22. Let D be a domain and l be a closed line segment lying in D. In elementary topology it is shown that l can be covered by a finite number of open disks that lie in D and have their centers on l. Use this fact to prove that any two points in a domain D can be joined by a polygonal path in D having all its segments parallel to the coordinate axes.

23. The notion of "connectedness" also applies to closed sets. We say that a closed set $S \subset \mathbf{C}$ is *connected* if it cannot be written as the union of two nonempty disjoint (nonintersecting) closed sets. A closed connected set is called a *continuum*. Determine which of the following sets is a continuum:

   (a) $\{z : |z-3| = 4\}$
   (b) $\{z : |z| = 1\} \cup \{z : |z| = 3\}$
   (c) $\{1, -1, i\}$
   (d) $\{z : |z-1| \geq 2\}$

24. Prove Theorem 1 by completing the following steps.

   (a) Show that any line segment can be parametrized by
   $$x = at + b, \qquad y = ct + d,$$
   where a, b, c, and d are real constants and t ranges between 0 and 1. Hence the values of u along a line segment lying in D are given by
   $$U(t) := u(at + b, ct + d), \qquad 0 \leq t \leq 1.$$
   (b) Use assumption (1) of Theorem 1 and the chain rule to show that $dU/dt = 0$ for $0 \leq t \leq 1$ and thus conclude that u is constant on any line segment in D.
   (c) By appealing to the definition of connectedness, argue that u must have the same value at any two points of D.

1.7 The Riemann Sphere and Stereographic Projection

For centuries cartographers have struggled with the problem of how to represent the spherical-like surface of Earth on a flat sheet of paper, and a variety of useful projections have resulted. In this section we describe one such method that identifies points on the surface of a sphere with points in the complex plane; namely, the so-called *stereographic projection*. While this projection is not one typically found in atlases, it is of considerable importance in the theory of complex variables.

To describe the stereographic projection, we consider the unit sphere in 3 dimensions (x_1, x_2, x_3) whose equation is given by
$$x_1^2 + x_2^2 + x_3^2 = 1.$$
A sketch of this sphere and its equatorial plane is given in Fig. 1.21.

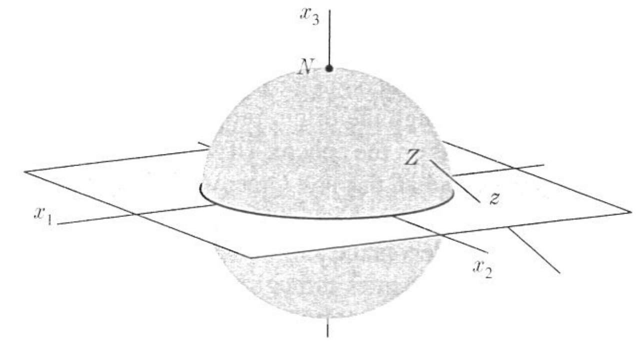

Figure 1.21 The Riemann sphere.

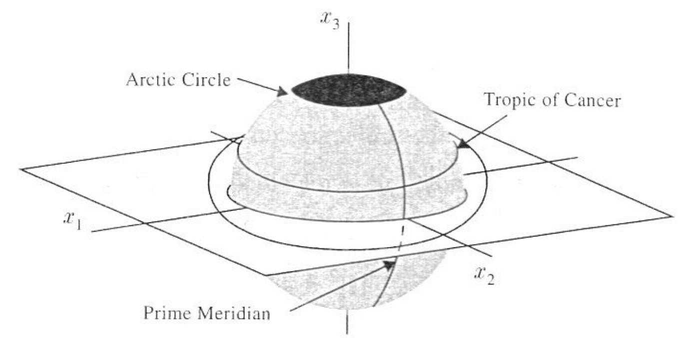

Figure 1.22 The Riemann geosphere.

Our goal is to associate with each point z in the equatorial plane a unique point Z on the sphere. For this purpose we construct the line that passes through the north pole $N = (0, 0, 1)$ of the sphere and the given point z in the x_1x_2-plane. This line pierces the spherical surface in exactly one point Z as shown in Fig. 1.21, and we say that the point Z is the stereographic projection of the point z. If we identify the equatorial plane as the complex plane (or z-plane), the unit sphere is called the *Riemann sphere*.†

Continuing with this geographical interpretation of the Riemann sphere, we note that under stereographic projection, points on the unit circle $|z| = 1$ (in the z-plane) remain fixed (that is, $z = Z$), forming the equator. Points outside the unit circle (for which $|z| > 1$) project to points in the northern hemisphere, while those inside the unit circle (for which $|z| < 1$) project to the southern hemisphere. In particular, the origin of the z-plane projects to the south pole of the Riemann sphere.

Figure 1.22 displays other features of the stereographic projection. Circles of latitude (parallel to the equator) on the sphere are the projections of circles centered at the origin in the z plane; the Arctic Circle and the Tropic of Cancer are projections

†Bernhard Riemann (1826–1866).

12. Show that the n points $z_0^{1/n}$ form the vertices of a regular n-sided polygon inscribed in the circle of radius $\sqrt[n]{|z_0|}$ about the origin.

13. Show that $\omega_3 = (-1 + \sqrt{3}i)/2$ and that $\omega_4 = i$. Use these values to verify identity (5) for the special cases $n = 3$ and $n = 4$.

14. Let m and n be positive integers that have no common factor. Prove that the set of numbers $(z^{1/n})^m$ is the same as the set of numbers $(z^m)^{1/n}$. We denote this common set of numbers by $z^{m/n}$. Show that

$$z^{m/n} = \sqrt[n]{|z|^m}\left[\cos\frac{m}{n}(\theta + 2k\pi) + i\sin\frac{m}{n}(\theta + 2k\pi)\right] \tag{9}$$

for $k = 0, 1, 2, \ldots, n-1$.

15. Use the result of Prob. 14 to find all the values of $(1-i)^{3/2}$.

16. Show that the real part of any solution of $(z+1)^{100} = (z-1)^{100}$ must be zero.

17. Let m be a fixed positive integer and let l be an integer that is not divisible by m. Prove the following generalization of Eq. (5):

$$1 + \omega_m^l + \omega_m^{2l} + \cdots + \omega_m^{(m-1)l} = 0.$$

18. Show that if α and β are nth and mth roots of unity, respectively, then the product $\alpha\beta$ is a kth root of unity for some integer k.

19. (*Electric Field*) A uniformly charged infinite rod, standing perpendicular to the z-plane at the point z_0, generates an electric field at every point in the plane. The intensity of this field varies inversely as the distance from z_0 to the point and is directed along the line from z_0 to the point.

(a) Show that the (vector) field at the point z is given by the function $F(z) = 1/(\bar{z} - \overline{z_0})$, in appropriate units. (Recall Fig. 1.13, Sec 1.3.)

(b) If three such rods are located at the points $1+i$, $-1+i$, and 0, find the positions of equilibrium (that is, the points where the vector sum of the fields is zero).

20. Write a computer program for solving the quadratic equation

$$az^2 + bz + c = 0, \qquad a \neq 0.$$

Use as inputs the real and imaginary parts of a, b, c and print the solutions in both rectangular and polar form.

21. Some complex integer square roots can be obtained by a modification of the polynomial factoring strategy. For example, if $3 + 4i$ equals $(a + bi)^2$, then $4 = 2ab$ and $3 = a^2 - b^2$. A little mental experimentation yields the answer $a = 2$, $b = 1$; of course $-2 - i$ is the other square root. Use this strategy to find the square roots of the following numbers:

(a) $8 + 6i$	(b) $5 + 12i$	(c) $24 + 10i$
(d) $3 - 4i$	(e) $-8 + 6i$	(f) $8 - 6i$

1.6 Planar Sets

In the calculus of functions of a real variable, the main theorems are typically stated for functions defined on an *interval* (open or closed). For functions of a complex variable the basic results are formulated for functions defined on sets that are 2-dimensional "domains" or "closed regions." In this section we give the precise definition of these point sets. We begin with the meaning of a "neighborhood" in the complex plane.

The set of all points that satisfy the inequality

$$|z - z_0| < \rho,$$

where ρ is a positive real number, is called an *open disk* or *circular neighborhood of* z_0. This set consists of all the points that lie inside the circle of radius ρ about z_0. In particular, the solution sets of the inequalities

$$|z - 2| < 3, \qquad |z + i| < \frac{1}{2}, \qquad |z| < 8$$

are circular neighborhoods of the respective points 2, $-i$, and 0. We shall make frequent reference to the neighborhood $|z| < 1$, which is called the *open unit disk*.

A point z_0 which lies in a set S is called an *interior point of* S if there is some circular neighborhood of z_0 that is completely contained in S. For example, if S is the right half-plane Re $z > 0$ and $z_0 = 0.01$, then z_0 is an interior point of S because S contains the neighborhood $|z - z_0| < 0.01$ (see Fig. 1.18).

If every point of a set S is an interior point of S, we say that S is an *open set*. Any open disk is an open set (Prob. 1). Each of the following inequalities also describes an open set: (a) $\rho_1 < |z - z_0| < \rho_2$, (b) $|z - 3| > 2$, (c) Im $z > 0$, and (d) $1 <$ Re $z < 2$. These sets are sketched in Fig. 1.19. Note that the solution set T of the inequality $|z - 3| \geq 2$ is *not* an open set since no point on the circle $|z - 3| = 2$ is an interior point of T. Note also that an open interval of the real axis is *not* an open set since it contains no open disk.

Let $w_1, w_2, \ldots, w_{n+1}$ be $n + 1$ points in the plane. For each $k = 1, 2, \ldots, n$, let l_k denote the line segment joining w_k to w_{k+1}. Then the successive line segments $l_1, l_2, \ldots, l_n$ form a continuous chain known as a *polygonal path* that joins w_1 to w_{n+1}.

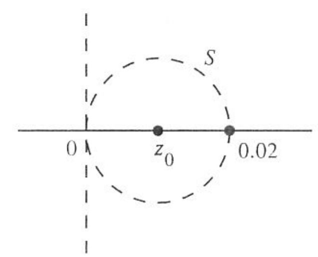

Figure 1.18 Interior point.

14. Does De Moivre's formula hold for negative integers n?

15. **(a)** Show that the multiplicative law (1) follows from Definition 5.

(b) Show that the division rule (9) follows from Definition 5.

16. Let $z = re^{i\theta}$, $(z \neq 0)$. Show that $\exp(\ln r + i\theta) = z$.†

17. Show that the function $z(t) = e^{it}$, $0 \le t \le 2\pi$, describes the unit circle $|z| = 1$ traversed in the counterclockwise direction (as t increases from 0 to 2π). Then describe each of the following curves.

(a) $z(t) = 3e^{it}$, $0 \le t \le 2\pi$ (b) $z(t) = 2e^{it} + i$, $0 \le t \le 2\pi$
(c) $z(t) = 2e^{i2\pi t}$, $0 \le t \le 1/2$ (d) $z(t) = 3e^{-it} + 2 - i$, $0 \le t \le 2\pi$

18. Sketch the curves that are given for $0 \le t \le 2\pi$ by

(a) $z(t) = e^{(1+i)t}$ (b) $z(t) = e^{(1-i)t}$
(c) $z(t) = e^{(-1+i)t}$ (d) $z(t) = e^{(-1-i)t}$

19. Let n be a positive integer greater than 2. Show that the points $e^{2\pi ik/n}$, $k = 0, 1, \ldots, n-1$, form the vertices of a regular polygon.

20. Prove that if $z \neq 1$, then

$$1 + z + z^2 + \cdots + z^n = \frac{z^{n+1} - 1}{z - 1}.$$

Use this result and De Moivre's formula to establish the following identities.

(a) $1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \dfrac{1}{2} + \dfrac{\sin[(n+\frac{1}{2})\theta]}{2\sin(\theta/2)}$

(b) $\sin\theta + \sin 2\theta + \cdots + \sin n\theta = \dfrac{\sin(n\theta/2)\sin((n+1)\theta/2)}{\sin(\theta/2)}$, where $0 < \theta < 2\pi$.

21. Prove that if n is a positive integer, then

$$\left|\frac{\sin(n\theta/2)}{\sin(\theta/2)}\right| \le n \qquad (\theta \neq 0, \pm 2\pi, \pm 4\pi, \ldots).$$

[HINT: Argue first that if $z = e^{i\theta}$, then the left-hand side equals $|(1 - z^n)/(1 - z)|$.]

22. Show that if n is an integer, then

$$\int_0^{2\pi} e^{in\theta}\, d\theta = \int_0^{2\pi} \cos(n\theta)\, d\theta + i\int_0^{2\pi} \sin(n\theta)\, d\theta = \begin{cases} 2\pi & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

23. Compute the following integrals by using the representations (11) or (12) together with the binomial formula.

(a) $\int_0^{2\pi} \cos^8\theta\, d\theta$ (b) $\int_0^{2\pi} \sin^6(2\theta)\, d\theta$.

†As a convenience in printing we sometimes write $\exp(z)$ instead of e^z.

1.5 Powers and Roots

In this section we shall derive formulas for the nth power and the mth roots of a complex number.

Let $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$ be the polar form of the complex number z. By taking $z_1 = z_2 = z$ in Eq. (13) of Sec. 1.4, we obtain the formula

$$z^2 = r^2 e^{i2\theta}.$$

Since $z^3 = zz^2$, we can apply the identity a second time to deduce that

$$z^3 = r^3 e^{i3\theta}.$$

Continuing in this manner we arrive at the formula for the nth power of z:

$$z^n = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta). \tag{1}$$

Clearly this is just an extension of De Moivre's formula, discussed in Example 3 of Sec. 1.4.

Equation (1) is an appealing formula for raising a complex number to a positive integer power. It is easy to see that the identity is also valid for negative integers n (see Prob. 2). The question arises whether the formula will work for $n = 1/m$, so that $\zeta = z^{1/m}$ is an mth root of z satisfying

$$\zeta^m = z. \tag{2}$$

Certainly if we define

$$\zeta = \sqrt[m]{r}\, e^{i\theta/m} \tag{3}$$

(where $\sqrt[m]{r}$ denotes the customary, positive, mth root), we compute a complex number ζ satisfying Eq. (2) [as is easily seen by applying Eq. (1)]. But the matter is more complicated than this; the number 1, for instance, has *two* square roots: 1 and -1. And each of these has, in turn, two square roots—generating *four* fourth roots of 1, namely, $1, -1, i$, and $-i$.

To see how the additional roots fit into the scheme of things, let's work out the polar description of the equation $\zeta^4 = 1$ for each of these numbers:

$$1^4 = (1e^{i0})^4 = 1^4 e^{i0} = 1,$$
$$i^4 = (1e^{i\pi/2})^4 = 1^4 e^{i2\pi} = 1,$$
$$(-1)^4 = (1e^{i\pi})^4 = 1^4 e^{i4\pi} = 1,$$
$$(-i)^4 = (1e^{i3\pi/2})^4 = 1^4 e^{i6\pi} = 1.$$

It is instructive to trace the consecutive powers of these roots in the Argand diagram. Thus Fig. 1.16 shows that i, i^2, i^3, and i^4 complete one revolution before landing on 1; $(-1), (-1)^2, (-1)^3$, and $(-1)^4$ go around twice; the powers of $(-i)$ go around three times counterclockwise, and of course $1, 1^2, 1^3$, and 1^4 never move.

21. If $r \operatorname{cis}\theta = r_1 \operatorname{cis}\theta_1 + r_2 \operatorname{cis}\theta_2$, determine r and θ in terms of r_1, r_2, θ_1, and θ_2. Check your answer by applying the law of cosines.

22. Use mathematical induction to prove the *generalized triangle inequality*:

$$\left|\sum_{k=1}^{n} z_k\right| \le \sum_{k=1}^{n} |z_k|.$$

23. Let m_1, m_2, and m_3 be three positive real numbers and let z_1, z_2, and z_3 be three complex numbers, each of modulus less than or equal to 1. Use the generalized triangle inequality (Prob. 22) to prove that

$$\left|\frac{m_1 z_1 + m_2 z_2 + m_3 z_3}{m_1 + m_2 + m_3}\right| \le 1,$$

and give a physical interpretation of the inequality.

24. Write computer programs for converting between rectangular and polar coordinates (using the principal value of the argument).

25. Recall that the dot (scalar) product of two planar vectors $\mathbf{v}_1 = (x_1, y_1)$ and $\mathbf{v}_2 = (x_2, y_2)$ is given by

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = x_1 x_2 + y_1 y_2 .$$

Show that the dot product of the vectors represented by the complex numbers z_1 and z_2 is given by

$$z_1 \cdot z_2 = \operatorname{Re}(\bar{z}_1 z_2) .$$

26. Use the formula for the dot product in Prob. 25 to show that the vectors represented by the (nonzero) complex numbers z_1 and z_2 are orthogonal if and only if $z_1 \cdot z_2 = 0$. [HINT: Recall from the discussion following Eq. (9) that orthogonality holds precisely when $z_1 = icz_2$ for some real c.]

27. Recall that in three dimensions the cross (vector) product of two vectors $\mathbf{v}_1 = (x_1, y_1, 0)$ and $\mathbf{v}_2 = (x_2, y_2, 0)$ in the xy-plane is given by

$$\mathbf{v}_1 \times \mathbf{v}_2 = (0, 0, x_1 y_2 - x_2 y_1) .$$

(a) Show that the third component of the cross product of vectors in the xy-plane represented by the complex numbers z_1 and z_2 is given by $\operatorname{Im}(\bar{z}_1 z_2)$.

(b) Show that the vectors represented by the (nonzero) complex numbers z_1 and z_2 are parallel if and only if $\operatorname{Im}(\bar{z}_1 z_2) = 0$. [HINT: Observe that these vectors are parallel precisely when $z_1 = cz_2$ for some real c.]

28. This problem demonstrates how complex notation can simplify the kinematic analysis of planar mechanisms.

Consider the crank-and-piston linkage depicted in Fig. 1.14. The crank arm a rotates about the fixed point O while the piston arm c executes horizontal motion. (If this were a gasoline engine, combustion forces would drive the piston and the connecting arm b would transform this energy into a rotation of the crankshaft.)

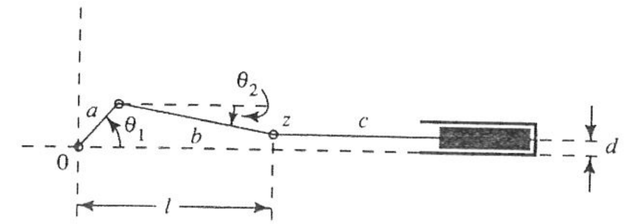

Figure 1.14 Crank-and-piston linkage.

For engineering analysis it is important to be able to relate the crankshaft's angular coordinates—position, velocity, and acceleration—to the corresponding linear coordinates for the piston. Although this calculation can be carried out using vector analysis, the following complex variable technique is more "automatic."

Let the crankshaft pivot O lie at the origin of the coordinate system, and let z be the complex number giving the location of the base of the piston rod, as depicted in Fig. 1.14,

$$z = l + id,$$

where l gives the piston's (linear) excursion and d is a fixed offset. The crank arm is described by $A = a(\cos\theta_1 + i\sin\theta_1)$ and the connecting arm by $B = b(\cos\theta_2 + i\sin\theta_2)$ (θ_2 is negative in Fig. 1.14). Exploit the obvious identity $A+B = z = l+id$ to derive the expression relating the piston position to the crankshaft angle:

$$l = a\cos\theta_1 + b\cos\left[\sin^{-1}\left(\frac{d - a\sin\theta_1}{b}\right)\right].$$

29. Suppose the mechanism in Prob. 28 has the dimensions

$$a = 0.1 \text{ m}, \qquad b = 0.2 \text{ m}, \qquad d = 0.1 \text{ m}$$

and the crankshaft rotates at a uniform velocity of 2 rad/s. Compute the position and velocity of the piston when $\theta_1 = \pi$.

30. For the linkage illustrated in Fig. 1.15, use complex variables to outline a scheme for expressing the angular position, velocity, and acceleration of arm c in terms of those of arm a. (You needn't work out the equations.)

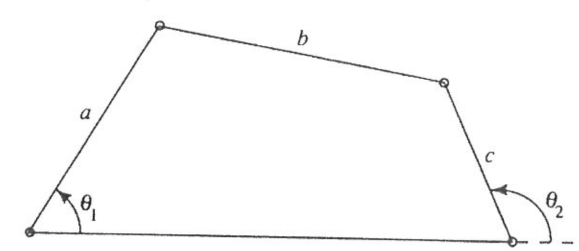

Figure 1.15 Linkage in Prob. 30.

For an n by n matrix $\mathbf{A} = [a_{ij}]$ with complex entries, prove the following:

(a) If $\mathbf{u}^\dagger \mathbf{A}\mathbf{u} = \mathbf{0}$ for all n by 1 column vectors $\mathbf{u}$ with complex entries, then $\mathbf{A}$ is the zero matrix (that is, $a_{ij} = 0$ for all i, j). [HINT: To show $a_{ij} = 0$, take $\mathbf{u}$ to be a column vector with all zeros except for its i^{th} and j^{th} entries.]

(b) Show by example that the conclusion ("$\mathbf{A}$ is the zero matrix") can fail if the hypothesis for part (a) only holds for vectors $\mathbf{u}$ with *real* number entries. [HINT: Try to find a 2 by 2 real *nonzero* matrix $\mathbf{A}$ such that $\mathbf{u}^\dagger \mathbf{A}\mathbf{u} = 0$ for all real 2 by 1 vectors $\mathbf{u}$.]

21. Let $\mathbf{A}$ be an n by n matrix with complex entries. We say that $\mathbf{A}$ is *Hermitean* if $\mathbf{A}^\dagger = \mathbf{A}$ (see Prob. 20).

(a) Show that if $\mathbf{A}$ is Hermitean, then $\mathbf{u}^\dagger \mathbf{A}\mathbf{u}$ is real for any n by 1 column vector $\mathbf{u}$ with complex entries.

(b) Show that if $\mathbf{B}$ is any m by n matrix with complex entries, then $\mathbf{B}^\dagger \mathbf{B}$ is Hermitean.

(c) Show that if $\mathbf{B}$ is any n by n matrix and $\mathbf{u}$ is any n by 1 column vector (each with complex entries), then $\mathbf{u}^\dagger \mathbf{B}^\dagger \mathbf{B}\mathbf{u}$ must be a nonnegative real number.

1.3 Vectors and Polar Forms

With each point z in the complex plane we can associate a *vector*, namely, the directed line segment from the origin to the point z. Recall that vectors are characterized by length and direction, and that a given vector remains unchanged under translation. Thus the vector determined by $z = 1 + i$ is the same as the vector from the point $2 + i$ to the point $3 + 2i$ (see Fig. 1.5). Note that every vector parallel to the real axis corresponds to a real number, while those parallel to the imaginary axis represent pure imaginary numbers. Observe, also, that the length of the vector associated with z is $|z|$.

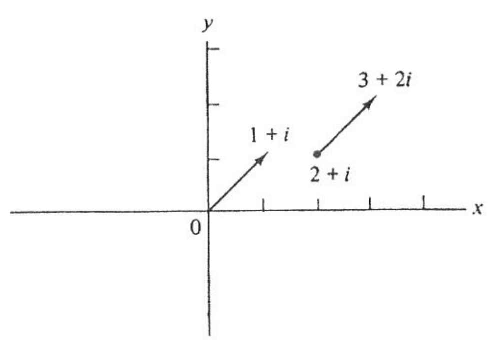

Figure 1.5 Complex numbers as vectors.

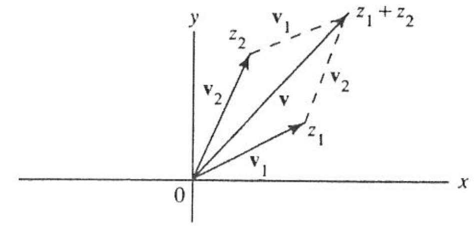

Figure 1.6 Vector addition.

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ denote the vectors determined by the points z_1 and z_2, respectively. The vector sum $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ is given by the parallelogram law, which is illustrated in Fig. 1.6. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then the terminal point of the vector $\mathbf{v}$ in Fig. 1.6 has the coordinates $(x_1 + x_2, y_1 + y_2)$; that is, it corresponds to the point $z_1 + z_2$. Thus we see that *the correspondence between complex numbers and planar vectors carries over to the operation of addition.*

Hereafter, the vector determined by the point z will be simply called *the vector z.*

Recall the geometric fact that the length of any side of a triangle is less than or equal to the sum of the lengths of the other two sides. If we apply this theorem to the triangle in Fig. 1.6 with vertices 0, z_1, and $z_1 + z_2$, we deduce a very important law relating the magnitudes of complex numbers and their sum:

Triangle Inequality. For any two complex numbers z_1 and z_2, we have

$$|z_1 + z_2| \le |z_1| + |z_2|. \tag{1}$$

The triangle inequality can easily be extended to more than two complex numbers, as requested in Prob. 22.

The vector $z_2 - z_1$, when added to the vector z_1, obviously yields the vector z_2. Thus $z_2 - z_1$ can be represented as the directed line segment from z_1 to z_2 (see Fig. 1.7). Applying the geometric theorem to the triangle in Fig. 1.7, we deduce another form of the triangle inequality:

$$|z_2| \le |z_1| + |z_2 - z_1|$$

or

$$|z_2| - |z_1| \le |z_2 - z_1|. \tag{2}$$

Inequality (2) states that the difference in the lengths of any two sides of a triangle is no greater than the length of the third side.

Example 1

Prove that the three distinct points z_1, z_2, and z_3 lie on the same straight line if and only if $z_3 - z_2 = c(z_2 - z_1)$ for some real number c.