# MATH 163-001 - CALCULUS II - Spring 1999

## Solutions to Review #3

The third midterm is on Wednesdat April 14th. It will cover only sections 8.1-8.9.
Sections 8.10 and some applications in section 8.11 will be evaluated in the final and in the quiz next week.
Remember that my office hours are from 3:30-5pm on Mondays and Wednesdays. I will also be available tomorrow Tuesday after 2pm. You can find me in my office: Hum 459.
Here are the answers for the review problems:

**1.** (a) $-2/3$ ; (b) $\cos 0 = 1$, (c) 0 (by l'Hopital).

**2.** (a) Convergent to 1/2 by geometric series with $r = 5/7$ and $a = 1/7$ .
(b) Divergent by n-th term test and 1(a).
(c) Conditionally convergent: convergent by alternating series test, not absolutely convergent by p-series with $p = 1/2 \leq 1$.

**3.** (a) By either ratio or root test.

- $|x| < 1$ absolute convergence,
- $|x| > 1$ divergence.

Hence radius of convergence R=1.
At the end points $|x| = 1$ :

- $x = 1$ divergent (substituting in series get harmonic series),
- $x = -1$ conditionally convergent (substituting get alternating harmonic series)

Hence the interval of convergence is: $-1 \leq x < 1$ (including $x = -1$, excluding $x = 1$).

(b) Observe that this is a geometric series with $r = \frac{x-2}{3}$, hence:

- $|x - 2| < 3$ (same as $|r| < 1$) absolute convergent,
- $|x - 2| \geq 3$ (same as $|r| \geq 1$) divergent.

We get for free the divergent behaviour at the endpoints: $|x - 2| = 3$.
The radius of convergence is $R = 3$.
The interval of convergence is: $|x - 2| < 3$, which is the same as: $-1 < x < 5$ (excluding both endpoints!).

**Note:** you can proceed as in (a). By root or ratio test you will conclude that $R = 3$, but you will have to analyze separately the endpoints by evaluating the series at $x = -1$ and $x = 5$ (if you do it properly you can see that both are divergent!).

**4.** We did this one carefully in class today. Taylor expansion generated by the function $f(x) = \sin(2x - 4)$ at $a = 2$

$$2(x-2) - \frac{2^3}{3!}(x-2)^3 + \frac{2^5}{5!})(x-2)^5 - \frac{2^7}{7!}(x-2)^7 + \ldots \quad (*)$$

(don't forget the dots at the end!).

**Second solution:** Another way of answering this problem is to notice that: for $x = 2$ then $2x-4 = 0$. You can use the Taylor expansion generated by $f(y) = \sin y$ at $y = 0$, which is known to converge for $|x| \leq \pi$: $\sin y = y - \frac{1}{3!}y^3 + \frac{1}{5!}y^5 - \frac{1}{7!}y^7 + \ldots$, then substitute in $y = 2x - 4$, observe that $y^n = (2x-4)^n = 2^n(x-2)^n$, and you are back in $(*)$.

5. (a) $P_4(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4$ (no dots this time!!!!!!)

Disregard for the exam on Wednesday the formula for the remainder, and 5(b).

6. (a) Use ratio test carefully (since there are factorials, root test is not appropiate). Doing some algebra we obtain:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)n!(n+1)(n+1)^n(2n)!}{(2n+2)(2n+1)(2n)!n!n^n} = \frac{(n+1)^2(n+1)^n}{2(n+1)(2n+1)n^n} =$$

$$= \frac{(n+1)}{2(2n+1)}\frac{(n+1)^n}{n^n} = \frac{n+1}{4n+2}\left(\frac{n+1}{n}\right)^n = \frac{n+1}{4n+2}\left(1+\frac{1}{n}\right)^n$$

Calculating the limit:

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \frac{1}{4}e = \frac{e}{4} = \rho < 1$$

Therefore by ratio test you deduce that the series is convergent.

(b) Substitute in the Maclaurin series for $\cos x$,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} - \ldots$$

then multiply term by term by x, add the first terms and you get:

$$1 + 3x - x^2 + x\cos x = 1 + 3x - x^2 + x\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - ...\right)$$

$$= 1 + 3x - x^2 + x - \frac{1}{2!}x^3 + \frac{1}{4!}x^5 - \frac{1}{6!}x^7 + \ldots$$

$$= 1 + 4x - x^2 - \frac{1}{2!}x^3 + \frac{1}{4!}x^5 - \frac{1}{6!}x^7 + \ldots$$

(c) Integrate term by term the series expansion in $t$, from $t = 0$ up to $t = x$, you get a power expansion for $\tan^{-1} x$ at $a = 0$, which coincides with the Maclaurin series:

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... \qquad |x| < 1$$