So that (we should expect that)

$$X_t^2 = X_0^2 + \int_0^t X_s dX_s + (dX_s)X_s + t.$$

Simple adapted biprocesses

Definition

Let $(\mathcal{A}, (\mathcal{A}_t)_{t>0}, \tau)$ be a filtered noncommutative probability space. A simple biprocess $(Y_t)_{t>0}$ is a piecewise constant function

$$Y : [0, \infty) \to \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}.$$

where $\mathcal{A}^{\mathrm{op}}$ is the opposite algebra. We say that Y is a simple adapted biprocess if $Y_t \in \mathcal{A}_t \otimes \mathcal{A}_t^{\mathrm{op}}$ for all $t > 0$. Here $\otimes$ is the *algebraic* tensor product.

More concretely, if Y is a simple biprocess then there exist simple processes A^j, B^j valued in $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ respectively such that

$$Y = \sum_{j=1}^{n} A^j \otimes B^j.$$

If A_t^j and B_t^j belong to $\mathcal{A}_t$ and $\mathcal{A}_t^{\mathrm{op}}$ respectively, then Y is adapted.

Integration of simple adapted biprocesses

The multiplication mapping $\sharp$, is defined by

$$(a \otimes b)\sharp x := axb, \quad a \in \mathcal{A}, b \in \mathcal{A}^{\mathrm{op}}, x \in L_2(\mathcal{A}, \tau).$$

This has a unique linear extension to an algebra homomorphism

$$\sharp : \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \to \mathcal{B}(L_2(\mathcal{A})).$$

(This is why the second factor has an op.) Note that $\sharp$ does *not* extend to the completed von Neumann tensor product

$$\mathcal{A} \overline{\otimes} \mathcal{A}^{\mathrm{op}}.$$

Instead, it only extends to the integral tensor product

$$\mathcal{A} \hat{\otimes}_i \mathcal{A}^{\mathrm{op}}.$$

Integration of adapted biprocesses

Let Y be a simple adapted biprocess. It is possible to choose a decompostion

$$Y = \sum_{j=1}^{n} A^j \otimes B^j$$

where A^j and B^j are $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$-valued processes respectively such that there exist times $0 = t_0 \leq t_1 \leq \cdots \leq t_m$ with

$$A^j_t = A^j_{t_k} \in \mathcal{A}_{t_k}, \quad B^j_t \in \mathcal{A}^{\mathrm{op}}_{t_k}, \quad t_k \leq t < t_{k+1}$$

and $A_t = 0, B_t = 0$ when $t > t_m$.

Definition

If Y is a simple adapted biprocess, let

$$\int_0^\infty Y_s \sharp dX_s := \sum_{j=1}^{n} \sum_{k=0}^{m} A^j_{t_k}(X_{t_{k+1}} - X_{t_k}) B^j_{t_k}.$$

The Hilbert space $\mathscr{B}_2$.

Recall how the Itô integral works:

- First we define the integral $\int_0^T X_t \, dW_t$ for simple processes adapted to $\{W_t\}_{t>0}$,
- Then the fundamental Itô isometry is proved,

$$\mathbb{E}\left(\int_0^T X_t \, dW_t \int_0^T Y_t \, dW_t\right) = \int_0^T \mathbb{E}(X_t Y_t) \, dt.$$

- The Itô integral is then extended by continuity to a linear map

$$\int_0^T \cdot \, dW_t : L_{2,\mathrm{ad}}((0,T) \times \Omega, (W_t)_{t>0}) \to L_2(\Omega).$$

If we want to mimic this, we need to figure out a version of the space $L_{2,\mathrm{ad}}((0,T) \times \Omega, (W_t)_{t>0})$.

The Hilbert space $\mathscr{B}_2((0,T),(\mathcal{A}_t)_{t>0})$

Biane and Speicher propose the following:

Definition

Let Y and Z be simple adapted biprocesses. Define the inner product:

$$\langle Y, Z\rangle_{\mathscr{B}_2(0,T)} := \int_0^T \langle Y_t, Z_t\rangle_{L_2(\tau\otimes\tau^{\mathrm{op}})}\,dt.$$

The space $\mathscr{B}_2((0,T),(\mathcal{A}_t)_{t>0})$ is the completion of the linear space of all simple adapted biprocesses with respect to the norm induced by this inner product.

The free Itô isometry

Let Y be a simple adapted biprocess, adapted to the filtration $\{\mathcal{A}_t\}_{t>0}$. For $T>0$, define

$$\mathcal{I}_T(Y) := \int_0^T Y_t \sharp dX_t \in \mathcal{A}_T.$$

Theorem (Free Itô isometry)

For all simple adapted biprocesses Y and Z, we have

$$\langle \mathcal{I}_T(Y), \mathcal{I}_T(Z)\rangle_{L_2(\tau)} = \langle Y, Z\rangle_{\mathscr{B}_2((0,T),(\mathcal{A}_t)_{t>0})}.$$

Hence, there is a unique continuous extension

$$\mathcal{I}_T : \mathscr{B}_2((0,T),(\mathcal{A}_t)_{t>0}) \to L_2(\mathcal{A}_T,\tau).$$

This is the *free Itô integral* with respect to the free Wiener process.

Proof of the free Itô isometry

By bilinearity, it suffices to prove the assertion when Y and Z are of the form

$$Y_t = (A \otimes B)\chi_{(t_1,t_2)}(t), Z_t = (C \otimes D)\chi_{(t_3,t_4)}(t)$$

where $A \otimes B \in \mathcal{A}_{t_1} \otimes \mathcal{A}_{t_1}^{\mathrm{op}}$ and $C \otimes D \in \mathcal{A}_{t_3} \otimes \mathcal{A}_{t_3}^{\mathrm{op}}$. Then

$$\mathcal{I}_T(Y) = A(X_{t_1} - X_{t_2})B, \quad \mathcal{I}_T(Z) = C(X_{t_3} - X_{t_4})D.$$

So

$$\begin{aligned}\langle \mathcal{I}_T(Y), \mathcal{I}_T(Z)\rangle &= \tau(B^*(X_{t_1} - X_{t_2})A^*C(X_{t_3} - X_{t_4})D) \\ &= \tau((X_{t_1} - X_{t_2})A^*C(X_{t_3} - X_{t_4})DB^*).\end{aligned}$$

Proof of the free Itô isometry

Free independence implies that

$$
\begin{aligned}
&\tau((X_{t_1}-X_{t_2})A^*C(X_{t_3}-X_{t_4})DB^*) \\
&= \tau((X_{t_1}-X_{t_2})(X_{t_3}-X_{t_4}))\tau(A^*C)\tau(DB^*) \\
&= \lambda((t_1,t_2)\cap(t_3,t_4))\tau(A^*C)\tau(DB^*) \\
&= \int_0^T \chi_{(t_1,t_2)}(t)\chi_{(t_3,t_4)}(t)\tau(A^*C)\tau(DB^*)\,dt \\
&= \int_0^T \chi_{(t_1,t_2)}(t)\chi_{(t_3,t_4)}(t)(\tau\otimes\tau^{\mathrm{op}})((A\otimes B^*)(C\otimes D))\,dt \\
&= \langle Y,Z\rangle_{\mathscr{B}_2((0,T),(\mathcal{A}_t)_{t>0})}.
\end{aligned}
$$

where λ is Lebesgue measure. This completes the proof.

Itô calculus

The next step in developing a free stochastic calculus is to have an Itô lemma. Recall that the original Itô lemma states that if F is a good enough function, then

$$F(W_t) = F(0) + \int_0^t F'(W_s)dW_s + \frac{1}{2}\int_0^t F''(W_s)\,ds.$$

Often remembered by the mnemonic $dW^2 = dt$.

The traditional Itô formula is proved by writing

$$F(W_t) = F(W_0) + \sum_{k=0}^{n-1} F(W_{t_{k+1}}) - F(W_{t_k})$$

and using the 2-term Taylor formula with remainder term

$$\begin{aligned}
F(W_t) = F(W_0) &+ \sum_{k=0}^{n-1} F'(W_{t_k})(W_{t_{k+1}} - W_{t_k}) \\
&+ \sum_{k=0}^{n-1} \frac{1}{2} F''(W_{t_k})(W_{t_{k+1}} - W_{t_k})^2 \\
&+ \sum_{k=0}^{n-1} \frac{1}{2} \int_0^1 (1-\theta)^2 F'''(W_{t_k}(1-\theta) + W_{t_{k+1}}\theta)(W_{t_{k+1}} - W_{t_k})^3 \, d\theta
\end{aligned}$$

Itô calculus

Finally we replace $(W_{t_{k+1}} - W_{t_k})^2$ with $t_{k+1} - t_k$.

$$
\begin{aligned}
F(W_t) = F(W_0) &+ \sum_{k=0}^{n-1} F'(W_{t_k})(W_{t_{k+1}} - W_{t_k}) + \sum_{k=0}^{n-1} \frac{1}{2} F''(W_{t_k})(t_{k+1} - t_k) \\
&+ \sum_{k=0}^{n-1} F''(W_{t_k})((W_{t_{k+1}} - W_{t_k})^2 - (t_{k+1} - t_k)) \\
&+ \sum_{k=0}^{n-1} \frac{1}{2} \int_0^1 (1-\theta)^2 F'''(W_{t_k}(1-\theta) + W_{t_{k+1}}\theta)(W_{t_{k+1}} - W_{t_k})^3 \, d\theta
\end{aligned}
$$

The final two terms vanish, in the $L_2(\Omega)$ and almost sure sense respectively, as the partition $\{0 \leq t_1 \leq t_2 \leq \cdots \leq t_n\}$ becomes finer. We're going to try and replicate this in the free setting.

Duhamel's formula

For technical reasons, I am going to prove the free Itô formula for $F(x) = \exp(i\xi x)$. The general case can be recovered by Fourier's inversion formula. So we need to compute

$$e^{i\xi X_t} - e^{i\xi X_0}.$$

This is done with *Duhamel's formula*. Let A and B be bounded linear operators. We have

$$e^{i\xi(A+B)} - e^{i\xi A} = i\xi \int_0^1 e^{i\xi(A+B)(1-\theta)} B e^{i\xi A\theta}\, d\theta.$$

Applying this twice yields

$$\begin{aligned} e^{i\xi(A+B)} - e^{i\xi A} &= i\xi \int_0^1 e^{i\xi A(1-\theta)} B e^{i\xi A\theta}\, d\theta \\ &- \xi^2 \int_{\theta_0+\theta_1+\theta_2=1} e^{i\xi A\theta_0} B e^{i\xi A\theta_1} B e^{i\xi A\theta_2}\, d\theta \\ &- i\xi^3 \int_{\theta_0+\theta_1+\theta_2+\theta_3=1} e^{i\xi(A+B)\theta_0} B e^{i\xi A\theta_1} B e^{i\xi A\theta_2} B e^{i\xi\theta_3}\, d\theta. \end{aligned}$$

The free Itô formula

Now write

$$e^{i\xi X_t} = e^{i\xi X_0} + \sum_{k=0}^{n-1} e^{i\xi X_{t_{k+1}}} - e^{i\xi X_{t_k}}$$

and apply Duhamel's formula. Let us abbreviate

$$\Delta X_{t_k} := X_{t_{k+1}} - X_{t_k}$$

This leads to

$$e^{i\xi X_t} = e^{i\xi X_0} + \sum_{k=0}^{n-1} i\xi \int_0^1 e^{i\xi X_{t_k}(1-\theta)} \Delta X_{t_k} e^{i\xi X_{t_k}\theta}\, d\theta$$

$$+ \sum_{k=0}^{n-1} -\xi^2 \int_{\theta_0+\theta_1+\theta_2=1} e^{i\xi X_{t_k}\theta_0} \Delta X_{t_k} e^{i\xi X_{t_k}\theta_1} \Delta X_{t_k} e^{i\xi\theta_2 X_{t_k}}\, d\theta$$

$$+ \text{ third order term.}$$

The third order term is an integral over a 3-simplex, and it vanishes as the partition becomes finer (for reasons I will not explain).

The free Itô formula

The difficult term is

$$\sum_{k=0}^{n-1} -\xi^2 \int_{\theta_0+\theta_1+\theta_2=1} e^{i\xi X_{t_k}\theta_0}\Delta X_{t_k} e^{i\xi X_{t_k}\theta_1}\Delta X_{t_k} e^{i\xi\theta_2 X_{t_k}}\, d\theta.$$

In the commutative case, this corresponds to $\sum_{k=0}^{n-1} F''(W_{t_k})(\Delta W_{t_k})^2$.
This is approximated by $(\Delta W_{t_k})^2 \approx (t_{k+1}-t_k)$.
Here, what happens is that

$$(\Delta X_{t_k})e^{i\xi\theta_2 X_{t_k}}(\Delta X_{t_k}) \approx \tau(e^{i\xi\theta_2 X_{t_k}})(t_{k+1}-t_k).$$

Using free independence, it can be proved that

$$\left\| \sum_{k=0}^{n-1} e^{i\xi X_{t_k}\theta_0}\Delta X_{t_k} e^{i\xi X_{t_k}\theta_1}\Delta X_{t_k} e^{i\xi X_{t_k}\theta_2} - e^{i\xi(\theta_0+\theta_2)X_{t_k}}\tau(e^{i\xi\theta_1 X_{t_k}})(t_{k+1}-t_k) \right\|_{L_2(}$$

vanishes as the partition becomes finer, uniformly in θ. So we get

$$\begin{aligned} e^{i\xi X_t} = e^{i\xi X_0} &+ \sum_{k=0}^{n-1} i\xi \int_0^1 e^{i\xi X_{t_k}(1-\theta)}\Delta X_{t_k} e^{i\xi X_{t_k}\theta}\,d\theta \\ &+ \sum_{k=0}^{n-1} -\xi^2 \int_{\theta_0+\theta_1+\theta_2=1} e^{i\xi\theta_0 X_{t_k}}\tau(e^{i\xi\theta_1 X_{t_k}})e^{i\xi\theta_2 X_{t_k}}(t_{k+1}-t_k)\,d\theta \\ &+ \text{ terms which vanish in the limit.} \end{aligned}$$

The free Itô formula

Taking the limit, we get

$$e^{i\xi X_t} = e^{i\xi X_0} + \int_0^t A_t \sharp dX_t + \frac{1}{2}\int_0^t B_t\,dt$$

where A_t is the biprocess

$$A_t = i\xi \int_0^1 e^{i\xi X_t(1-\theta)} \otimes e^{i\xi X_t\theta}\,d\theta$$

and B_t is the process

$$\begin{aligned} B_t &= -2\xi^2 \int_{\theta_0+\theta_1+\theta_2=1} e^{i\xi\theta_0 X_t}\tau(e^{i\xi\theta_1 X_t})e^{i\xi\theta_2 X_t})\,d\theta \\ &= -\xi^2 \int_0^1 e^{i\xi(1-\theta)X_t}\tau(e^{i\xi\theta X_t})\,d\theta. \end{aligned}$$

This implies (via Fourier inversion) the formula stated at the beginning.

The Itô correction term

That is, if F is an extremely nice function (say, Schwartz class) we have

$$F(X_t) = F(X_0) + \int_0^t \partial F(X_s)\sharp dX_s + \frac{1}{2}\int_0^t \Delta_s F(X_s)\,ds.$$

where

$$\partial F(X_s) = (2\pi)^{-1}\int_{-\infty}^{\infty}\int_0^1 i\xi\widehat{F}(\xi)e^{i\xi(1-\theta)X_s}\otimes e^{i\xi\theta X_s}\,d\theta d\xi$$

and

$$\begin{aligned}\Delta_s F(X_s) &= (2\pi)^{-1}\int_{-\infty}^{\infty}\int_0^1 -i\xi^2\widehat{F}(\xi)e^{i\xi(1-\theta)X_s}\tau(e^{i\xi\theta X_s})\,d\theta d\xi \\ &= \int_{-\infty}^{\infty}\int_0^1 F''((1-\theta)X_s+\theta\lambda)\,d\theta d\nu_s(\lambda)\end{aligned}$$

where ν_s is the distribution of X_s (semicircular distribution with radius $2s$).

Suggested further reading

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Thank you for listening!