

SASAKIAN GEOMETRY ON SPHERE BUNDLES

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ABSTRACT. The purpose of this paper is to study the Sasakian geometry on odd dimensional sphere bundles over a smooth projective algebraic variety N with the ultimate, but probably unachievable goal of understanding the existence and non-existence of extremal and constant scalar curvature Sasaki metrics. We apply the fiber join construction of Yamazaki [Yam99] for K-contact manifolds to the Sasaki case. This construction depends on the choice of $d + 1$ integral Kähler classes $[\omega_j]$ on N that are not necessarily colinear in the Kähler cone. We show that the colinear case is equivalent to a subclass of a different join construction originally described in [BG00a, BGO07], applied to the spherical case by the authors in [BTF14, BTF16] when $d = 1$, and known as cone decomposable [BHLTF18]. The non-colinear case gives rise to infinite families of new inequivalent cone indecomposable Sasaki contact CR on certain sphere bundles. We prove that the Sasaki cone for some of these structures contains an open set of extremal Sasaki metrics and, for certain specialized cases, the regular ray within this cone is shown to have constant scalar curvature. We also compute the cohomology groups of all such sphere bundles over a product of Riemann surfaces.

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Date: July 21, 2020.

The first author was partially supported by grant #519432 from the Simons Foundation. The second author was partially supported by grant #422410 from the Simons Foundation.

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1. INTRODUCTION

In 1999 T. Yamazaki [Yam99] gave a construction of K-contact structures on certain odd dimensional sphere bundles over a symplectic manifold. It is easy to see that if one takes the base symplectic manifold to be a smooth projective algebraic variety N , Yamazaki's construction gives a family of Sasakian structures on these odd dimensional sphere bundles. More precisely they are the unit sphere bundles of a complex vector bundle that splits as a sum of line bundles. Such structures have recently become of interest in the study of Calabi-Yau A_∞ algebras used in topological conformal field theory [Cos07]. In particular, it was shown in [TT18] that the Calabi-Yau A_∞ algebras studied in [TTY16] are equivalent to the standard de Rham differential graded algebra on certain odd dimensional sphere bundles. For this among other reasons, we feel that a concerted study of Sasaki geometry on sphere bundles is warranted.

In Theorem 4.8 of [BHLTF18] a partial classification under a fairly restrictive condition was given in the case of S^3 bundles over a smooth projective algebraic variety. Moreover, several examples where the restrictive condition is violated were given some of which used Yamazaki's fiber join construction. These conditions involve invariants known as cone decomposability. It is the purpose of the present paper to give

a further study of this decomposability issue within the context of Yamazaki's construction for Sasakian structures on odd dimensional sphere bundles. An ultimate (perhaps unachievable) goal is to classify the Sasaki CR structures on such sphere bundles as well as understand the existence or non-existence of extremal and constant scalar curvature Sasaki metrics on such sphere bundles. This paper takes the first few steps toward this goal.

The fiber join construction requires choosing $d + 1$ not necessarily distinct Kähler classes in the Kähler cone $\mathcal{K}(N)$ of a smooth projective algebraic variety N which produces a Sasakian structure on an S^{2d+1} sphere bundle over N . We can divide these into two types: 1. all the Kähler classes are colinear; 2. not all Kähler classes are colinear. We show that this dichotomy corresponds to the notions of *cone decomposable* and *cone indecomposable* introduced in [BHLTF18] and which are invariants of the underlying Sasaki CR structure. In Section 3 we prove Proposition 3.8 that the colinear fiber joins are equivalent to special cases of the Sasaki joins described in [BGO07]. We refer to the latter as *regular* Sasaki joins. They have been studied in detail in [BTF16] when $d = 1$; however, here we also present some new results when $d > 1$. It is easy to see that if (N, ω_N) is an extremal Kähler structure the colinear case always has an open set of extremal Sasaki metrics in its Sasaki cone. Our main focus will thus be on the non-colinear or cone indecomposable case where our proof of extremality is not a priori, but depends on the applicability of the admissible construction as described in [ACGTF08]. This requires a splitting of the positive integer $d = d_0 + d_\infty$. For the definition of admissible and super admissible see Section 4.1.

Theorem 1.1. *Let $M_{\mathbb{w}}$ be a super admissible cone indecomposable fiber join whose regular quotient is a ruled manifold of the form $\mathbb{P}(E_0 \oplus E_\infty) \rightarrow N$ where E_0, E_∞ are projectively flat hermitian holomorphic vector bundles on N of complex dimension $(d_0+1), (d_\infty+1)$ respectively, and N is a local Kähler product of non-negative CSC metrics. Then the Sasaki cone of $M_{\mathbb{w}}$ has an open set of extremal Sasaki metrics.*

It is possible to be more explicit and also to obtain CSC Sasaki metrics as well to relax the non-negativity assumption. However, to do so we need to understand the admissible polynomial, whose degree grows linearly with d , in more detail. Therefore, we do the analysis for some special cases of sphere bundles over certain products of Riemann surfaces.

Theorem 1.2. *Let M_K be a cone indecomposable admissible fiber join over $\Sigma_{g_1} \times \Sigma_{g_2}$. Then the Sasaki cone of M_K has an open set of extremal Sasaki metrics containing a regular ray generated by ξ_1*

- (1) *for all $d \geq 1$ and for all 2 by 2 matrices K with values in \mathbb{Z}^+ if $g_1 = 0, g_2 = 0, 1$;*
- (2) *if $g_1 = 0, g_2 = g > 1, d_0 = d_\infty = 1$ and*

$$K = \begin{pmatrix} 2 & g \\ 1 & 1 \end{pmatrix};$$

- (3) *with $d_0 = d_\infty = 0, g_1 = g_2 = g$, and ξ_1 having constant scalar curvature*
 - (a) *if $g = 0$ and $K = \begin{pmatrix} k & l \\ l & k \end{pmatrix}$ with $k, l \in \mathbb{Z}^+$ and $k \neq l$, or*
 - (b) *$g = 1$, and $K = \begin{pmatrix} k_0^1 & k_0^2 \\ k_\infty^1 & k_\infty^2 \end{pmatrix}$ is such that $\frac{k_0^1 - k_\infty^1}{k_0^1 + k_\infty^1} = \frac{k_\infty^2 - k_0^2}{k_0^2 + k_\infty^2}$,*
or
 - (c) *$g > 1$, if $K = \begin{pmatrix} k+1 & k \\ k & k+1 \end{pmatrix}$ is such that $k \in \mathbb{Z}^+$ satisfies*
$$k > \lfloor \frac{2g-3+\sqrt{4g^2-8g+5}}{2} \rfloor.$$

In the case of a product of Riemann spheres more can be said. For example we can prove the existence of a countable infinity of inequivalent Sasaki contact structure on the same (up to diffeomorphism) sphere bundle.

Theorem 1.3. *Consider the set $\{M_K\}$ of cone indecomposable admissible fiber joins over $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ in case (1) of Theorem 1.2. There exist at least one diffeomorphism type within this set that admits a countable infinity of inequivalent Sasaki contact structures all of which have an open set of extremal Sasaki metrics in their Sasaki cone.*

Remark 1.4. A similar result holds in the cone decomposable case.

In Section 5.3 we describe the topology of the manifolds M_K . If d_0 and d_∞ are both odd, M_K is a spin manifold for any K , otherwise, both spin and non-spin can occur. In Proposition 5.8 we compute the cohomology groups of a general Sasaki fiber join on S^{2d+1} bundles over the product of Riemann surfaces $\Sigma_{g_1} \times \Sigma_{g_2}$. More can be said when $g_1 = g_2 = 0$. For case (a) of Theorem 1.2 we prove in Proposition 5.25 that each ordered pair (k, l) of positive integers with $k > l$ determines a unique homeomorphism type of S^3 bundle over $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$, so there exists a countable infinity of such homeomorphism types. Furthermore, in each case the diffeomorphism type is known up to finite ambiguity. Since $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ has a 2-torus of Hamiltonian automorphisms, all the

M_K of type (1) are toric, and of Koiso-Sakane type, that is, the quotient manifold of the regular ray is the Bott manifold

$$M_3(0, k-l, -(k-l)) = \mathbb{P}(\mathbb{1} \oplus \mathcal{O}(k-l, -(k-l))).$$

The Kähler geometry of these quotients has been studied in detail elsewhere (see for example [BCTF19] and references therein).

Also in Section 5.4 we present some more extremality results in special cases when $g_1 \neq g_2$ as well as cases with negative constant transverse scalar curvature. Finally in Section 5.5 we present the existence results of extremal Sasaki metrics for special cases with higher dimensional base space N .

Acknowledgements. *We thank Eveline Legendre and Hongnian Huang for their interest in our work.*

2. BRIEF REVIEW OF SASAKI GEOMETRY

Recall that a Sasakian structure on a contact manifold M^{2n+1} of dimension $2n+1$ is a special type of contact metric structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with underlying almost CR structure (\mathcal{D}, J) where η is a contact form such that $\mathcal{D} = \ker \eta$, ξ is its Reeb vector field, $J = \Phi|_{\mathcal{D}}$, and $g = d\eta \circ (\mathbb{1} \times \Phi) + \eta \otimes \eta$ is a Riemannian metric. \mathcal{S} is a *K-contact* structure if ξ is a Killing vector field and it is *Sasakian* if in addition the almost CR structure is integrable, i.e. (\mathcal{D}, J) is a CR structure. We refer to [BG08] for the fundamentals of Sasaki geometry. We call (\mathcal{D}, J) a *Sasaki CR structure* or *CR structure of Sasaki type*, and \mathcal{D} a *Sasaki contact structure* or *contact structure of Sasaki type*. We shall always assume that the Sasaki manifold M^{2n+1} is compact and connected.

Definition 2.1. *Let (M, \mathcal{D}, J) and (M', \mathcal{D}', J') be Sasaki CR structures. We say that (M, \mathcal{D}, J) and (M', \mathcal{D}', J') are **equivalent**, denoted $(M', \mathcal{D}', J') \approx (M, \mathcal{D}, J)$, if there exists a diffeomorphism $\psi : M \rightarrow M'$ such that*

$$\psi_*\mathcal{D} = \mathcal{D}', \quad J' = \psi_*J\psi_*^{-1}.$$

2.1. Invariants and the Classification of Sasaki CR Structures.

The classification of Sasaki CR structures on a given manifold is of major importance to us. An important invariant is the conical family of Sasakian structures within a fixed contact CR structure (\mathcal{D}, J) known as the (*unreduced*) *Sasaki cone* and denoted by \mathfrak{t}^+ . We are also interested in a variation within this family. To describe the Sasaki cone we fix a Sasakian structure $\mathcal{S}_o = (\xi_o, \eta_o, \Phi_o, g_o)$ on M whose underlying CR structure is (\mathcal{D}, J) and let \mathfrak{t} denote the Lie algebra of the maximal

torus in the automorphism group of \mathcal{S}_o . The (*unreduced*) *Sasaki cone* [BGS08] is defined by

$$(1) \quad \mathfrak{t}^+(\mathcal{D}, J) = \{\xi \in \mathfrak{t} \mid \eta_o(\xi) > 0 \text{ everywhere on } M\},$$

which is a cone of dimension $k \geq 1$ in the Lie algebra \mathfrak{t} . When the underlying CR structure (\mathcal{D}, J) is understood we often write \mathfrak{t}^+ instead of $\mathfrak{t}^+(\mathcal{D}, J)$. The *reduced Sasaki cone* $\kappa(\mathcal{D}, J)$ is $\mathfrak{t}^+(\mathcal{D}, J)/\mathcal{W}$ where \mathcal{W} is the Weyl group of the maximal compact subgroup of $\mathfrak{CR}(\mathcal{D}, J)$ which is viewed as the *moduli space* of Sasakian structures with underlying CR structure (\mathcal{D}, J) , and it is an important invariant of the Sasaki CR structure, that is if $(\mathcal{D}', J') \approx (\mathcal{D}, J)$ are equivalent Sasaki CR structures, then $\kappa(\mathcal{D}, J) = \kappa(\mathcal{D}', J')$. In particular, the reducibility structure [BHLTF18] of $\kappa(\mathcal{D}, J)$ which gives rise to a multi-foliate structure on the manifold M . This multifoliate structure is an invariant, up to order, of the CR Sasaki structure. On the tangent space level it corresponds to decomposing \mathcal{D} into irreducible subspaces up to order. A cruder invariant is the sequence of ranks of the individual pieces. However, in the present paper it is enough to consider just two types of Sasaki CR structures, namely, cone decomposable and cone indecomposable. Examples of such Sasaki CR structures were given in Section 4 of [BHLTF18].

Another important invariant of (\mathcal{D}, J) is its first Chern class $c_1(\mathcal{D})$. In fact it is an invariant of the underlying contact structure. So if the contact structures \mathcal{D} and \mathcal{D}' are equivalent, then $c_1(\mathcal{D}) = c_1(\mathcal{D}')$. The converse is not true. There are more subtle invariants related to ‘contact homology’ which were explored to some extent for Sasaki contact structures in [BP14, BMvK16], but will not concern us here.

In practice it is more convenient to work with the unreduced Sasaki cone $\mathfrak{t}^+(\mathcal{D}, J)$. It is also clear from the definition that $\mathfrak{t}^+(\mathcal{D}, J)$ is a cone under the transverse scaling defined by

$$(2) \quad \mathcal{S} = (\xi, \eta, \Phi, g) \mapsto \mathcal{S}_a = (a^{-1}\xi, a\eta, g_a), \quad g_a = ag + (a^2 - a)\eta \otimes \eta, \quad a \in \mathbb{R}^+$$

So Sasakian structures in \mathfrak{t}^+ come in rays, and since the Reeb vector field ξ is Killing $\dim \mathfrak{t}^+ \geq 1$, and it follows from contact geometry that $\dim \mathfrak{t}^+ \leq n + 1$. When $\dim \mathfrak{t}^+(\mathcal{D}, J) = n + 1$ we have a toric contact manifold of Reeb type studied in [BM93, BG00b, Ler02, Ler04b, Leg11, Leg16]. In this case there is a strong connection between the geometry and topology of (M, \mathcal{S}) and the combinatorics of $\mathfrak{t}^+(\mathcal{D}, J)$. Much can also be said in the complexity 1 case ($\dim \mathfrak{t}^+(\mathcal{D}, J) = n$) [AH06].

Recall [BGS08] that a Sasakian structure (ξ, η, Φ, g) is *extremal* if the $(1, 0)$ gradient of the scalar curvature is transversely holomorphic.

A main focus of the present paper is existence proofs of extremal and constant scalar curvature Sasaki metrics in certain cone indecomposable CR Sasaki structures on sphere bundles. The essential technique is the admissible construction described in [ACGTF08]. We shall often use the openness theorem of [BGS08] which says that if an extremal Sasaki metric exists, there is an open set of extremal Sasaki metrics in the Sasaki cone $\mathfrak{t}^+(\mathcal{D}, J)$. We emphasize that both extremal and constant scalar curvature Sasaki metrics come in rays; however, it is the transverse scalar curvature $s_g^T = s_g + 2n$, which is the scalar curvature of the transverse Kähler metric g^T , that rescales not the scalar curvature s_g of the Sasaki metric g . Under scaling the latter transforms as $s_{g_a} = a^{-1}(s_g + 2n) - 2n$.

2.2. Sphere Bundles. In this paper we are interested in $2d+1$ dimensional sphere bundles¹ M over a smooth projective algebraic variety N of complex dimension n such that a Sasakian structure on M restricts to a weighted Sasakian structure on each fiber which is the standard sphere S^{2d+1} :

$$(3) \quad \begin{array}{ccc} S^{2d+1} & \longrightarrow & M \\ & & \downarrow \\ & & N. \end{array}$$

The dimension of the Sasaki manifold M is $2n + 2d + 1$, and note that if N is toric, so is M with $\dim \mathfrak{t}^+(\mathcal{D}, J) = n + d + 1$. More generally, the complexity of M equals the complexity of N , and we have

$$(4) \quad d + 1 \leq \dim \mathfrak{t}^+(\mathcal{D}, J) \leq n + d + 1.$$

It is well known that the Sasaki cone of the standard Sasaki CR structure on S^{2d+1} is the $(d+1)$ -st orthant, and this defines a subcone of \mathfrak{t}^+ of dimension $d+1$ which we denote by $\mathfrak{t}_{sph}^+(\mathcal{D}, J)$. The sphere bundles studied in this paper all have a regular Reeb vector field $\xi_1 \in \mathfrak{t}_{sph}^+(\mathcal{D}, J)$ which plays an important role for us, and whose quotient is described in more detail in Section 3.3 below. However, in contrast there are sphere bundles with CR Sasaki structures having no regular Reeb field in their Sasaki cone. In [BG06] infinitely many such Sasaki CR structures are given on the trivial sphere bundles $S^{2d} \times S^{2d+1}$ for $d > 1$ which have Sasaki metrics of positive Ricci curvature. They are represented by Brieskorn manifolds belonging to infinitely many inequivalent contact

¹By a sphere bundle we mean an oriented sphere bundle with the linear structure group $SO(d+1)$.

structures [BMvK16] and it is shown in [BvC18] that their Sasaki cones admit no extremal Sasaki metrics whatsoever.

We briefly discuss some topology of sphere bundles over smooth projective algebraic varieties. From the long exact homotopy sequence of the bundle (3) we have

$$(5) \quad \pi_i(M) \approx \pi_i(N) \text{ for } i \leq 2d.$$

Yamazaki treated the special case of sphere bundles over a Riemann surface Σ_g of genus g . In this case there are precisely two diffeomorphism types, namely the trivial bundle $\Sigma_g \times S^{2d+1}$, and the nontrivial bundle $\Sigma_g \widetilde{\times} S^{2d+1}$. These are distinguished by the second Stiefel-Whitney class of the complex vector bundle E .

Consider the Leray-Hirsch Theorem of the fibration (3). The condition that there is a global $(2d+1)$ -class that restricts to the fundamental class of S^{2d+1} implies the vanishing of the Euler class of the bundle. Then from the Leray-Serre spectral sequence of (3) we get

Proposition 2.2. *Let N be a compact symplectic manifold and M an S^{2d+1} bundle over N . Then*

$$H^p(M, \mathbb{Z}) \approx H^p(N, \mathbb{Z})$$

for all $p < 2d + 1$. In particular, if $n = \dim_{\mathbb{C}} N \leq d$ then M has the integer cohomology groups of the product $N \times S^{2d+1}$.

Remark 2.3. In the last statement of Proposition 2.2 we have an isomorphism of groups, but not necessarily an isomorphism of rings. It is an interesting and important problem to determine the cohomology ring structure.

We are also interested in the possible diffeomorphism types of M for which we apply Sullivan's rational homotopy theory [Sul77] to our sphere bundles (3). If N is simply connected² the rational homotopy type of M is well understood, see Example 2.69 in [FOT08]. For the sphere bundle (3) Sullivan's relative minimal model is

$$(6) \quad (\wedge V, d) \longrightarrow (\wedge V \otimes u, d) \longrightarrow (\wedge u, 0)$$

where $(\wedge V, d)$ is a differential commutative graded algebra (cdga) that is a model for N , $(\wedge u, 0)$ a model for S^{2d+1} , and $du = \mathbf{e}$ is a cocycle in $(\wedge V)^{2d+2}$ representing the Euler class of the sphere bundle M . The minimal model is completely determined by the Euler class. Recall

²More generally, we can assume that $\pi_1(N)$ is nilpotent and that it acts nilpotently on the higher homotopy groups of N . Then N is said to be a nilpotent space.

that a cdga is said to be *formal* if there exists a quasi-isomorphism³ $(\wedge V, d) \approx (H^*(M, \mathbb{Q}), 0)$. Then

Lemma 2.4. *A sphere bundle M over a nilpotent base N is formal when $n = \dim_{\mathbb{C}} N \leq d$.*

Proof. When $n = \dim_{\mathbb{C}} N \leq d$ the Euler class ϵ vanishes giving the isomorphism. \square

3. YAMAZAKI'S FIBER JOIN CONSTRUCTION

A procedure of Yamazaki which he called the ‘fiber join’ allows the construction of infinitely many Sasakian structures on the total space of odd dimensional sphere bundles over a projective algebraic variety. Yamazaki [Yam99] does this for K-contact structures, but as we shall see the restriction to Sasakian structures works equally well. For $j = 1, \dots, d+1$ let $\mathcal{S}_j = (\xi_j, \eta_j, \Phi_j, g_j)$ be regular K-contact structures on the manifolds M_j with the same smooth manifold N as quotient but possibly different integral symplectic forms ω_j . We assume that $d > 0$ throughout. The manifold M_j is the total space of a principal S^1 bundle over N . Let L_j denote the complex line bundle on N associated to M_j such that $c_1(L_j) = [\omega_j]$. We identify L_j with $M_j \times_{S^1} \mathbb{C}$. Then Yamazaki shows that the unit sphere bundle M in the complex vector bundle $E = \bigoplus_{j=1}^{d+1} L_j^*$ has natural K-contact structures which we describe below. We should mention here that there is a generalization of Yamazaki’s construction due to Lerman [Ler04a] where the sphere bundle is replaced by a ‘contact fiber bundle’. This generalization was used in [BGO07] (Theorem 3.5) to construct toric Sasaki manifolds that are contact fiber bundles with a toric Sasaki fiber, and a toric symplectic base. Here is Yamazaki’s fiber join:

Definition 3.1. *The smooth manifold $M = M_1 *_f \dots *_f M_{d+1}$ defined to be the unit sphere in the complex vector bundle $E = \bigoplus_{j=1}^{d+1} L_j^*$ is called the **fiber join** of the M_j .*

The line bundle L_j^* has a Hermitian metric which defines a ‘norm’ $r_j : L_j^* \rightarrow \mathbb{R}_{\geq 0}$. We let (r_j, θ_j) denote polar coordinates on the fiber of the line bundle L_j^* . Then M is an S^{2d+1} -bundle over N whose fibers are defined by the equation $\sum_{j=1}^{d+1} r_j^2 = 1$. The fibers S^{2d+1} can be thought of as the topological join

$$S^1 * \overset{d+1 \text{ times}}{\dots} * S^1 = S^{2d+1}.$$

³that is a morphism of cdga’s that induces an isomorphism in cohomology.

3.1. The K-Contact Structure. Recall Definition 6.4.7 of [BG08] that a *K-contact structure* is a contact metric structure (ξ, η, Φ, g) such that ξ is Killing vector field. We note that this condition is equivalent to $\mathcal{L}_\xi \Phi = 0$.

The contact bundle is $\mathcal{D} = \ker \eta$ and the real line bundle generated by ξ is denoted by $\mathbb{R}\xi$. It is convenient to decompose the tangent bundle of M into horizontal and vertical parts as $TM = \mathcal{H} + \mathcal{V}$. We then see that $\mathbb{R}\xi \subset \mathcal{V}$ and $\mathcal{H} \subset \mathcal{D}$ and we have the decompositions

$$TM = \mathcal{H} + \mathcal{V}/\mathbb{R}\xi + \mathbb{R}\xi, \quad \mathcal{D} = \mathcal{H} + \mathcal{V}/\mathbb{R}\xi.$$

Let $\mathbf{a} = (a_1, \dots, a_{d+1}) \in (\mathbb{R}^+)^{d+1}$ and consider the 1-form on $\bigoplus_{j=1}^{d+1} L_j^*$ defined by⁴

$$(7) \quad \eta_{\mathbf{a}} = \sum_{j=1}^{d+1} \frac{1}{a_j} r_j^2 (\eta_j + d\theta_j).$$

We can check that this restricts to a contact form on M and its Reeb vector field is

$$(8) \quad \xi_{\mathbf{a}} = \frac{1}{2} \sum_{j=1}^{d+1} a_j (\xi_j + \partial_{\theta_j}).$$

As a complex line bundle on N we can identify the total space of the line bundle L_j with the quotient $M_j \times_{S^1} \mathbb{C}$ where the S^1 action on M_j is the flow of the Reeb vector field ξ_j and its action on \mathbb{C} is induced by the vector field $-\partial_{\theta_j}$. Thus, on L_j we can identify ∂_{θ_j} with ξ_j which implies that on M we have the identification

$$\xi_{\mathbf{a}} = \sum_{j=1}^{d+1} a_j \partial_{\theta_j}$$

in terms of the polar coordinates $\{(r_j, \theta_j)\}_{j=1}^{d+1}$ on each fiber S^{2d+1} satisfying $\sum_j r_j^2 = 1$. The geometry transverse to the Reeb foliation of $\mathcal{S}_{\mathbf{a}} = (\xi_{\mathbf{a}}, \eta_{\mathbf{a}}, \Phi_{\mathbf{a}}, g_{\mathbf{a}})$ on the fiber join M is given by the transverse symplectic form

$$(9) \quad d\eta_{\mathbf{a}} = \sum_{j=1}^{d+1} \frac{1}{a_j} (r_j^2 d\eta_j + 2r_j dr_j \wedge (\eta_j + d\theta_j))$$

⁴Note that $a_j = 1/\lambda_j$ and the two Equations (7) and (8) have relative minus signs in Yamazaki which can be absorbed into the definition of θ_j .

This transverse symplectic form splits as a sum of a nondegenerate 2-form on $\mathcal{H} = \sum_j \mathcal{H}_j$ and a nondegenerate 2-form on $\mathcal{V} = \sum_j \mathcal{V}_j$ as

$$(10) \quad d\eta_{\mathbf{a}} = \sum_j \frac{1}{a_j} r_j^2 d\eta_j + 2 \sum_j \frac{1}{a_j} (r_j dr_j \wedge (\eta_j + d\theta_j)).$$

The transverse Riemannian metric is then given by

$$(11) \quad g_{\mathbf{a}}^T = d\eta_{\mathbf{a}} \circ (\mathbb{1} \otimes \Phi_{\mathbf{a}}) = \sum_{j=1}^{d+1} \frac{1}{a_j} d(r_j^2 (\eta_j + d\theta_j)) \circ (\mathbb{1} \otimes \Phi_{\mathbf{a}}),$$

so the K-contact metric is

$$(12) \quad g_{\mathbf{a}} = d\eta_{\mathbf{a}} \circ (\mathbb{1} \otimes \Phi_{\mathbf{a}}) + \eta_{\mathbf{a}} \otimes \eta_{\mathbf{a}}.$$

Theorem 3.2 of Yamazaki states that this $\mathcal{S}_{\mathbf{a}}$ gives the fiber join $M_1 *_f \cdots *_f M_{d+1}$ a K-contact structure. It will be convenient to consider the integral symplectic forms within the set of symplectic forms $\{\omega_j\}_{j=1}^{d+1}$. The following result is implicit in [Yam99]

Proposition 3.2. *Let M be the $d+1$ fiber join of Yamazaki with its induced K-contact structures. Then M admits an action of the $(d+1)$ -torus T^{d+1} of Reeb type that leaves the K-contact structure invariant. Furthermore, the K-contact structure on M restricts to the standard toric contact structure on the fiber $F_x \approx S^{2d+1}$ for each $x \in N$.*

Proof. Following [Yam99] we see that the vector fields $\xi_j + \partial_{\theta_j}$ span an $(d+1)$ -dimensional Abelian Lie algebra \mathfrak{t}_{d+1} which generate the T^{d+1} action. Furthermore, it is clear from Equation (8) that $\xi_{\mathbf{a}} \in \mathfrak{t}_{d+1}$, so the action is of Reeb type. Note also that the vector fields $\xi_j - \partial_{\theta_j}$ restricted to a fiber $F_x \approx S^{2d+1}$ are tangent to F_x . So the torus T^{d+1} acts on each fiber. Thus, the Reeb vector field $\xi_{\mathbf{a}}$ restricted to F_x is tangent to F_x . It follows that $\ker \eta_{\mathbf{a}} \cap TF_x$ is a codimension 1 distribution on F_x . So the restriction of $\eta_{\mathbf{a}}$ to F_x is a contact form on F_x with a T^{d+1} action of Reeb type which implies that $\eta_{\mathbf{a}}|_{F_x}$ defines the standard contact structure on $F_x \approx S^{2d+1}$. We also note that Yamazaki shows that his K-contact structure on the sphere is contact equivalent to the weighted sphere structure of Takahashi [Tak78] (This is described by Example 7.1.12 of [BG08]). \square

Remark 3.3. By defining $R = \sqrt{\sum_{j=1}^{d+1} r_j^2}$ we can identify $\bigoplus_{j=1}^{d+1} L_j^* \setminus \{\underline{0}\}$ with the cone $C(M) = M \times \mathbb{R}^+$ where $\underline{0}$ denotes the zero section and $R \in \mathbb{R}^+$.

3.2. The Sasaki Case. In this paper we are concerned with applying Yamazaki's fiber join construction to the case of Sasaki manifolds. Thus, we assume that N is a smooth projective algebraic variety and that the symplectic forms are Kähler with respect to some complex structure⁵. So the Kähler classes $[\omega_j]$ are integral classes, i.e. $[\omega_j] \in H^2(N, \mathbb{Z}) \cap H^{1,1}(N, \mathbb{R}) = \mathcal{K}_{NS}(N)$ called the Neron-Severi lattice of the Kähler cone. In this case the total space M_j of the principal S^1 bundle has a regular Sasakian structure, so the corresponding holomorphic line bundles L_j are positive satisfying $c_1(L_j) = [\omega_j]$. It is important to realize that the Kähler forms ω_j are not necessarily distinct and on the fiber join we get deformations of Sasakian structures for each $(d+1)$ -tuple $(\omega_1, \dots, \omega_{d+1})$ of Kähler forms on N such that $[\omega_j] \in \mathcal{K}_{NS}(N)$ for all $j = 1, \dots, d+1$.

So in the Sasaki case the fiber join construction requires a choice of $d+1$ elements (not necessarily distinct) of $\mathcal{K}_{NS}(N)$. This defines the set \mathfrak{S}_{d+1} consisting of $d+1$ elements of $\mathcal{K}_{NS}(N)$, and each element of $\mathfrak{S}_{d+1} \otimes \mathbb{Z}^+ \approx (\mathbb{Z}^+)^{d+1}$ gives rise to a fiber join. For each fiber join we have a Sasaki cone's worth of isotopy classes of Sasakian (K-contact) structures. We choose an ordering on $\mathfrak{S}_{d+1} \otimes \mathbb{Z}^+$ as follows: let r denote the number of distinct elements of \mathfrak{S}_{d+1} with s_j choices of $[\omega_j]$ with $j = 1, \dots, r$. Then we have $s_1 + \dots + s_r = d+1$, and we give a partial ordering by $s_1 \geq s_2 \geq \dots \geq s_r$. If $s_i = s_j$ for $i \neq j$ we make a choice to give an ordering. Within the subset consisting of elements s_k copies of $[\omega_k]$ we choose the order according to $b_1 \geq b_2 \geq \dots \geq b_{s_k}$ where $b_j \in \mathbb{Z}^+$, and again if $b_k = b_j$ we simply make a choice. So we consider $\mathfrak{S}_{d+1} \otimes \mathbb{Z}^+$ to be an ordered set whose elements are denoted by \mathfrak{w} . This gives an (almost) CR structure $(\mathcal{D}_{\mathfrak{w}}, J)$ of Sasaki (K-contact) type. We shall denote the Sasaki (K-contact) manifold constructed by this fiber join by $M_{\mathfrak{w}}$. We denote the underlying contact bundle on $M_{\mathfrak{w}}$ by $\mathcal{D}_{\mathfrak{w}}$ and write the contact manifold as $(M_{\mathfrak{w}}, \mathcal{D}_{\mathfrak{w}})$ and the corresponding CR manifold as $(M_{\mathfrak{w}}, \mathcal{D}_{\mathfrak{w}}, J)$. When $M_{\mathfrak{w}}$ is understood, we denote the underlying (almost) CR structure by $(\mathcal{D}_{\mathfrak{w}}, J)$. From the fiber join construction one sees that the contact manifold $(M, \mathcal{D}_{\mathfrak{w}})$ is independent of the order of the (M_j, ω_j) , so we shall simply choose a fixed order for our set \mathfrak{w} , regarding a different order as equivalent. A special case of interest is when $\omega_j = \omega$ for a fixed symplectic form ω and all j , that is when the integer $r = 1$. In this case we write the contact structure as $\mathcal{D}_{\mathfrak{b}}$ where $\mathfrak{b} = (b_1, \dots, b_{d+1}) \in \mathbb{Z}^{d+1}$.

⁵The fiber join can be generalized to the orbifold category, but we do not do so here

Theorem 3.4. *Let N be a smooth projective algebraic variety of complex dimension n with integral Kähler forms ω_j for $j = 1, \dots, d+1$. Let L_j be positive holomorphic line bundles on N defined by $c_1(L_j) = [\omega_j]$. Then the unit sphere bundle $M_{\mathfrak{w}}$ in $\bigoplus_j L_j^*$ with its underlying contact structure $\mathcal{D}_{\mathfrak{w}}$, has a natural Sasaki CR structure $(\mathcal{D}_{\mathfrak{w}}, J)$ with a $d+1$ -dimensional family of Sasakian structures denoted by $\mathcal{S}_{\mathbf{a}} = (\xi_{\mathbf{a}}, \eta_{\mathbf{a}}, \Phi_{\mathbf{a}}, g_{\mathbf{a}})$ for each $\mathbf{a} \in (\mathbb{R}^+)^{d+1}$ such that the Sasaki automorphism group $\mathfrak{Aut}(\mathcal{S}_{\mathbf{a}})$ contains the torus \mathbb{T}^{d+1} .*

Proof. Yamazaki shows that the structure $\mathcal{S}_{\mathbf{a}} = (\xi_{\mathbf{a}}, \eta_{\mathbf{a}}, \Phi_{\mathbf{a}}, g_{\mathbf{a}})$ is K-contact for each $\mathbf{a} \in (\mathbb{R}^+)^{d+1}$. So it suffices to show that the underlying almost CR structure on the contact bundle $\mathcal{D}_{\mathfrak{w}}$ is integrable. For each Reeb field $\xi_{\mathbf{a}} \in \mathfrak{t}_{sph}^+$ we have the decomposition $\mathcal{D}_{\mathfrak{w}} \approx \mathcal{H} \oplus (\mathcal{V}/\mathbb{R}\xi_{\mathbf{a}})$. The isomorphism $\mathcal{H} \approx \pi^*TN$ shows that the almost complex structure on \mathcal{H} is the lifted integrable complex structure on TN . Moreover, as in the proof of Proposition 3.2 the almost complex structure on $\mathcal{V}/\mathbb{R}\xi_{\mathbf{a}}$ is that of the weighted sphere. But it follows from Equation 7.1.3 of [BG08] that this is independent of the weights \mathbf{a} and coincides with the standard transverse complex structure on the sphere S^{2d+1} . Thus, for any choice of weight vector \mathbf{a} the K-contact structure $\mathcal{S}_{\mathbf{a}}$ is Sasakian. Moreover, since the Sasakian structure restricted to a fiber belongs to the standard toric contact structure of Reeb type, the Sasaki automorphism group $\mathfrak{Aut}(\mathcal{S}_{\mathbf{a}})$ contains the torus \mathbb{T}^{d+1} for each Reeb vector field $\xi_{\mathbf{a}}$ in the Sasaki cone \mathfrak{t}^+ . \square

Remark 3.5. Note that when the symplectic forms ω_j are Kähler, Yamazaki's assumption that $\sum_{j=0}^n \frac{1}{a_j} r_j^2 \pi^* \omega_j$ is non-degenerate is automatically satisfied.

3.3. The Orbifold Quotients. For each quasi-regular Reeb field $\xi_{\mathbf{a}}$ we have an S^1 orbi-bundle over the projective orbifold $\mathbb{P}_{\mathbf{a}}[\bigoplus L_j^*]$ which is an orbi-bundle over N with fiber the weighted projective space $\mathbb{C}\mathbb{P}^d[\mathbf{a}]$. The S^1 action is generated by a quasi-regular Reeb vector field lying in the Sasaki subcone \mathfrak{t}_{sph}^+ of M . First consider the fiber-wise \mathbb{C}^* action with weight vector $\mathbf{a} = (a_1, \dots, a_{d+1}) \in (\mathbb{Z}^+)^{d+1}$, viz. $\mathcal{A} : \bigoplus_{j=1}^{d+1} L_j^* \rightarrow \bigoplus_{j=1}^{d+1} L_j^*$ defined by

$$(13) \quad \mathcal{A}(v_1, \dots, v_{d+1}) = (\lambda^{a_1} v_1, \dots, \lambda^{a_{d+1}} v_{d+1}).$$

We denote the group of this action by $\mathbb{C}_{\mathbf{a}}^*$. Restricting this to the circle subgroup $S_{\mathbf{a}}^1 \subset \mathbb{C}_{\mathbf{a}}^*$ action on the unit sphere bundle M gives the identification

$$(14) \quad (\bigoplus_{j=1}^{d+1} L_j^* \setminus \underline{0}) / \mathbb{C}_{\mathbf{a}}^* = M / S_{\mathbf{a}}^1.$$

Here and hereafter we assume that $\gcd(a_1, \dots, a_{d+1}) = 1$. The circle group $S_{\mathbf{a}}^1 \subset \mathbb{T}^{d+1}$ acts only on the fibers $F_x \approx S^{2d+1}$ as a weighted circle action. It thus acts locally freely on $M_{\mathbf{w}}$, so the quotient $M_{\mathbf{w}}/S_{\mathbf{a}}^1$ is a projective algebraic orbifold. In fact, it follows from Equation (14) that we have the weighted projectivization:

$$M_{\mathbf{w}}/S_{\mathbf{a}}^1 = (\oplus_{j=1}^{d+1} L_j^* \setminus \underline{0})/\mathbb{C}_{\mathbf{a}}^* =: \mathbb{P}_{\mathbf{a}}[\oplus_{j=1}^{d+1} L_j^*]$$

which is a ‘fiber bundle’ over N whose fibers are orbifolds, specifically a weighted projective space $\mathbb{C}\mathbb{P}^d[\mathbf{a}]$ with its canonical orbifold structure. So for each weighted $\mathbb{C}_{\mathbf{a}}^*$ action we have the commutative diagram

$$(15) \quad \begin{array}{ccccc} S_{\mathbf{a}}^1 & \longrightarrow & S^{2d+1} & \longrightarrow & \mathbb{C}\mathbb{P}^d[\mathbf{a}] \\ \downarrow id & & \downarrow & & \downarrow \\ S_{\mathbf{a}}^1 & \longrightarrow & M_{\mathbf{w}} & \longrightarrow & \mathbb{P}_{\mathbf{a}}(\oplus_{j=1}^{d+1} L_j^*) \\ & & \downarrow & & \downarrow \\ & & N & \xrightarrow{id} & N. \end{array}$$

The regular case when $\mathbf{a} = (1, \dots, 1)$ is of particular interest to us where we have the standard projectivization [BT82].

$$(16) \quad \begin{array}{ccccc} S^1 & \longrightarrow & S^{2d+1} & \longrightarrow & \mathbb{C}\mathbb{P}^d \\ \downarrow id & & \downarrow & & \downarrow \\ S^1 & \longrightarrow & M_{\mathbf{w}} & \longrightarrow & \mathbb{P}(\oplus_{j=1}^{d+1} L_j^*) \\ & & \downarrow & & \downarrow \\ & & N & \xrightarrow{id} & N. \end{array}$$

3.4. Reduction of the Structure Group. Since the vector bundle E splits as a sum of nontrivial complex line bundles, the transition functions of E takes its values in the complex torus $\mathbb{T}_{\mathbb{C}}^{d+1}$. They can be represented by a $d+1$ by $d+1$ diagonal matrix $\gamma = (\gamma_1, \dots, \gamma_{d+1})$. Now each holomorphic line bundle corresponds to an element of $H^1(N, \mathcal{O}^*)$, i.e. equivalence classes of Čech cocycles, represented by transition functions $g_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha} \cap U_{\beta})$ where equivalence is given by

$$g'_{\alpha\beta} = \phi_{\alpha} g_{\alpha\beta} \phi_{\beta}^{-1}$$

for some $\phi_\alpha \in \mathcal{O}^*(U_\alpha)$. Then the transition functions⁶ of $\mathbb{T}_{\mathbb{C}}^{d+1}$ takes the form

$$\begin{pmatrix} g_{\alpha\beta}^{(1)} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & g_{\alpha\beta}^{(d+1)} \end{pmatrix}$$

where $g_{\alpha\beta}^{(j)} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$. If we let $g_{\alpha\beta}$ denote the transitions functions for a positive line bundle L , the transition functions for L^* are $g_{\alpha\beta}^{-1}$. The question is when can we reduce these transition functions to a \mathbb{C}^* subgroup? The transition functions for such a subgroup, denoted by $\mathbb{C}_{\mathbf{b}}^*$, takes the form

$$(17) \quad G_{\alpha\beta}(\mathbf{b}) = \begin{pmatrix} g_{\alpha\beta}^{-b_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & g_{\alpha\beta}^{-b_{d+1}} \end{pmatrix}$$

where $b_j \in \mathbb{Z}^+$.

We make use of the following lemma whose proof is easy.

Lemma 3.6. *Let $M_{\mathbb{w}} = M_1 *_f \cdots *_f M_{d+1}$ be a fiber join with a collection \mathfrak{S}_{d+1} of $(d+1)$ Kähler classes $[\omega_j]$ (not necessarily distinct). Then the following are equivalent:*

- (1) *All $[\omega_j] \in \mathfrak{S}_{d+1} \subset \mathcal{K}_{NS}(N)$ are colinear;*
- (2) *there exists a primitive Kähler class $[\omega_N] \in \mathcal{K}_{NS}(N)$ and $b_j \in \mathbb{Z}^+$ such that $[\omega_j] = b_j[\omega_N]$ for all $j = 1, \dots, d+1$;*
- (3) *after possibly tensoring L_j by a flat line bundle, the line bundles L_j take the form $L_j = L^{b_j}$ with $b_j \in \mathbb{Z}^+$ where $c_1(L) = [\omega_N]$ for some positive holomorphic line bundle $L \in H^1(N, \mathcal{O}^*)$.*

We have the exact sequence

$$0 \longrightarrow H^1(N, \mathbb{Z}) \longrightarrow H^1(N, \mathcal{O}) \longrightarrow H^1(N, \mathcal{O}^*) \xrightarrow{c_1} H^2(N, \mathbb{Z}) \longrightarrow H^2(N, \mathcal{O})$$

so generally the positive line bundles L_j are not unique. Moreover, for positive line bundles $L \in H^1(N, \mathcal{O}^*)$, the image $c_1(L)$ lies in the Neron-Severi lattice $\mathcal{K}_{NS}(N) \subset H^2(N, \mathbb{Z})$.

Lemma 3.7. *For a fiber join $M_{\mathbb{w}} = M_1 *_f \cdots *_f M_{d+1}$, the group $\mathbb{T}_{\mathbb{C}}^{d+1}$ of the bundle E reduces to the \mathbb{C}^* subgroup defined by $G_{\alpha\beta}(\mathbf{b})$ if and only if all $[\omega_j] \in \mathfrak{S}_{d+1}$ are colinear in $\mathcal{K}_{NS}(N)$.*

⁶The parentheses in the superscript indicates an independent labelling; whereas, without the parentheses such as $g_{\alpha\beta}^b$ will indicate a power of $g_{\alpha\beta}$.

Proof. Since $M_{\mathfrak{w}}$ is a fiber join, there exist positive line bundles L_j on N such that $c_1(L_j) = [\omega_j]$. Suppose further that all $[\omega_j]$ are colinear. Then by Lemma 3.6 there exist $b_j \in \mathbb{Z}$ and a positive holomorphic line bundle L such that $L_j = L^{b_j}$ for all $j = 1, \dots, d+1$. So the transition functions of L_j are $g_{\alpha\beta}^{b_j}$ where $g_{\alpha\beta}$ are the transition functions of L . Thus, the transition functions of $E = \bigoplus_j L_j^*$ reduce to the 1-parameter subgroup $G_{\alpha\beta}(\mathbf{b})$ of Equation (17).

Conversely, if we can reduce $\mathbb{T}_{\mathbb{C}}^{d+1}$ to the 1-parameter subgroup $G_{\alpha\beta}(\mathbf{b})$, the transition functions will take the form in Equation (17) up to re-ordering. Since the Sasaki manifold $M_{\mathfrak{w}}$ is a fiber join, each L_j is a positive line bundle. But then the holomorphic line bundle L defined by the cocycle $\{g_{\alpha\beta}\}$ is also positive. So there exists a primitive Kähler class $[\omega_N] \in \mathcal{K}_{NS}(N)$ such that $c_1(L) = [\omega_N]$ and $L_j = L^{b_j}$. It follows from Lemma 3.6 that all $[\omega_j] \in \mathfrak{S}_{d+1}$ are colinear. \square

We see that the set of Sasaki fiber joins $M_{\mathfrak{w}}$ divides naturally into 2 classes:

- (1) All the Kähler classes $[\omega_j] \in \mathfrak{S}_{d+1}$ are colinear, in which case we say that the fiber join $M_{\mathfrak{w}}$ is *colinear*;
- (2) not all Kähler classes $[\omega_j] \in \mathfrak{S}_{d+1}$ are colinear, in which case we say that the fiber join is *non-colinear*.

We have

Proposition 3.8. *Consider the fiber join $M_{\mathfrak{w}} = M_1 *_f \dots *_f M_{d+1}$ defined by the collection \mathfrak{S}_{d+1} . Then*

- (1) *when all Kähler classes $[\omega_j] \in \mathfrak{S}_{d+1}$ are colinear, in which case $\omega_j = b_j \omega_N$ for some positive line bundle L with $c_1(L) = [\omega_N]$ and $b_j \in \mathbb{Z}^+$, the orbifold quotient $\mathbb{P}_{\mathbf{a}}(\bigoplus_{j=1}^{d+1} L_j^*)$ is a product $N \times \mathbb{C}\mathbb{P}^d[\mathbf{a}]$ if and only if $\mathbf{b} = b\mathbf{a}$ where $b = \gcd(b_1, \dots, b_{d+1})$. Hence, the Sasaki CR structure $(\mathcal{D}_{\mathfrak{w}}, J)$ is cone decomposable;*
- (2) *when not all Kähler classes $[\omega_j] \in \mathfrak{S}_{d+1}$ are colinear, the quotient orbifold $\mathbb{P}_{\mathbf{a}}(\bigoplus_{j=1}^{d+1} L_j^*)$ is not a product for any $\mathbf{a} \in (\mathfrak{t}_{sph}^+)^* \cap \mathbb{Q}$, and $(\mathcal{D}_{\mathfrak{w}}, J)$ is cone indecomposable.*

Proof. Case (1): By Lemma 3.6 we can write $L_i = L^{b_i}$ where L satisfies $c_1(L) = [\omega_N]$. So we are interested in the weighted projectivization

$$(18) \quad \mathbb{P}_{\mathbf{a}}(\bigoplus_{j=1}^{d+1} L_j^*) = ((\bigoplus_{j=1}^{d+1} L_j^*) \setminus \{0\}) / \mathbb{C}_{\mathbf{a}}^*.$$

The $\mathbb{C}_{\mathbf{a}}^*$ action on $\bigoplus_{j=1}^{d+1} L_j^*$ is given by

$$(19) \quad (v_1, \dots, v_{d+1}) \mapsto (\lambda^{a_1} v_1, \dots, \lambda^{a_{d+1}} v_{d+1})$$

for some $\lambda \in \mathbb{C}^*$ and $a_j \in \mathbb{Z}^+$. By Lemma 3.7 the transition functions for the vector bundle $\bigoplus_{j=1}^{d+1} L_j^*$ take the form of the diagonal matrix

$(\gamma^{b_1}, \dots, \gamma^{b_{d+1}})$. Since these \mathbb{C}^* actions commute, we have a global $\mathbb{C}_{\mathbf{a}}^*$ action on $\bigoplus_{j=1}^{d+1} L_j^*$, so the projectivization given by Equation (18) is well defined.

Note that if we choose \mathbf{a} such that $\mathbf{b} = b\mathbf{a}$, then we can choose the matrix of the $\mathbb{C}_{\mathbf{a}}^*$ action to coincide with the transition function matrix, by choosing $\lambda = \gamma^b$ (restricting to an open set of a good cover, for example). This reduces the induced transition functions on the projectivization $\mathbb{P}_{\mathbf{a}}(\bigoplus_{j=1}^{d+1} L_j^*)$ to the identity. Thus, it is the product

$$\mathbb{P}_{\mathbf{a}}(\bigoplus_{j=1}^{d+1} L_j^*) = N \times \mathbb{C}\mathbb{P}^d[\mathbf{a}].$$

Furthermore, for any other choice of \mathbf{a} the $\mathbb{C}_{\mathbf{a}}^*$ action does not coincide with the transition functions, and will not reduce them to the identity.

Case (2): Since the $[\omega_j] \in \mathfrak{S}_{d+1}$ are not all colinear, the group of the bundle E does not reduce to a \mathbb{C}^* subgroup by Lemma 3.7, but only to a $\mathbb{T}_{\mathbb{C}}^k$ subgroup for $k = 2, \dots, d+1$. But then the action (19) cannot coincide with the transition functions when restricted to open sets. So the projective group does not reduce to the identity. \square

Remark 3.9. Note that the unit sphere bundle of L_j^* is identified with M_j and is the Sasaki submanifold of M obtained by setting $r_i = 0$ for $i \neq j$. The fact that for the Sasaki case the submanifolds M_j are all Sasaki force the integers a_j in the projectivization to be all positive integers.

As described in [BGS08] the T^{d+1} action gives rise to the (unreduced) Sasaki cone \mathfrak{t}_{sph}^+ on M . It follows from Proposition 4.2 of [Yam99] that any other Reeb vector field $\xi_{\mathbf{a}'}$ of the form (8) with \mathbf{a} replaced by $\mathbf{a}' \in (\mathbb{R}^+)^{d+1}$ is an element of \mathfrak{t}_{sph}^+ . It follows that

Proposition 3.10. *The contact structure $\mathcal{D}_{\mathbf{a}} = \ker \eta_{\mathbf{a}}$ defined by Equation (7) is independent of \mathbf{a} with $\xi_{\mathbf{a}} \in \mathfrak{t}_{sph}^+$, and only depends on the regular Sasaki manifolds M_j , or equivalently on the collection \mathfrak{S}_{d+1} (up to order). Accordingly, we denote the underlying contact structure by $\mathcal{D}_{\mathfrak{w}}$.*

3.5. The Colinear Case. We consider the special case when $\omega_j = b_j \omega_N$ for some Kähler form on N . Proposition 3.8 says that this will be a product precisely when $\mathbf{b} = b\mathbf{a}$. We now compute the induced Kähler form on $N \times \mathbb{C}\mathbb{P}^d[\mathbf{a}]$.

Proposition 3.11. *The Kähler form on $N \times \mathbb{C}\mathbb{P}^d[\mathbf{a}]$ induced by the quasiregular Sasakian structure $\mathcal{S}_{\mathbf{a}} = (\xi_{\mathbf{a}}, \eta_{\mathbf{a}}, \Phi_{\mathbf{a}}, g_{\mathbf{a}})$ on the fiber join $M_{\mathfrak{w}}$ is $b\omega_N + \omega_{\mathbf{a}}$. Equivalently, the transverse Kähler structure on $M_{\mathfrak{w}}$*

is

$$(20) \quad \pi_{\mathbf{a}}^*(b\omega_N + \omega_{\mathbf{a}}) = d\eta_{\mathbf{a}}$$

where $\eta_{\mathbf{a}}$ is given in Equation (7).

Proof. The Kähler form on the product $N \times \mathbb{C}\mathbb{P}^d[\mathbf{a}]$ takes the form $l_1\omega_N + l_2\omega_{\mathbf{a}}$ for some positive integers l_1, l_2 . We have the following commutative diagram

$$(21) \quad \begin{array}{ccc} M_i & \xrightarrow{\iota} & M_{\mathfrak{w}} \\ & \searrow^{\tilde{\pi}_i} & \downarrow \pi_{\mathbf{a}} \\ & & N \times \mathbb{C}\mathbb{P}^d[\mathbf{a}] \\ & \searrow^{\pi_i} & \downarrow pr_1 \\ & & N \end{array}$$

where pr_1 and the π 's are the obvious projections, and ι is the natural inclusion obtained by setting $r_j = 0$ for $j \neq i$. It follows from Equation (7) that the restriction of the Sasaki structure $\mathcal{S}_{\mathbf{a}} = (\xi_{\mathbf{a}}, \eta_{\mathbf{a}}, \Phi_{\mathbf{a}}, g_{\mathbf{a}})$ on $M_{\mathfrak{w}}$ to the submanifold M_j is the Sasakian structure $a_j^{-1}\mathcal{S}_j$ not \mathcal{S}_j . Indeed, we have $d\eta_{\mathbf{a}}|_{M_i} = a_i^{-1}d\eta_i$, so by Diagram (21)

$$\tilde{\pi}_i^*(l_1\omega_N + l_2\omega_{\mathbf{a}}) = d\eta_{\mathbf{a}}|_{M_i} = a_i^{-1}d\eta_i = a_i^{-1}b_i\pi_i^*\omega_N$$

which implies $l_1 = \frac{b_i}{a_i}$ and $\tilde{\pi}_i^*\omega_{\mathbf{a}} = 0$. But then Proposition 3.8 gives $l_1 = b$. We also know that the Kähler form $\omega_{\mathbf{a}}$ on $\mathbb{C}\mathbb{P}^d[\mathbf{a}]$ is induced by the weighted Sasakian structure $\mathcal{S}_{\mathbf{a}}$ on each fiber S^{2d+1} ; hence, $l_2 = 1$. \square

3.6. The Non-colinear Case. In the case where the ω_j are not generated by a single Kähler form ω_N , there exists at least two line bundles L_1 and L_2 whose transition functions are independent, say γ_1 and γ_2 ; they generate a 2-torus \mathbb{T}^2 that does not reduce to a circle action.

The contact bundle is $\mathcal{D}_{\mathfrak{w}} = \ker \eta_{\mathbf{a}}$ for all $a \in (\mathfrak{t}_{sph}^+)^*$ and the real line bundle generated by $\xi_{\mathbf{a}}$ is denoted by $\mathbb{R}\xi_{\mathbf{a}}$. It is convenient to decompose the tangent bundle of M into horizontal and vertical parts as $TM = \mathcal{H} + \mathcal{V}$. We then see that $\mathbb{R}\xi_{\mathbf{a}} \subset \mathcal{V}$ and $\mathcal{H} \subset \mathcal{D}_{\mathbf{a}}$ and we have the decompositions

$$TM = \mathcal{H} + \mathcal{V}/\mathbb{R}\xi_{\mathbf{a}} + \mathbb{R}\xi_{\mathbf{a}}, \quad \mathcal{D}_{\mathfrak{w}} = \mathcal{H} + \mathcal{V}/\mathbb{R}\xi_{\mathbf{a}}.$$

In fact, Yamazaki constructs a K-contact structure $(\xi_{\mathbf{a}}, \eta_{\mathbf{a}}, \Phi_{\mathbf{a}}, g_{\mathbf{a}})$ with integral symplectic forms ω_j satisfying $\pi_j^*\omega_j = d\eta_j$ as long as the 2-form $\sum_{j=1}^{d+1} \frac{r_j^2}{a_j} \pi^*\omega_j$ is non-degenerate. However, in the Sasaki case this non-degeneracy condition is automatic.

3.7. Topology of the Projectivization. For each j the principal S^1 bundle M_j over N is classified by homotopy classes of maps $[N, BS^1] = [N, \mathbb{C}\mathbb{P}^\infty]$. Since the fiber join is constructed from the split complex vector bundle $E = \bigoplus_j L_j^*$, the group of the bundle reduces to the maximal complex torus $\mathbb{T}_{\mathbb{C}}^{d+1} = (\mathbb{C}^*)^{d+1}$ as we have previously discussed. So equivalence classes of such complex vector bundles are in one-to-one correspondence with homotopy classes of maps $[N, B(S^1)^{d+1}] = [N, (BS^1)^{d+1}] = [N, (\mathbb{C}\mathbb{P}^\infty)^{d+1}]$. The cohomology ring of the latter is the polynomial ring $\mathbb{Z}[x_1, \dots, x_{d+1}]$ where $x_j = c_1(\mathcal{O}(1))$ is the universal first Chern class of the dual of the tautological bundle. Being the universal classes each x_j pullsback to a negative integral Kähler class $-b_j[\omega_j]$.

Consider now the projectivization $\mathbb{P}(\bigoplus_j L_j^*)$ which is defined by projecting the bundle E fiberwise. So the fibers of $\mathbb{P}(\bigoplus_j L_j^*)$ are projective spaces $\mathbb{C}\mathbb{P}^d$. That is, $\mathbb{P}(\bigoplus_j L_j^*)$ is a $\mathbb{C}\mathbb{P}^d$ bundle over N . The transition functions for $\mathbb{P}(\bigoplus_j L_j^*)$ are equivalence classes \mathcal{G} of matrices where $\mathcal{G}' \sim \mathcal{G}$ if and only if there exists $\lambda \in \mathbb{C}^*$ such that $\mathcal{G}' = \lambda \mathcal{G}$. Since $c_1(L_j) = -b_j[\omega_j]$, we see that the cohomology ring is (cf. [BT82] page 270)

$$H^*(\mathbb{P}(\bigoplus_{j=1}^{d+1} L_j^*), \mathbb{Z}) = H^*(N, \mathbb{Z})[x] / \left(\prod_{j=1}^{d+1} (x + b_j[\omega_j]) \right).$$

Note that x is a 2-class and in the denominator the coefficient of x^{d+1-k} is the k th elementary symmetric function $\sigma_k(b_1[\omega_1], \dots, b_{d+1}[\omega_{d+1}])$. So x^{d+1} is a linear combination of $\{1, x, \dots, x^d\}$ with coefficients in $H^*(N, \mathbb{Z})$ which are precisely the Chern classes of E . This gives

$$x^{d+1} = - \sum_{k=1}^d \sigma_k(b_1[\omega_1], \dots, b_{d+1}[\omega_{d+1}]) x^{d+1-k}.$$

If we consider the unit sphere bundle M in $\bigoplus_j L_j^*$ with its canonical contact structure $\mathcal{D}_{\mathfrak{w}}$, the projectivized bundle $\mathbb{P}(\bigoplus_j L_j^*)$ coincides with the quotient of M by the S^1 action generated by the regular Reeb vector field ξ_1 on M obtained by choosing $\mathbf{a} = (1, \dots, 1)$. Note in the special case that $c_1(L_j) = b_j[\omega_N]$ for projective bundle $\mathbb{P}(\bigoplus_{j=1}^{d+1} L_j^*)$ we can, without loss of generality, order the b_j s as $b_1 \leq b_2 \leq \dots \leq b_{d+1}$ and then take $b_1 = 1$ in which case $\mathbb{P}(\bigoplus_{j=1}^{d+1} L_j^*)$ becomes $\mathbb{P}(\mathbb{1} \oplus \bigoplus_{j=2}^{d+1} (L^*)^{b_j})$ where $c_1(L) = [\omega_N]$.

3.8. The First Chern Class of $\mathcal{D}_{\mathfrak{w}}$. The first Chern class $c_1(\mathcal{D})$ of the contact bundle \mathcal{D} is a fundamental invariant of contact structures. For the fiber join we can express $c_1(\mathcal{D}_{\mathfrak{w}})$ in terms of invariants on N .

Proposition 3.12. *Let $M_{\mathfrak{w}}$ be the unit sphere bundle in the vector bundle $E = \bigoplus_j L_j^*$ over N obtained as the fiber join $M_{\mathfrak{w}} = M_1 *_f \cdots *_f M_{d+1}$ with its canonical family of Sasakian structures with contact bundle $\mathcal{D}_{\mathfrak{w}}$. Then*

$$c_1(\mathcal{D}_{\mathfrak{w}}) = \pi_M^* \left(\sum_{j=1}^{d+1} c_1(L_j^*) + c_1(N) \right) = \pi_M^* \left(- \sum_{j=1}^{d+1} [\omega_j] + c_1(N) \right).$$

where $[\omega_j]$ are the Kähler classes in $\mathfrak{S}_{d+1} \subset \mathcal{K}_{NS}(N)$ defined up to order.

Proof. First we set $E_0 = E - \{\text{zero section}\}$ and we identify the zero section with N . We let $\pi_0 : E_0 \rightarrow N$ denote the restriction of $\pi : E \rightarrow N$ to E_0 . Let h be a Hermitian metric on E . Following Milnor and Stasheff [MS74] we construct a “canonical” complex d -plane bundle \mathcal{E} over E_0 as follows. A point of E_0 is specified by a fiber F of E together with a non-zero complex vector v in F . Then the fiber of \mathcal{E} over $v \in E_0$ is the orthogonal complement of v in the vector space F .

Now since M is the unit sphere bundle in E and hence in E_0 , we can identify E_0 with the cone $C(M) = M \times \mathbb{R}^+$ by thinking of a point of M as the pair (x, u) with $x \in N$ and $\|u\| = 1$. Then the pair (x, v) with $\|v\| = r$ gives the triple $(x, \frac{v}{\|v\|}, r)$ as a point of $C(M)$ with $u = \frac{v}{\|v\|}$. In terms of the coordinates (r_j, θ_j) on E_0 we identify the vector v with complex coordinates $z_j = r_j e^{i\theta_j}$ and $r = \sqrt{\sum_{j=1}^{d+1} r_j^2}$.

Let $\tilde{\mathcal{D}}$ denote the pullback of \mathcal{D} to $C(M)$. We claim that under the identification of $C(M)$ and E_0 , $\tilde{\mathcal{D}}$ is identified with $\pi_0^*TN + \mathcal{E}$ as real vector bundles. To see this we note that at the point $p = (x, \frac{v}{\|v\|}, r)$ in the cone $C(M)$ the vector $\Psi_p = r(\partial_r)_p$ is perpendicular to the sphere S^{2d+1} in the fiber $\pi_0^{-1}(x) \subset E_0 = C(M)$. Now fix a Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ on M in the family with underlying contact structure \mathcal{D} . This induces a polarized Kählerian structure $(d(r^2\eta), \tilde{g} = dr^2 + r^2g, I, \xi)$ on $C(M)$. Using the complex structure I on $C(M)$ we obtain a complex vector $\Psi_p + I\Psi_p = \Psi_p - i\xi_p$. This gives the Reeb vector field $-\xi$ which belongs to the conjugate Sasakian structure $(-\xi, -\eta, -\Phi, g)$. This implies that the complex structure I on $C(M)$ is the complex conjugate to the complex structure on E_0 under the identification described above.

The orthogonal complement of $\Psi_p - i\xi_p$ in $T_p C(M)^{\mathbb{C}} = T_p E_0^{\mathbb{C}}$ with respect to the Hermitian metric \tilde{g} is $\tilde{\mathcal{D}}$. Thus, we can identify $\tilde{\mathcal{D}}$ with $\pi_0^*TN + \mathcal{E}$. This identification is independent of the choice of

Sasakian structure in the family since all members correspond to the same contact bundle \mathcal{D} .

With the above identification and the definition of Chern classes given in [MS74], we have for $n \geq 1$

$$\sum_{j=1}^{d+1} c_1(L_j^*) = c_1(\oplus_{j=1}^{d+1} L_j^*) = c_1(E) = (\pi_0^*)^{-1} c_1(\mathcal{E}) = (\pi_0^*)^{-1} c_1(\tilde{\mathcal{D}}) - c_1(TN).$$

This implies $c_1(\tilde{\mathcal{D}}) = c_1(\mathcal{E}) + \pi_0^* c_1(TN)$. But $c_1(L_j^*) = -[\omega_j]$. \square

Remark 3.13. This proof works for the k th Chern class when $2k < 2d + 1$, since then $\pi_0^* : H^{2k}(N, \mathbb{Z}) \rightarrow H^{2k}(E_0, \mathbb{Z})$ is an isomorphism. This gives the k th Chern class in terms of the k th elementary symmetric polynomial, viz

$$(22) \quad c_k(\mathcal{D}) = \pi_M^* (\sigma_k(-[\omega_1], \dots, -[\omega_{d+1}]) + c_k(N)).$$

If $d_0 = d_\infty = 0$ it does not hold for c_2 in which case c_2 is the Euler class of the real bundle $E_{\mathbb{R}}$.

We can rewrite the equation of Proposition 3.12 in terms of the ordering described previously.

Corollary 3.14. *For the general fiber join $M_{\mathfrak{w}}$ we have*

(1) $\pi_M^* : H^2(N, G) \rightarrow H^2(M_{\mathfrak{w}}, G)$ is an isomorphism for any coefficients G .

(2) $c_1(\mathcal{D}_{\mathfrak{w}}) =$

$$\pi_M^* \left(-(\sigma_1(b_1, \dots, b_{s_1})[\omega_1] + \dots + \sigma_1(b_{s_{r-1}+1}, \dots, b_{s_{r-1}+s_r})[\omega_r]) + c_1(N) \right),$$

where σ_1 is the first elementary symmetric function.

(3) $w_2(M_{\mathfrak{w}}) = \pi_M^* (w_2(E) + w_2(N))$.

(4) $M_{\mathfrak{w}}$ is a spin manifold if and only if E and N are either both spin or both non-spin.

Proof. Since $d \geq 1$ item (1) follows from the Gysin sequence while items (2), (3), and (4) are immediate. \square

In the colinear case it follows directly that

$$(23) \quad c_1(\mathcal{D}_{\mathfrak{b}}) = \pi_M^* (-|\mathfrak{b}|[\omega_N] + c_1(N))$$

3.9. Relation between Fiber Joins and Regular Sphere Joins.

In this section we prove that a colinear fiber join is equivalent to a regular sphere join. We refer to the join $M \star_{l_1, 1} S_{\mathfrak{w}}^{2d+1}$ as studied in [BTF16], but here with a $2d + 1$ dimensional sphere, as a *regular sphere join*.

Proposition 3.15. *Let M be the Sasaki manifold which as a principal S^1 bundle over N has Euler class $[\omega_N]$ and let*

$$M_{\mathbf{b}} = M_1 *_f \overset{k \text{ times}}{\dots} *_f M_{d+1}$$

be a colinear fiber join where $M_j = M/\mathbb{Z}_{b_j}$ has Euler class $b_j[\omega_N]$ for $j = 1, \dots, d+1$. Then the fiber join $M_{\mathbf{b}}$ can be identified with the regular sphere join $M \star_{l,1} S_{\mathbf{w}}^{2d+1}$, where $\mathbf{b} = b\mathbf{w}$ and $l = b = \gcd(b_1, \dots, b_{d+1})$.

Proof. Since the fiber join $M_{\mathbf{b}}$ is colinear, it has a product projectivization $N \times \mathbb{C}\mathbb{P}^d(\mathbf{w})$ of the Reeb vector field defined by $\xi_{\mathbf{w}}$ precisely when $\mathbf{b} = b\mathbf{w}$ by (1) of Proposition 3.8, and by Proposition 3.11 the Kähler form on $N \times \mathbb{C}\mathbb{P}^d(\mathbf{w})$ is $b\omega_N + \omega_{\mathbf{w}}$.

Now one easily sees from [BTF16] that the regular $S_{\mathbf{w}}^{2d+1}$ join also has the projectivization $N \times \mathbb{C}\mathbb{P}^d(\mathbf{w})$ with Kähler form $l\omega_N + \omega_{\mathbf{w}}$. Thus, to identify the principal S^1 bundles we must take $l = b$. This identifies the transverse Kähler structures of the fiber join $M_{\mathbf{b}}$ and the regular sphere join $M \star_{l,1} S_{\mathbf{w}}^{2d+1}$. However, this will not uniquely determine a Sasakian structure [Noz14] up to isomorphism unless the first Betti number $b_1(M) = 0$. Nevertheless, we can always choose the connection 1-forms to coincide which will identify the joins. \square

Recall [BHLTF18] that a Sasaki CR structure (\mathcal{D}, J) is *cone decomposable* if there is a decomposable Sasakian structure $\mathcal{S}_{\mathbf{a}} \in \mathfrak{t}^+$. This implies by construction that $\mathcal{S}_{\mathbf{a}}$ is quasiregular and its quotient orbifold is a product. Hence, by (2) of Proposition 3.8 a non-colinear fiber join is cone indecomposable. Thus, we have

Corollary 3.16. *A fiber join $M_{\mathbf{w}}$ is cone decomposable if and only if it is colinear.*

We can now rephrase Proposition 3.8 as

Proposition 3.17. *A fiber join $M_{\mathbf{w}}$ has a quasiregular Reeb vector field $\xi_{\mathbf{a}} \in \mathfrak{t}_{sph}^+$ whose quotient is the product orbifold $N \times \mathbb{C}\mathbb{P}^1[\mathbf{a}]$ if and only if it is cone decomposable. Furthermore, if the Picard number of N is 1 any fiber join $M_{\mathbf{w}}$ over N is cone decomposable.*

Remark 3.18. Proposition 3.15 says that a colinear fiber join is equivalent to a regular fiber join of the form $M \star_{l,1} S_{\mathbf{w}}^{2d+1}$; nevertheless, there are sphere bundles that are regular joins with $l_2 > 1$. So sphere bundles that are regular joins are more general than colinear fiber joins. Here is a family of counterexamples giving regular joins that are not fiber joins.

Example 3.1. *The well known $Y^{p,q}$ first studied in [GMSW04] and described as a regular $S^3_{\mathbf{w}}$ join in [BTF16] are such counterexamples for all $p > 1$. Indeed Example 6.8 of [BTF16] shows that*

$$Y^{p,q} = S^3 \star_{l_1, l_2} S^3_{\mathbf{w}}$$

where

$$l_1 = \gcd(p+q, p-q), \quad l_2 = p, \quad \mathbf{w} = \frac{1}{l_1}(p+q, p-q),$$

and it is diffeomorphic to $S^2 \times S^3$ for $1 \leq q \leq p$ with $\gcd(p, q) = 1$. Since $l_2 = p > 1$, $Y^{p,q}$ is a regular S^3 join but not a fiber join.

4. EXTREMAL, CSC, AND EINSTEIN SASAKI METRICS ON FIBER JOINS

In this section we will apply Proposition 3.15 and known existence results to explore examples of fiber joins that admit Sasaki structures that are extremal, CSC, or even Sasaki Einstein (SE). We begin by describing important admissible conditions from [ACGTF08].

4.1. The Admissible Conditions and proof of Theorem 1.1.

For a fiber join as defined in Theorem 3.4 we have that the complex manifold arising as the quotient of the regular Reeb vector field ξ_1 is equal to $\mathbb{P}(\bigoplus_j L_j^*) \rightarrow N$. Thus, in some cases this will be an *admissible* projective bundle as defined in [ACGTF08]. Specifically, following Section 1.2 of [ACGTF08] this happens exactly when the following all hold true:

- (1) The base N is a local product of Kähler manifolds (N_a, Ω_a) , $a \in \mathcal{A} \subset \mathbb{N}$, where \mathcal{A} is a finite index set.
- (2) There exist $d_0, d_\infty \in \mathbb{N} \cup \{0\}$, with $d = d_0 + d_\infty + 1$, such that $E_0 := \bigoplus_{j=1}^{d_0+1} L_j^*$ and $E_\infty := \bigoplus_{j=d_0+2}^{d_0+d_\infty+2} L_j^*$ are both projectively flat hermitian holomorphic vector bundles. This would, for example, be true if $L_j^* = L_0$ for $j = 1, \dots, d_0 + 1$ and $L_j^* = L_\infty$ for $j = d_0 + 2, \dots, d_0 + d_\infty + 2$, where L_0 and L_∞ are some holomorphic line bundles. That is, $E_0 = L_0 \otimes \mathbb{C}^{d_0+1}$ and $E_\infty = L_\infty \otimes \mathbb{C}^{d_\infty+1}$. More generally, $c_1(L_1^*) = \dots = c_1(L_{d_0+1}^*)$ and $c_1(L_{d_0+2}^*) = \dots = c_1(L_{d_0+d_\infty+2}^*)$ would be sufficient.
- (3) $\frac{c_1(E_\infty)}{d_\infty+1} - \frac{c_1(E_0)}{d_0+1} = \sum_{a \in \mathcal{A}} [\epsilon_a \Omega_a]$, where $\epsilon_a = \pm 1$.

The Kähler cone of the total space of an admissible bundle $\mathbb{P}(E_0 \oplus E_\infty) \rightarrow N$ has a subcone of so-called *admissible Kähler classes* (defined in Section 1.3 of [ACGTF08]). This subcone has dimension $|\mathcal{A}| + 1$ and, in general, this is not the entire Kähler cone. However, by Remark 2 in

[ACGTF08], if $b_2(N_a) = 1$ for all $a \in \mathcal{A}$ and $b_1(N_a) \neq 0$ for at most one $a \in \mathcal{A}$, then the entire Kähler cone is indeed admissible.

Definition 4.1. *Any fiber join $M_{\mathfrak{w}}$ where the quotient of the regular Reeb vector field ξ_1 is admissible will also be called **admissible**. If further every Kähler class in the Kähler cone of this quotient is admissible, then we call $M_{\mathfrak{w}}$ **super admissible**.*

We now recall Proposition 11 of [ACGTF08], which in turn is just a slight generalization of the the works by Guan [Gua95], Hwang [Hwa94], and Hwang-Singer [HS02].

Proposition 4.2. [ACGTF08] *Suppose that $S = \mathbb{P}(E_0 \oplus E_\infty) \rightarrow N$ is admissible where N is a local Kähler product of non-negative CSC metrics. Then every admissible Kähler class contains an (admissible) extremal Kähler metric.*

In Appendix A we shall describe what we mean by *admissible metrics*. As a corollary to Proposition 4.2 we then have the following result which then proves Theorem 1.1.

Proposition 4.3. *Suppose $M_{\mathfrak{w}}$ is a super admissible fiber join where the local Kähler product N is a product of Kähler metrics of non-negative constant scalar curvature. Then the Sasakian structure defined by the regular Reeb vector field ξ_1 has an extremal representative which is contained in an open set of extremal Sasakian structures.*

Remark 4.4. In [ACGTF08] and related papers, more specific results occur regarding the existence of smooth extremal Kähler metrics, CSC Kähler metrics and KE metrics on the total spaces of admissible projective bundles. In general, these existence results - especially for the CSC and KE cases - depend on the Kähler classes⁷. Thus, it is not feasible to apply these results to the regular quotient of admissible fiber joins unless we identify the Kähler class of the quotient Kähler structure. Excluding the trivial case where $c_1(L_1) = \cdots = c_1(L_d)$ (and thus the regular quotient is a (local) product), in this paper we will only explore this in the admissible case with $d = 1$.

In the case $d = 1$ Proposition 3.15 tells us that the colinear fiber join is identical to the regular sphere join $M \star_{l,1} S_{\mathfrak{w}}^3$, as explored thoroughly in [BTF16], where $\mathbf{b} = b\mathfrak{w}$ and $l = b = \gcd(b_1, b_2)$ with $b_1 \geq b_2$. This allows us to apply Theorems 1.1, 1.2, and 1.4 of [BTF16] as follows:

⁷Proposition 4.3 above and Proposition 5.7 below exhibit applications of exceptions to this rule.

Theorem 4.5. *Let $M_{\mathfrak{w}}$ be the Sasaki manifold which as a principal S^1 bundle over N has primitive Euler class $[\omega_N]$ and let*

$$M_{\mathfrak{w}} = M_{\mathfrak{b}} = M_1 *_f M_2$$

be a cone decomposable fiber join where $M_j = M/\mathbb{Z}_{b_j}$ has Euler class $b_j[\omega_N]$ for $j = 1, 2$. Assume that the Kähler structure of (N, ω_N) has constant scalar curvature. Then

- (1) *there exist a Reeb vector field in the Sasaki subcone $\mathfrak{t}_{sph}^+(\mathcal{D}_{\mathfrak{b}}, J)$ of $M_{\mathfrak{b}}$ such that the corresponding ray of Sasaki structures has constant scalar curvature;*
- (2) *if the scalar curvature of ω_N is non-negative, $\mathfrak{t}_{sph}^+(\mathcal{D}_{\mathfrak{b}}, J)$ is exhausted by extremal Sasaki metrics;*
- (3) *assuming further that ω_N is positive Kähler-Einstein, there exists a Reeb vector field in the Sasaki subcone $\mathfrak{t}_{sph}^+(\mathcal{D}_{\mathfrak{b}}, J)$ of $M_{\mathfrak{b}}$ such that the corresponding Sasaki metric is Sasaki-Einstein if and only if $b_1 + b_2 = \mathcal{J}_N$. Moreover, up to equivalence the number of such SE metrics equals the number of partitions of \mathcal{J}_N into the unordered sum of two positive integers.*

For $d = 1$, admissible, and non-colinear we work out the Kähler class of the regular quotient in Appendix A.3. We will then apply this to the special case explored in Section 5.3.

As mentioned above in the case $d > 1$ known existence results here are somewhat sporadic; nevertheless we do have

Proposition 4.6. *Let M be the Sasaki manifold which as a principal S^1 bundle over N has primitive Euler class $[\omega_N]$ and let*

$$M_{\mathfrak{b}} = M_1 *_f \cdots *_f M_{d+1}$$

be a cone decomposable fiber join where $M_j = M/\mathbb{Z}_{b_j}$ has Euler class $b_j[\omega_N]$ for $j = 1, \dots, d+1$. Assume that the Kähler structure of (N, ω_N) is extremal. Then the Sasaki structure corresponding to the quasi-regular Reeb field $\xi_{\mathfrak{a}}$ with \mathfrak{a} such that $\mathfrak{b}\mathfrak{a} = \mathfrak{b}$, where $b = \gcd(b_1, \dots, b_n)$ is extremal (up to isotopy). Thus, as long as (N, ω_N) is extremal, both the Sasaki cone $\mathfrak{t}^+(\mathcal{D}_{\mathfrak{b}}, J)$ and its subcone $\mathfrak{t}_{sph}^+(\mathcal{D}_{\mathfrak{b}}, J)$ of $M_{\mathfrak{b}}$ will always contain an open set of Sasaki extremal structures.

Proof. This result follows directly from Proposition 3.8, Proposition 3.11, the fact that $\mathbb{C}\mathbb{P}_{\mathfrak{a}}$ admits a canonical extremal Kähler metric, and the fact that a product of two extremal Kähler metrics is again extremal. Openness holds by [BGS08]. \square

Remark 4.7. Notice that we do not assume that $M_{\mathfrak{b}}$ is admissible in Proposition 4.6.

4.2. Sasaki-Einstein Metrics on Fiber Joins. We briefly consider the Fano case with Fano index \mathcal{J}_N . Since here N is a smooth projective Fano variety of complex dimension n , we recall the well known result of Kobayashi and Ochia [KO73] that $\mathcal{J}_N \leq n + 1$ and equality holds if and only if $N = \mathbb{C}\mathbb{P}^n$. Moreover, when $\mathcal{J}_N = n$ the variety N is a quadric. So the case when N is smooth is quite restrictive. This is not at all so restrictive when N is an orbifold [BTF19b, BTF19a]. However, here we have the following obstructions

Proposition 4.8. *Let $M_{\mathbf{w}} = M_1 \star_f \cdots \star_f M_{d+1}$ be a Sasaki fiber join with $\mathfrak{S}_{d+1} = \{[\omega_j]\}_{j=1}^{d+1}$, and suppose $M_{\mathbf{w}}$ admits an SE metric.*

- (1) *Then $c_1(N) = \sum_{j=1}^{d+1} [\omega_j]$.*
- (2) *If $M_{\mathbf{w}}$ is cone decomposable with $\omega_j = b_j \omega_N$ for some primitive Kähler form ω_N , which is equivalent to the regular join $M \star_{l,1} S_{\mathbf{w}}^{2d+1}$, then*
 - (a) *$n \geq d$ and $n = d$ if and only if $\mathbf{w} = (1, \dots, 1)$,*
 - (b) *$\mathcal{J}_N = l|\mathbf{w}|$ which implies $\mathcal{J}_N \geq l(d+1) \geq 2$ and that $|\mathbf{w}|$ must divide \mathcal{J}_N .*
 - (c) *If $M_{\mathbf{w}}$ is also admissible then*

$$\mathcal{J}_N = (d_0 + 1)b_0 + (d_\infty + 1)b_\infty \leq n + 1.$$

- (3) *If $M_{\mathbf{w}}$ is cone indecomposable and admissible, then*

$$(24) \quad c_1(N) = (d_0 + 1)[\omega_0] + (d_\infty + 1)[\omega_\infty].$$

In particular, if $d_\infty = d_0 \geq n$ there are no SE metrics for any N .

Proof. A necessary condition to have a Sasaki-Einstein (SE) metric is that the real first Chern class of the contact bundle must vanish (we will ignore any torsion). Then the first statement follows immediately from Proposition 3.12 from which it also follows that N must be Fano. Applying this to the colinear fiber join and using Proposition 3.15, we have from (23) that $\mathcal{J}_N = |\mathbf{b}| = l|\mathbf{w}|$. Thus, we have

$$(25) \quad d + 1 \leq l(d + 1) \leq l|\mathbf{w}| = |\mathbf{b}| = \mathcal{J}_N \leq n + 1$$

which implies both (a) and (b). To prove (c) we note that the admissible conditions imply (24), so setting

$$\omega_0 = b_0 \omega_N, \quad \omega_\infty = b_\infty \omega_N$$

for some $b_0, b_\infty \in \mathbb{Z}^+$ and some primitive Kähler form ω_N gives the result. The proof of the first statement in (3) is straightforward. The last statement then follows from [KO73]. \square

Remark 4.9. In the colinear case $M_{\mathbf{b}}$ it follows from (2) of Proposition 4.8 that the subcone \mathfrak{t}_{sph}^+ contains no SE metric when the Fano index \mathcal{J}_N is 1.

5. APPLICATIONS

We now apply the results of the previous sections to describe some explicit examples where the hypotheses of Propositions 5.7 and 4.3 hold.

5.1. $N = \mathbb{C}\mathbb{P}^n$. Any fiber join $M_{\mathbf{b}} = M_{b_1} \star_f \cdots \star_f M_{b_{d+1}}$ over $\mathbb{C}\mathbb{P}^n$ is cone decomposable and is determined by a vector \mathbf{b} with coefficients $b_j \in \mathbb{Z}^+$ such that M_j has Euler class $b_j[\omega_{FS}]$ and is the lens space $S^{2n+1}/\mathbb{Z}_{b_j}$. The corresponding Sasaki CR structure on $M_{\mathbf{b}}$ is denoted by $(\mathcal{D}_{\mathbf{b}}, J)$. Now Sasaki contact structures can be distinguished by their first Chern class for which we have

$$(26) \quad c_1(\mathcal{D}_{\mathbf{b}}) = (n + 1 - |\mathbf{b}|)\pi_M^*[\omega_{FS}].$$

Moreover, from Proposition 3.15 it is equivalent to the regular join $S^{2n+1} \star_{b,1} S_{\mathbf{w}}^{2d+1}$ with $\mathbf{b} = b\mathbf{w}$ and $b = \gcd(b_1, \dots, b_{d+1})$, and for $d = 1$ the much stronger Theorem 4.5 applies giving constant scalar curvature Sasaki metrics in $\mathfrak{t}_{sph}^+(\mathcal{D}_{\mathbf{b}}, J)$ as done in [BTF16]. When $d > 1$ the best we can currently do comes from Proposition 4.6. In any case we also have

Proposition 5.1. *Let $\{M_{\mathbf{b}}\}$ be a collection of fiber joins over $\mathbb{C}\mathbb{P}^n$ with fixed integral cohomology ring and fixed Pontrjagin classes. Then for each associated Sasaki CR structure $(\mathcal{D}_{\mathbf{b}}, J)$ its Sasaki cone and as well as the subcone $\mathfrak{t}_{sph}^+(\mathcal{D}_{\mathbf{b}}, J)$ contains an open set of extremal Sasaki metrics. If we assume also that $n \leq d$, then there are a finite number of diffeomorphism types within the collection $\{M_{\mathbf{b}}\}$. Hence, for some diffeomorphism type there exists a countable infinity of inequivalent Sasaki contact structures all of which have an open set of extremal Sasaki metrics.*

Proof. The first statement follows directly from Proposition 4.6. The second statement follows since when $n \leq d$ Lemma 2.4 implies that the minimal model (6) is formal, so the hypothesis of Theorem 13.1 of [Sul77] is satisfied. The last statement then follows immediately from Equation (26). \square

Remark 5.2. On the other hand if $n > d$ the Euler class \mathfrak{e} of the sphere bundle cannot vanish, otherwise a single odd class of degree

less than one half the dimension would survive to E_∞ in the Leray-Serre spectral sequence, so by Leray's Theorem M could not have a Sasakian structure which it does by Theorem 3.4. The Euler class produces torsion \mathbb{Z}_e in $H^{2d+2}(M, \mathbb{Z})$ where $\mathfrak{e} = e[\omega_{FS}]^{d+1}$ when $e > 1$.

A particular case of interest is the cohomological Einstein case $c_1(\mathcal{D}_{\mathbf{b}}) = 0$. Using (1) of Corollary 3.14 we have solutions to $c_1(\mathcal{D}_{\mathbf{b}}) = 0$ if and only if

$$(27) \quad n + 1 = |\mathbf{b}| = \sum_{j=1}^{d+1} b_j.$$

As stated in Proposition 4.8 there are solutions if and only if $n \geq d$ and if $n = d$ there is a unique solution, namely $b_j = 1$ for all $j = 1, \dots, n+1$. Generally the number of solutions is given by the number of partitions of $n + 1$ as the unordered sum of $d + 1$ positive integers.

As a specific example of (3) of Theorem 4.5 we have

Corollary 5.3. *Let $(\mathcal{D}_{\mathbf{b}}, J)$ be a Sasaki CR structure of a rank 2 co-linear fiber join on an S^3 bundle over $\mathbb{C}\mathbb{P}^n$. Then \mathfrak{t}_{sph}^+ admits an SE metric if and only if $b_1 + b_2 = n + 1$. Moreover, up to equivalence the number of such Sasaki CR structures equals the number of partitions of $n + 1$ into the unordered sum of two positive integers.*

For $N = \mathbb{C}\mathbb{P}^1$ and $d = 2$, we can apply a recent result by Legendre [Leg19] and state the following result about the regular ray.

Proposition 5.4. *Let M be the Sasaki manifold which as a principal S^1 bundle over $\mathbb{C}\mathbb{P}^1$ has primitive Euler class $[\omega_{FS}]$ and let*

$$M_{\mathbf{b}} = M_1 *_f M_2 *_f M_3$$

be a (necessarily cone decomposable) fiber join where $M_j = S^3/\mathbb{Z}_{b_j}$ has Euler class $b_j[\omega_{FS}]$ for $j = 1, 2, 3$ where ω_{FS} denotes the standard Fubini-Study Kähler form. Then the Sasaki structure corresponding to the regular Reeb field ξ_1 is extremal (up to isotopy), so both \mathfrak{t}^+ and its subcone \mathfrak{t}_{sph}^+ have open sets of extremal Sasaki metrics.

Proof. Since the regular quotient equals $\mathbb{P}(\mathcal{O}(-b_1) \oplus \mathcal{O}(-b_2) \oplus \mathcal{O}(-b_3)) \rightarrow \mathbb{C}\mathbb{P}^1$ this follows directly from Corollary 1.6 in [Leg19]. The key is that Legendre proves existence of extremal Kähler metrics in *every* Kähler class of $\mathbb{P}(\mathcal{O}(-b_1) \oplus \mathcal{O}(-b_2) \oplus \mathcal{O}(-b_3)) \rightarrow \mathbb{C}\mathbb{P}^1$ and thus we have avoided the shortcomings of not knowing which of those classes is the Kähler class of the transverse Kähler metric. \square

Remark 5.5. It is interesting that the proof of Proposition 5.4 does not use the admissible construction. In fact, the projective bundle associated to $M_{\mathbf{b}}$ is not admissible in the sense of [ACGTF08].

Remark 5.6. By a famous theorem of Grothendieck every complex vector bundle on $\mathbb{C}\mathbb{P}^1$ splits as a sum of line bundles. So when $N = \mathbb{C}\mathbb{P}^1$ every sphere bundle with structure group $SO(d+1)$ reduces to a $(d+1)$ -dimensional torus \mathbb{T}^{d+1} and admits a fiber join construction.

5.2. N is a Riemann surface Σ_g of genus g . In this case all Sasaki CR structures are cone decomposable, and there are precisely two diffeomorphism types, the trivial bundle $\Sigma_g \times S^{2d+1}$, and the non-trivial bundle $\Sigma_g \tilde{\times} S^{2d+1}$ which are distinguished by their second Stiefel-Whitney class as discussed by Yamazaki. Here we consider those related to admissible projective bundles as described in Section 4.1, namely

$$(28) \quad \bigoplus_{j=1}^{d+1} L_j^* = E_0 \oplus E_\infty = L_0 \otimes \mathbb{C}^{d_0+1} \oplus L_\infty \otimes \mathbb{C}^{d_\infty+1}$$

where L_0, L_∞ are line bundles that satisfy $c_1(L_0) = -b_1[\Omega_{\Sigma_g}]$ and $c_1(L_\infty) = -b_2[\Omega_{\Sigma_g}]$ where Ω_{Σ_g} is the standard area form on Σ_g , which has constant scalar curvature. Then $d = d_0 + d_\infty + 1$, $L_0 = L_j^*$ for $j = 1, \dots, d_0 + 1$, and $L_\infty = L_j^*$ for $j = d_0 + 2, \dots, d_0 + d_\infty + 2$. Note that

$$\frac{c_1(E_\infty)}{(d_\infty + 1)} - \frac{c_1(E_0)}{(d_0 + 1)} = (b_1 - b_2)[\Omega_{\Sigma_g}].$$

The quotient of the regular Reeb vector field equals $\mathbb{P}(E_0 \oplus E_\infty) \rightarrow \Sigma_g$. Since the fiber join in this case is super admissible, we can apply Theorem 6 of [ACGTF08] to arrive at the following.

Proposition 5.7. *Let M be the Sasaki manifold which as a principal S^1 bundle over a compact Riemann surface Σ_g has primitive Euler class $[\Omega_{\Sigma_g}]$ and let*

$$M_{\mathbf{b}} = M_1 *_f \cdots *_f M_{d+1}$$

be a (necessarily cone decomposable) fiber join where $M_j = M/\mathbb{Z}_{b_j}$ has Euler class $b_1[\Omega_{\Sigma_g}]$ for $j = 1, \dots, d_0 + 1$ and Euler class $b_2[\Omega_{\Sigma_g}]$ for $j = d_0 + 2, \dots, d_0 + d_\infty + 2$, where $d = d_0 + d_\infty + 1$.

- (1) *For genus g of Σ_g equal to 0 or 1, the Sasaki structure on $M_{\mathbf{b}}$ corresponding to the regular Reeb field ξ_1 is extremal (up to isotopy).*
- (2) *For genus $g > 1$ and $-d_0(d_0 + 1) \leq \frac{2(1-g)}{b_1 - b_2} \leq d_\infty(d_\infty + 1)$, the Sasaki structure on $M_{\mathbf{b}}$ corresponding to the regular Reeb field ξ_1 is extremal (up to isotopy).*

In both cases there is an open set of extremal Sasaki metrics in both the Sasaki cone \mathfrak{t}^+ and its subcone \mathfrak{t}_{sph}^+ .

The case $d = 1$ ($d_0, d_\infty = 0$) was treated in detail in [BTF13, BTF14].

5.3. N is a product of Riemann Surfaces. Consider $N = \Sigma_{g_1} \times \Sigma_{g_2}$ where Σ_g is a Riemann surface of genus g , and let $M_{\mathfrak{w}}$ be a Sasaki fiber join on an S^{2d+1} bundle over $\Sigma_{g_1} \times \Sigma_{g_2}$. We choose a complex structure on Σ_g and as before let Ω_i denote the standard area form on Σ_{g_i} . With slight abuse of notation, we denote the pull-back of their Kähler classes to $H^2(N, \mathbb{Z})$ by $[\Omega_1]$ and $[\Omega_2]$. The Kähler cone of N then equals $span_{\mathbb{R}^+}\{[\Omega_1], [\Omega_2]\}$. The fiber join $M_{\mathfrak{w}}$ is formed from the set \mathfrak{S}_{d+1} of Kähler classes which are judiciously represented by Kähler forms

$$(29) \quad \omega_j = k_j^1 \Omega_1 + k_j^2 \Omega_2, \quad k_j^1, k_j^2 \in \mathbb{Z}^+.$$

So the choices of Kähler forms is given by the 2 by $d + 1$ matrix

$$(30) \quad K = \begin{pmatrix} k_1^1 & k_1^2 \\ \vdots & \vdots \\ k_{d+1}^1 & k_{d+1}^2 \end{pmatrix}$$

with values in \mathbb{Z}^+ . From the choice of $d + 1 = d_0 + d_\infty + 2$ Kähler forms, at most two are linearly independent which we denote by ω_0, ω_∞ and which we allow to be linearly independent. In order to apply the admissible conditions we assume that E takes the form of Equation (28). In this case we take

$$(31) \quad \omega_0 = k_0^1 \Omega_1 + k_0^2 \Omega_2, \quad \omega_\infty = k_\infty^1 \Omega_1 + k_\infty^2 \Omega_2,$$

so $c_1(L_0) = -[\omega_0]$ and $c_1(L_\infty) = -[\omega_\infty]$. We thus arrive at the Kähler forms

$$(32) \quad \omega_j = \begin{cases} \omega_0 & \text{for } j = 1, \dots, d_0 + 1, \\ \omega_\infty & \text{for } j = d_0 + 2, \dots, d_0 + d_\infty + 2. \end{cases}$$

The fiber join $M_{\mathfrak{w}}$ with $\mathfrak{S}_2 = \{[\omega_0], [\omega_\infty]\}$ gives rise to an infinite family of inequivalent Sasaki CR structures (\mathcal{D}_K, J) parameterized by the matrix with some obvious relations

$$(33) \quad K = \begin{pmatrix} k_0^1 & k_0^2 \\ k_\infty^1 & k_\infty^2 \end{pmatrix}, \quad k_i^j \in \mathbb{Z}^+.$$

The fiber join $M_{\mathfrak{w}}$ is colinear if and only if $\det K = 0$. Since there is nothing special about the order of ω_0 and ω_∞ we have equivalence under the interchange of the rows of K . Furthermore, interchanging the $\mathbb{C}\mathbb{P}^1$'s corresponds to interchanging the columns of K . Thus, we are interested in the set $\mathfrak{K}(M_{\mathfrak{w}})$ of Sasaki CR structures defined by the

equivalence classes of 2 by 2 matrices over \mathbb{Z}^+ where matrices K, K' are equivalent if they are equivalent under interchange of rows or interchange of columns. Note that under this equivalence $|\det K| = |\det K'|$, but this relation does not determine the equivalence class. However, it is clear from this relation that the cardinality of the set $\mathfrak{K}(M_{\mathbb{w}})$ is \aleph_0 . In what follows we choose a representative M_K in $\mathfrak{K}(M_{\mathbb{w}})$ and think of M_K as a manifold with its underlying Sasaki CR structure (\mathcal{D}_K, J) and its family of Sasakian structures parameterized by $\mathfrak{t}^+(\mathcal{D}_K, J)$.

5.3.1. *Topological analysis.* This splits naturally into two cases, $d = 1$ and $d > 1$.

Proposition 5.8. *Let M_K be a Sasaki fiber join on an S^{2d+1} bundle over $\Sigma_{g_1} \times \Sigma_{g_2}$. Then*

- (1) *when $d > 1$ $M_{\mathbb{w}}$ has the integral cohomology groups of the product $\Sigma_{g_1} \times \Sigma_{g_2} \times S^{2d+1}$,*
- (2) *when $d = 1$ the integral cohomology groups are*

$$(34) \quad H^p(M_K, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } p = 0, 7 \\ \mathbb{Z}^{2g_1+2g_2} & \text{if } p = 1, 3, 6 \\ \mathbb{Z}^{4g_1g_2+2} & \text{if } p = 2, 5 \\ \mathbb{Z}^{2g_1+2g_2} \oplus \mathbb{Z}_e & \text{if } p = 4 \\ 0 & \text{otherwise} \end{cases}$$

where $e = k_0^1 k_\infty^2 + k_0^2 k_\infty^1$.

Proof. In case (1) the Euler class of the bundle vanishes, so the spectral sequence collapses. In case (2) the E_2 term of the spectral sequence is

$$E_2^{p,q} = H^p(\mathcal{U}, \mathcal{H}^q(S^3)) = \begin{cases} H^p(\mathcal{U}) & \text{if } q = 0, 3 \\ 0 & \text{otherwise} \end{cases}$$

where $\mathcal{U} = \bigcup_\alpha U_\alpha$ is a good cover of $\Sigma_{g_1} \times \Sigma_{g_2}$ and \mathcal{H}^q is the derived functor sheaf $U_\alpha \mapsto H^q(\pi^{-1}(U_\alpha))$. Let u be the top class on the fiber. Then since $M_{\mathbb{w}}$ has a Sasakian structure $d_4(u)$ which is the Euler class of the bundle cannot vanish. Using (31) we compute the Euler class

$$\mathbf{e} = c_1(L_0) \cup c_1(L_\infty) = (-[\omega_0])(-[\omega_\infty]) = [\omega_0][\omega_\infty] = (k_0^1 k_\infty^2 + k_0^2 k_\infty^1)[\Omega_1][\Omega_2].$$

Then since $d_4(u) = \mathbf{e}$ is the only non-vanishing differential in the spectral sequence we get the result. \square

There are a countable infinity of distinct homotopy types of S^3 bundles over a product of Riemann surfaces that admit cone decomposable Sasaki fiber joins, and a countable infinity of distinct homotopy types

that admit cone indecomposable Sasaki fiber joins. We say more in the next section for the case $g_1 = g_2 = 0$. Here we mention one further topological invariant, namely the second Stiefel-Whitney class $w_2(M_{\mathfrak{R}})$ which is the mod 2 reduction of $c_1(\mathcal{D}_{\mathfrak{R}})$. Of course M_K is a spin manifold if and only if $w_2(M_K) = 0$. We now have

Proposition 5.9. *Let M_K be a Sasaki fiber join on an S^3 bundle over $\Sigma_{g_1} \times \Sigma_{g_2}$. Then*

- (1) *if both d_0 and d_∞ are odd, $w_2(M_K) = 0$;*
- (2) *if both d_0 and d_∞ are even, $w_2(M_K) = 0$ if and only if $k_0^i + k_\infty^i$ is even for $i = 1, 2$;*
- (3) *if one of d_0, d_∞ is odd and one is even, $w_2(M_K) = 0$ if and only if both components of one of the column vectors of K are even.*

Proof. $w_2(M_K)$ is the mod 2 reduction of $c_1(\mathcal{D}_K)$ which from Proposition 3.12 and Equation (31) is

$$(35) \quad c_1(\mathcal{D}_K) = (2 - 2g_1 - (d_0 + 1)k_0^1 - (d_\infty + 1)k_\infty^1)\pi_M^*[\Omega_1] \\ + (2 - 2g_2 - (d_0 + 1)k_0^2 - (d_\infty + 1)k_\infty^2)\pi_M^*[\Omega_2].$$

So the result easily follows. \square

We also have the following corollary of Propositions 5.8 and 5.9

Corollary 5.10. *Let M_K be a Sasaki fiber join on an S^3 bundle over $\Sigma_{g_1} \times \Sigma_{g_2}$. Then*

- (1) *if e is odd M_K is cone indecomposable and non-spin,*
- (2) *if $e = 2$ M_K is cone decomposable with $k_0^1 = k_\infty^1 = k_0^2 = k_\infty^2 = 1$.*

Proof. Since $e = k_0^1 k_\infty^2 + k_0^2 k_\infty^1$ and M_K is cone decomposable if and only if $0 = \det K = k_0^1 k_\infty^2 - k_0^2 k_\infty^1$, we have $e = 2k_0^1 k_\infty^2$ if and only if M_K is cone decomposable. Then using (2) of Proposition 5.9 shows that e must be even, so $w_2 \neq 0$ which proves (1). (2) is clear. \square

Remark 5.11. When $d = 1$ there is a classification of such sphere bundles due to Dold and Whitney [DW59]. For any d there can be topological twists such as $\Sigma_{g_1} \times (\Sigma_{g_2} \tilde{\times} S^{2d+1})$. The so-called cohomological rigidity problem is of much interest; however, most of what is known occurs in even dimension [CMS11], and lies beyond the scope of the present article.

5.3.2. *Extremal examples and proof of parts (1) and (2) of Theorem 1.2.* As a particular example we take N to be a product of the Riemann sphere with a Riemann surface of arbitrary genus g . We can consider three cases $g = 0, 1, > 1$. In the first two we can be quite general;

however, when $g > 1$ complications arise so we specialize further. We now give a result that proves parts (1) and (2) of Theorem 1.2.

Proposition 5.12. *Let M_K be an admissible fiber join with $N = \mathbb{C}\mathbb{P}^1 \times \Sigma_g$. Then*

- (1) *if $g = 0, 1$ the regular ray generated by ξ_1 in $\mathfrak{t}^+(\mathcal{D}_K, J)$ has an extremal representative for all d_0, d_∞ and all $K \in \mathfrak{K}$;*
- (2) *if $g > 1$, $d_0 = d_\infty = 1$ and we choose*

$$K = \begin{pmatrix} 2 & g \\ 1 & 1 \end{pmatrix},$$

the regular ray in $\mathfrak{t}^+(\mathcal{D}_K, J)$ has an extremal representative.

Thus, in both cases the Sasaki cone $\mathfrak{t}^+(\mathcal{D}_K, J)$ of $M_{\mathfrak{w}}$ as well as the subcone $\mathfrak{t}_{sph}^+(\mathcal{D}_K, J)$ contains an open set of extremal Sasaki metrics.

Proof. If $g = 0$ the quotient complex manifold of M_K arising from the regular Sasakian structure with Reeb vector field ξ_1 is equal to

$$\begin{aligned} & \mathbb{P}(L_0 \otimes \mathbb{C}^{d_0+1} \oplus L_\infty \otimes \mathbb{C}^{d_\infty+1}) = \mathbb{P}(\mathbb{C}^{d_0+1} \oplus L_0^* \otimes L_\infty \otimes \mathbb{C}^{d_\infty+1}) \\ (36) \quad & = \mathbb{P}(\mathbb{C}^{d_0+1} \oplus \mathcal{O}(k_0^1 - k_\infty^1, k_0^2 - k_\infty^2) \otimes \mathbb{C}^{d_\infty+1}) \rightarrow \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1. \end{aligned}$$

So the projectivization $\mathbb{P}(\mathbb{C}^{d_0+1} \oplus L_0^* \otimes L_\infty \otimes \mathbb{C}^{d_\infty+1})$ is a $\mathbb{C}\mathbb{P}^d$ bundle over $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ which is a generalized Bott manifold [CMS10], and we have the following commutative diagram

$$(37) \quad \begin{array}{ccccc} S^1 & \longrightarrow & S^{2d+1} & \longrightarrow & \mathbb{C}\mathbb{P}^d \\ \downarrow id & & \downarrow & & \downarrow \\ S^1 & \longrightarrow & M_K & \longrightarrow & \mathbb{P}(\mathbb{C}^{d_0+1} \oplus \mathcal{O}(k_0^1 - k_\infty^1, k_0^2 - k_\infty^2) \otimes \mathbb{C}^{d_\infty+1}). \\ & & \downarrow & & \downarrow \\ & & \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1 & \xrightarrow{id} & \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1 \end{array}$$

Now, every Kähler class on $\mathbb{P}(\mathbb{C}^{d_0+1} \oplus \mathcal{O}(k_0^1 - k_\infty^1, k_0^2 - k_\infty^2) \otimes \mathbb{C}^{d_\infty+1})$ is admissible in the broader sense of the definition given in [ACGTF08], (so the fiber join is super admissible). Thus, for $g = 0$ the result follows from Proposition 4.3. When $g \geq 1$ we note that by Remark 2 in [ACGT08] (or the comments above Definition 4.1) that the fiber joins are still super admissible. The $g = 1$ case then follows from Proposition 4.3, similarly to above, and the special case with $g > 1$ follows from Proposition A.2 in the Appendix. \square

Remark 5.13. When $g > 1$ we expect many other examples. However, the analysis in the Appendix that assures the positivity of the ‘extremal

polynomial' needs to be worked out in each case which becomes more complicated as d_0, d_∞ grow. Note also that for the example given in (2) of Proposition 5.12 if $g > 2$ the fiber join M_K is cone indecomposable, whereas, it is cone decomposable when $g = 2$.

5.3.3. CSC examples and proof of part (3) of Theorem 1.2. In the above setting we will now assume that $N = \Sigma_{g_1} \times \Sigma_{g_2}$ and $d = 1$. The quotient of the regular ray equals $\mathbb{P}(\mathbb{1} \oplus L_0^* \otimes L_\infty)$ with $c_1(L_0) = -[\omega_0]$, $c_1(L_\infty) = -[\omega_\infty]$, and ω_0 and ω_∞ are given by (31). In order to apply the admissible construction we assume $k_0^i \neq k_\infty^i$ for $i = 1, 2$. We will use our findings in Appendix A.3 together with results from Section 3.4 of [ACGTF08] to obtain admissible cone indecomposable fiber joins with a cscS regular ray (up to isotopy). Note that in the cone decomposable case, i.e., the colinear case, no true admissible case ($[\omega_0] \neq [\omega_\infty]$) would have a cscS regular ray (see e.g. Theorem 7 of [ACGTF08]), but in that case we have a regular join and so by Theorem 4.5 we have a cscS ray elsewhere in the Sasaki cone. Thus, below we assume non-colinearity.

Here we have

$$\omega_0 - \omega_\infty = (k_0^1 - k_\infty^1)\Omega_1 + (k_0^2 - k_\infty^2)\Omega_2$$

and

$$\omega_0 + \omega_\infty = (k_0^1 + k_\infty^1)\Omega_1 + (k_0^2 + k_\infty^2)\Omega_2$$

so (54) in Appendix A is satisfied with $\mathcal{A} = \{1, 2\}$, $\omega_{N_1} = 2\pi(k_0^1 - k_\infty^1)\Omega_1$, $\omega_{N_2} = 2\pi(k_0^2 - k_\infty^2)\Omega_2$. This means that the Kähler class of the regular quotient is admissible with the admissible data associated to

the matrix $K = \begin{pmatrix} k_0^1 & k_0^2 \\ k_\infty^1 & k_\infty^2 \end{pmatrix}$ given by

(38)

$$s_1 = \frac{2(1 - g_1)}{k_0^1 - k_\infty^1}, \quad s_2 = \frac{2(1 - g_2)}{k_0^2 - k_\infty^2}, \quad r_1 = \frac{k_0^1 - k_\infty^1}{k_0^1 + k_\infty^1}, \quad r_2 = \frac{k_0^2 - k_\infty^2}{k_0^2 + k_\infty^2}.$$

Note that $r_1 \neq r_2$ since we are assuming non-colinearity and we have a genuine $|\mathcal{A}| = 2$ admissible case. Compared to Section 3.4 of [ACGTF08] we have a slight notation change (using r_i instead of x_i), but aside from this we can directly use the findings in this section to get the following slight generalization of Lemma 7 of [ACGTF08]

Lemma 5.14. *With s_1, s_2, r_1, r_2 given by the admissible data (38), the Kähler class of the regular quotient has an admissible CSC Kähler metric iff*

$$(39) \quad r_1(s_1(r_1 - r_2) - 2 + (1 - s)r_1r_2) + 3(s - 1)r_2 = 0,$$

$$(40) \quad r_2(s_2(r_2 - r_1) - 2 + (1 - s)r_1r_2) + 3(s - 1)r_1 = 0,$$

for some $s \in \mathbb{R}$ (up to scale, s is the scalar curvature of the resulting Kähler metric), and

$$(41) \quad Q(\mathfrak{z}) = (1+r_1\mathfrak{z})(1+r_2\mathfrak{z}) + \left(1 - \frac{s}{2}\right)r_1r_2(1-\mathfrak{z}^2) > 0, \text{ for } -1 < \mathfrak{z} < 1.$$

Moreover, if $s \geq 0$ condition (41) automatically holds.

Proof. From Section 3.4 of [ACGTF08] we see, assuming $r_1 \neq r_2$, that a CSC Kähler metric exists if and only if s satisfies Equations (39) and (40) and the extremal polynomial $F_\Omega(\mathfrak{z})$ of [ACGTF08], which equals $(1-\mathfrak{z}^2)Q(\mathfrak{z})$, is positive on $-1 < \mathfrak{z} < 1$. This proves the first statement. For the second statement we note that since $s_a r_a < 2$, equations (39) and (40) imply that $r_1 r_2 < 0$, i.e., $(k_0^1 - k_\infty^1)(k_0^2 - k_\infty^2) < 0$. For a solution to (39) and (40) with $s \geq 0$, we see right away that $Q(\mathfrak{z})$ is concave down or linear. Hence, since $Q(\pm 1) > 0$, (41) holds. \square

While a systematic analysis of all the possible solutions of (39) and (40) with (38) seems not really to be tractable, we can obtain solutions by making a special ansatz. In particular, the first part of Lemma 8 of [ACGTF08] is:

Lemma 5.15. [ACGTF08] *With the admissible data (38) assumed, suppose in addition that $s_1 + s_2 = 0$ and $r_1 + r_2 = 0$. Then $s = \frac{1-r_1^2+2s_1r_1}{3-r_1^2}$ is a solution to Equations (39) and (40). In particular, if in addition $g_1 = 0, 1$, then $s \geq 0$ giving a CSC Kähler metric.*

Proof. The first statement is precisely the first case of Lemma 8 in [ACGTF08], and the second statement follows from Lemma 5.14 together with $s_1 r_1 = \frac{2(1-g_1)}{k_0^1 + k_\infty^1}$ since $s \geq 0$ will hold if $s_1 r_1 \geq 0$. \square

Note that the conditions in the hypothesis of Lemma 5.15 imply that $g_1 = 1$ if and only if $g_2 = 1$, and since $s_a = 0$ the only other condition is $r_1 + r_2 = 0$. However, if $g_1 = g_2 = 0$ the conditions $s_1 + s_2 = r_1 + r_2 = 0$ imply $k_0^1 = k_\infty^2$ and $k_\infty^1 = k_0^2$. So for $N = \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ and $N = T^2 \times T^2$ Lemma 5.15 gives

Lemma 5.16. *Assume the hypothesis of Lemma 5.15. If*

- (1) $g_1 = g_2 = 0$ and $K = \begin{pmatrix} k & l \\ l & k \end{pmatrix}$ with $k, l \in \mathbb{Z}^+$ and $k \neq l$, or,
- (2) $g_1 = g_2 = 1$, and $K = \begin{pmatrix} k_0^1 & k_0^2 \\ k_\infty^1 & k_\infty^2 \end{pmatrix}$ is such that $\frac{k_0^1 - k_\infty^1}{k_0^1 + k_\infty^1} = \frac{k_\infty^2 - k_0^2}{k_0^2 + k_\infty^2}$,

then, up to isotopy, the Kähler metric is CSC.

When $g_i > 1$ we can still look for solutions with positive scalar curvature, but generally it is somewhat tedious. So we make the additional assumption that the genera are equal.

Lemma 5.17. *Under the hypothesis of Lemma 5.15 with $g_1 = g_2 = g$, if $K = \begin{pmatrix} k+1 & k \\ k & k+1 \end{pmatrix}$ is such that $k \in \mathbb{Z}^+$ satisfies $k > \lfloor \frac{2g-3+\sqrt{4g^2-8g+5}}{2} \rfloor$, then, up to isotopy, the Kähler metric is CSC.*

Proof. Assuming $g_1 = g_2 = g$ and setting $k_0^1 = k+1$, $k_\infty^1 = k$, $k_0^2 = k$, and $k_\infty^2 = k+1$ for $k \in \mathbb{Z}^+$, we have $s_1 = 2(1-g)$, $s_2 = 2(g-1)$, $r_1 = \frac{1}{2k+1}$, and $r_2 = \frac{-1}{2k+1}$. Thus, by Lemma 5.15, we know that (39) and (40) are solved with $s = \frac{2(k^2+(3-2g)k+(1-g))}{6k^2+6k+1}$. We can now use Lemma 5.14, for any value of g , as long as $k^2 + (3-2g)k + (1-g) > 0$, that is, as long as $k > \lfloor \frac{2g-3+\sqrt{4g^2-8g+5}}{2} \rfloor$. \square

Putting these lemmas together gives Theorem 1.2 of the Introduction.

To end this subsection let us give a couple of examples of solutions to (39), (40), and (41), where the transverse scalar curvature $s = s_g^T < 0$:

Proposition 5.18. *In the above setting with $g_1 = 5$, $g_2 = 3$, and $K = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$, up to isotopy, the regular ray is cscS.*

Proof. From (38), we have

$$s_1 = -8, \quad s_2 = 2, \quad r_1 = 1/3, \quad r_2 = -1/2,$$

which is easily seen to solve (39) and (40) with $s = -1$. Now, $Q(\mathfrak{z})$ from (41) is given by $Q(\mathfrak{z}) = (1/12)(9 - 2\mathfrak{z} + \mathfrak{z}^2)$, which clearly satisfies the condition in (41). \square

Proposition 5.19. *In the above setting with $g_1 = 18$, $g_2 = 14$, and $K = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$, up to isotopy, the regular ray is cscS.*

Proof. From (38), we have

$$s_1 = -34, \quad s_2 = 13, \quad r_1 = 1/3, \quad r_2 = -1/2,$$

which is easily seen to solve (39) and (40) with $s = -6$. Now, $Q(\mathfrak{z})$ from (41) is given by $Q(\mathfrak{z}) = (1/6)(2 - \mathfrak{z} + 3\mathfrak{z}^2)$, which clearly satisfies the condition in (41). \square

Remark 5.20. Note that with K as in Propositions 5.18 and 5.19 we can also solve (39) and (40) with (38) when $g_1 = 31$, $g_2 = 25$, and

$s = -11$. Here, however, $Q(\mathfrak{z}) = (1/12)(-1 - 2\mathfrak{z} + 11\mathfrak{z}^2)$, and clearly (41) fails.

5.4. $N = \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ **and the Proof of Theorem 1.3.** We now consider the case that N is a product of Riemann spheres in more detail. Again we divide our analysis into the two cases $d > 1$ and $d = 1$.

5.4.1. $d > 1$. We are interested in the total Pontrjagin class of the manifolds $M_{\mathfrak{R}}$. It follows from Proposition 5.8 that the only non-vanishing Pontrjagin class of $M_{\mathfrak{R}}$ is p_1 which we note is not only a diffeomorphism invariant but also a homeomorphism invariant⁸ [Sha85]. So Sasaki CR manifolds $M_{\mathfrak{w}}$ with distinct p_1 cannot be homeomorphic.

5.4.2. *Proof of Theorem 1.3 when $d > 1$.* First it follows from Remark 3.13 that when $d > 1$ $p_1(M_{\mathfrak{R}}) = c_1(\mathcal{D}_{\mathfrak{R}})^2 - 2c_2(\mathcal{D}_{\mathfrak{R}})$ is equal to

$$\begin{aligned} &= c_1(\mathcal{D}_{\mathfrak{R}})^2 - 2\pi_M^*(\sigma_2(-[\omega_0], \dots, -[\omega_0], -[\omega_\infty], \dots, -[\omega_\infty]) + c_2(\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1)) \\ &= 2(2 - (d_0 + 1)k_0^1 - (d_\infty + 1)k_\infty^1)(2 - (d_0 + 1)k_0^2 - (d_\infty + 1)k_\infty^2)\gamma - 8\gamma \\ &\quad - 2(d_0(d_0 + 1)k_0^1k_0^2 + d_\infty(d_\infty + 1)k_\infty^1k_\infty^2 + (d_0 + 1)(d_\infty + 1)(k_0^1k_\infty^2 + k_\infty^1k_0^2))\gamma \\ &= (-4(k_0^1 + k_0^2 + k_\infty^1 + k_\infty^2) + 2k_0^1k_0^2 + 2k_\infty^1k_\infty^2 - 4d_0k_0^1 - 4d_0k_0^2 - 4d_\inftyk_\infty^1 \\ &\quad - 4d_\inftyk_\infty^2 + 2d_0k_0^1k_0^2 + 2d_\inftyk_\infty^1k_\infty^2)\gamma \end{aligned}$$

where $\gamma = \pi_M^*[\Omega_1 \wedge \Omega_2]$. Here we have used

$$\sigma_2 = (d_0(d_0 + 1)k_0^1k_0^2 + d_\infty(d_\infty + 1)k_\infty^1k_\infty^2 + (d_0 + 1)(d_\infty + 1)(k_0^1k_\infty^2 + k_\infty^1k_0^2))[\Omega_1 \wedge \Omega_2].$$

Proposition 5.21. *Consider the set $\{M_K\}$ of fiber joins with $K = \begin{pmatrix} k_0 & 2 \\ k_\infty & 2 \end{pmatrix}$, $k_0, k_\infty \in \mathbb{Z}^+$, $d > 1$, and fixed integral cohomology ring $R_{\mathbb{Z}}$.*

Then there are finitely many diffeomorphism types of both cone decomposable and cone indecomposable fiber joins within the set $\{M_K\}$ whose integral cohomology ring is isomorphic to $R_{\mathbb{Z}}$ has a finite number of diffeomorphism types. Therefore, at least one such Sasaki CR manifold of both types admits a countable infinity of inequivalent contact structures of Sasaki type.

Proof. Notice that if we choose $k_0^2 = k_\infty^2 = 2$ we obtain a two parameter family of sequences $M_{k_0^1, k_\infty^1}$ with constant first Pontrjagin class, namely from the equation above we see that the first Pontrjagin class is

$$(42) \quad p_1(M_{k_0^1, k_\infty^1}) = -8(d_0 + d_\infty + 2)\gamma.$$

⁸This appeared to be a folklore result, but Diarmuid Crowley provided us the reference [Sha85] for its proof.

So the set $\{M_K\}$ with K as in the Proposition has the same Pontrjagin class p_1 , and since $d > 1$ the Euler class of the sphere bundle vanishes. So by Lemma 2.4 these manifolds are formal. It follows from this and Proposition 5.8 that the hypothesis of Theorem 13.1 of [Sul77] is satisfied. Hence, there are at most a finite number of diffeomorphism types. When $k_0 = k_\infty = k$ the Sasaki CR structure is cone decomposable; otherwise, it is cone indecomposable. But also one easily sees from Equation (35) that $c_1(\mathcal{D}_K)$ depends linearly on the parameters k_0 and k_∞ giving rise to infinitely many inequivalent contact structures. \square

5.4.3. $d = 1$. When $d = 1$ Sullivan's minimal model is not formal [KT91, BFMT16] since a triple Massey product is non-vanishing; however, Corollary 2.3 of [KT91] says that, nevertheless, there is a finite number of diffeomorphism types if the equation

$$(43) \quad p_1 \equiv 2e \pmod{4}$$

holds.

5.4.4. *Proof of Theorem 1.3 when $d = 1$.*

Proposition 5.22. *Consider the set $\{M_K\}_{k_1, k_2}$ of fiber joins with $K = \begin{pmatrix} k_1 + b & k_2 + c \\ k_1 & k_2 \end{pmatrix}$, $d = 1$, and fixed integral cohomology ring $R_{\mathbb{Z}}$. Then for any fixed pair (b, c) there exists a countable subsequence $\{M_k\} \subset \{M_K\}$ of cone indecomposable fiber joins within the set $\{M_k\}$ and finitely many diffeomorphism types whose integral cohomology ring is isomorphic to $R_{\mathbb{Z}}$. Therefore, at least one such diffeomorphism class admits a countable infinity of inequivalent contact structures of Sasaki type for both spin and non-spin manifolds.*

Proof. Let M_0, M_∞ be the Sasaki 5-manifolds given by the S^1 bundles over $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ corresponding to the Kähler classes ω_0, ω_∞ of Equation (32), and consider 7 dimensional the fiber join $M_0 \star_f M_\infty$ with Sasaki CR structures parameterized by the Neron-Severi lattice $\mathcal{K}_{NS}(\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1) = \text{span}_{\mathbb{Z}^+} \{[\Omega_1], [\Omega_2]\}$. The projectivization of Equation (36) by the regular Reeb vector field ξ_1 is the twist 1 stage 3 Bott manifold $M_3(0, k_0^1 - k_\infty^1, k_0^2 - k_\infty^2)$ whose Kähler geometry was studied in [BCTF19]. The cohomology ring is

$$H^*(M_3(0, b, c), \mathbb{Z}) = \mathbb{Z}[x_1, x_2, x_3] / (x_1^2, x_2^2, x_3(bx_1 + cx_2 + x_3))$$

where x_1, x_2, x_3 are 2-classes. Since the base is untwisted, the x_i represent the volume form of the two $\mathbb{C}\mathbb{P}^1$ s. Moreover, its first Pontrjagin class is

$$(44) \quad p_1(M_3(0, b, c)) = 2bcx_1x_2.$$

where $b = k_0^1 - k_\infty^1$ and $c = k_0^2 - k_\infty^2$. Since M_K is an S^1 bundle over $M_3(0, b, c)$, the S^1 action trivializes the vertical piece of the tangent bundle M_K . Thus, the characteristic classes of M_K are the pullbacks of the characteristic classes of $M_3(0, b, c)$. So the first Pontrjagin class is

$$(45) \quad p_1(M_K) = 2(k_0^1 - k_\infty^1)(k_0^2 - k_\infty^2)\pi^*(x_1x_2) = 2bc\pi^*(x_1x_2).$$

The existence of the sequence $\{M_k\}$ depends on the parity of b and c . To satisfy the condition (43) we must choose the subsequence $\{M_k\}$ such that

$$(46) \quad bk_2 + ck_1 \equiv 0 \pmod{2}.$$

It is easy to see that such a countable sequence $\{M_k\}$ always exists. Now from (35) we have

$$(47) \quad c_1(\mathcal{D}_{K_{k,b,c}}) = (2 - 2k_1 + b)\pi_M^*[\Omega_1] + (2 - 2k_2 + c)\pi_M^*[\Omega_2].$$

So for each pair (b, c) there exists a countable infinity of inequivalent Sasaki contact structures. Moreover, the manifolds $M_{k,b,c}$ are spin when b and c are both even, and non-spin otherwise for all such k . Since (43) is satisfied Corollary 2.3 of [KT91] implies that for both spin and non-spin manifolds the sequence $\{M_{k,b,c}\}$ admits only a finite number of diffeomorphism types. \square

Then Propositions 5.21 and 5.22 together with part (1) of Theorem 1.2 prove Theorem 1.3. \square

Remark 5.23. For the contact invariant $c_1(\mathcal{D}_{k,b,c})$ there are three cases to consider

- (1) $c_1(\mathcal{D}_K)$ is negative definite which occurs if and only if no column of K is $(1, 1)^t$;
- (2) $c_1(\mathcal{D}_K)$ is a negative multiple of $\pi_M^*[\Omega_i]$ for some $i = 1, 2$ which occurs if and only if exactly one column of K is $(1, 1)^t$;
- (3) $c_1(\mathcal{D}_K) = 0$ which occurs if and only if $K = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

The only possible SE metric in the collection $\mathfrak{R}(M_{\mathbb{w}})$ occurs in case (3). This is the iterated join $S^3 \star S^3 \star S^3$ as described in [BG00a] and the SE metric is homogenous with a regular Reeb vector field.

Proposition 5.24. *For each $b, c \in \mathbb{Z}$ we consider the set $\{M_{K_{k^i, b, c}}\}_{k^1, k^2}$ of S^3 bundles over $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$. Then*

- (1) $M_{K_{k^i, b, c}}$ is spin if and only if b and c are both even
- (2) $M_{K_{k^i, b, c}}$ is cone decomposable if and only if $ck^1 = bk^2$.

Each type admits a countable infinity of inequivalent Sasaki contact structures such that the Sasaki CR structure (\mathcal{D}_K, J) admits an open set of extremal Sasaki structures in \mathfrak{t}_{sph}^+ . Moreover, among the elements of $\{M_{K_{k^i, b, c}}\}_{k^1, k^2}$ there are only a finite number of diffeomorphism types with the same integral cohomology ring.

Proof. From (47) M_K are spin if and only if b and c are both even, so infinite sequences of both occur. Since $c_1(\mathcal{D})$ is a contact invariant our construction gives a countable infinity of such structures. Moreover, by Proposition 5.21 there are finitely many diffeomorphism types among the set $\{M_{K_{k^i, b, c}}\}_{k^1, k^2}$. \square

Consider the cone indecomposable S^3 fiber joins described in case (1) of Theorem 1.2, namely those with matrix

$$(48) \quad K = \begin{pmatrix} k & l \\ l & k \end{pmatrix}$$

with $k, l \in \mathbb{Z}^+$ and $k \neq l$. Without loss of generality we can take $k > l$.

Proposition 5.25. *Let $\{M_K\}_{k, l}$ be the set of cone indecomposable fiber joins over $\mathbb{C}P^1 \times \mathbb{C}P^1$ with K given by Equation (48) and $k > l$. Each such pair (k, l) defines a unique homeomorphism type and the diffeomorphism type is defined up to finite ambiguity.*

Proof. From (45) we see that

$$(49) \quad p_1(M_K) = -2(k - l)^2 \pi^* x_1 x_2.$$

and from Proposition 5.8 we have

$$(50) \quad \mathfrak{e} = (k^2 + l^2) x_1 x_2.$$

For each such ordered pair, there is a unique pair of cohomology classes p_1 and \mathfrak{e} , and since p_1 is a homeomorphism invariant we have inequivalent homeomorphism types for distinct ordered pairs (k, l) . Then Corollary 2.3 of [KT91] implies that the set of possible diffeomorphism types is finite since $p_1 \equiv 2\mathfrak{e} \pmod{4}$. \square

Remark 5.26. Although distinct pairs (k, l) are never homeomorphic, if they have the same Euler class \mathfrak{e} , they could be homotopy equivalent; however, we do not determine this.

5.5. More General N . As seen from Proposition 4.3, our results generalize to the case where N is a local product with nonnegative cscK metrics. So we can replace $\mathbb{C}\mathbb{P}^1$ with any dimensional complex projective space (as the second Betti numbers remain equal to one) and consider arbitrary products of them. We summarize this in a statement below.

Proposition 5.27. *Let $N = \mathbb{C}\mathbb{P}^{n_1} \times \cdots \times \mathbb{C}\mathbb{P}^{n_k}$ or $N = \mathbb{C}\mathbb{P}^{n_1} \times \cdots \times \mathbb{C}\mathbb{P}^{n_k} \times T^2$ and let*

$$M_{\mathfrak{w}} = M_1 *_f \cdots *_f M_{d+1}$$

be a fiber join where M_j is a Sasaki manifold which as a principal S^1 bundle over N has Euler class $[\omega_0] \in H^2(N, \mathbb{Z})$ for $j = 1, \dots, d_0 + 1$ and Euler class $[\omega_\infty] \in H^2(N, \mathbb{Z})$ for $j = d_0 + 2, \dots, d_0 + d_\infty + 2$, where $d = d_0 + d_\infty + 1$, and the Kähler forms ω_0, ω_∞ have constant scalar curvature. Then the regular ray of $M_{\mathfrak{w}}$ is extremal (up to isotopy). Thus, the Sasaki subcone \mathfrak{t}_{sph}^+ of $M_{\mathfrak{w}}$ will always contain an open set of Sasaki extremal structures.

APPENDIX A. ADMISSIBLE PROJECTIVE BUNDLES

The purpose of this Appendix is two-fold. We want to augment Section 4.1 with a description of the so-called admissible extremal Kähler metrics from [ACGTF08], appearing in Proposition 4.2, and we want to state and prove Proposition A.2 which we need for part (2) of Proposition 5.12. Broadly speaking, we are keeping the notation faithful to the work in [ACGTF08] and therefore a tiny bit different from Section 4.1.

We are interested in the *admissible projective bundles* described in [ACGTF08]. These are projective bundles of the form

$$\mathbb{P}(E_0 \oplus E_\infty) \longrightarrow N$$

that satisfy the following conditions:

- For $\mathcal{A} \subset \mathbb{N}$ N is covered by a product $\tilde{N} = \prod_{a \in \mathcal{A}} N_a$ of simply connected Kähler manifolds $(N_a, \pm\Omega_{N_a}, g_a)$ of complex dimension d_a . The \pm means that either $+g_a$ or $-g_a$ is positive definite. We assume here that $\pm g_a$ has CSC with scalar curvature $\pm 2d_a s_a$.
- E_0 and E_∞ are holomorphic projectively flat Hermitian vector bundles over N of ranks $(d_0 + 1)$ and $(d_\infty + 1)$, respectively, which satisfy

$$\frac{c_1(E_\infty)}{(d_\infty + 1)} - \frac{c_1(E_0)}{(d_0 + 1)} = \frac{1}{2\pi}[\omega_N]$$

where $\omega_N = \sum_{a \in \mathcal{A}} \omega_{N_a}$.

Remark A.1. To connect with the notation in Section 4.1, we note that $\omega_{N_a} = 2\pi\epsilon_a\Omega_a$.

A.1. Admissible Metrics. On admissible projective bundles as defined above we can now construct the *admissible metrics*, [ACGTF08]. Here we recover the main points of this construction.

We let $\hat{\mathcal{A}}$ be the extended index set:

- $\hat{\mathcal{A}} = \mathcal{A}$, if $d_0 = d_\infty = 0$.
- $\hat{\mathcal{A}} = \mathcal{A} \cup \{0\}$, if $d_0 > 0$ and $d_\infty = 0$.
- $\hat{\mathcal{A}} = \mathcal{A} \cup \{\infty\}$, if $d_0 = 0$ and $d_\infty > 0$.
- $\hat{\mathcal{A}} = \mathcal{A} \cup \{0\} \cup \{\infty\}$, if $d_0 > 0$ and $d_\infty > 0$.

Now we set $r_0 = 1$, $r_\infty = -1$, $s_0 = d_0 + 1$ and $s_\infty = -(d_\infty + 1)$. This will correspond to (g_0, ω_0) (or $(g_\infty, \omega_\infty)$) being the induced Fubini-Study structure with scalar curvature $d_0(d_0 + 1)$ (or $d_\infty(d_\infty + 1)$) on each fiber of $P(E_0)$ (or $P(E_\infty)$).

Consider the circle action on $M = \mathbb{P}(E_0 \oplus E_\infty) \rightarrow N$ induced by the standard circle action on E_0 . It extends to a holomorphic \mathbb{C}^* action. The open and dense set M_0 of stable points with respect to the latter action has the structure of a principal circle bundle over the stable quotient. The hermitian norm on the fibers induces via a Legendre transform a function $\mathfrak{z} : M_0 \rightarrow (-1, 1)$ whose extension to M consists of the critical manifolds $e_0 := \mathfrak{z}^{-1}(1) = \mathbb{P}(E_0 \oplus 0)$ and $e_\infty := \mathfrak{z}^{-1}(-1) = \mathbb{P}(0 \oplus E_\infty)$. Letting θ be a connection one form for the Hermitian metric on M_0 , with curvature $d\theta = \sum_{a \in \hat{\mathcal{A}}} \omega_{N_a}$, an admissible Kähler metric and form are given (up to scale) by the respective formulas

$$(51) \quad g = \sum_{a \in \hat{\mathcal{A}}} \frac{1 + r_a \mathfrak{z}}{r_a} g_a + \frac{d\mathfrak{z}^2}{\Theta(\mathfrak{z})} + \Theta(\mathfrak{z})\theta^2, \quad \omega = \sum_{a \in \hat{\mathcal{A}}} \frac{1 + r_a \mathfrak{z}}{r_a} \omega_a + d\mathfrak{z} \wedge \theta,$$

valid on M_0 . Here Θ is a smooth function with domain containing $(-1, 1)$ and r_a , $a \in \mathcal{A}$ are real numbers of the same sign as g_a and satisfying $0 < |r_a| < 1$. The complex structure yielding this Kähler structure is given by the pullback of the base complex structure along with the requirement $Jd\mathfrak{z} = \Theta\theta$. The function \mathfrak{z} is hamiltonian with $K = J \text{grad } \mathfrak{z}$ a Killing vector field. In fact, \mathfrak{z} is the moment map on M for the circle action, decomposing M into the free orbits $M_0 = \mathfrak{z}^{-1}((-1, 1))$ and the special orbits $\mathfrak{z}^{-1}(\pm 1)$. Finally, θ satisfies $\theta(K) = 1$.

Now g is a (positive definite) Kähler metric which extends smoothly to all of M if and only if Θ satisfies the following positivity and boundary conditions

(52)

$$(i) \Theta(\mathfrak{z}) > 0, \quad -1 < \mathfrak{z} < 1, \quad (ii) \Theta(\pm 1) = 0, \quad (iii) \Theta'(\pm 1) = \mp 2.$$

The Kähler class $\Omega_{\mathbf{r}} = [\omega]$ of an admissible metric as in (51) is also called *admissible* and is uniquely determined by the parameters r_a , $a \in \mathcal{A}$, once the data associated with M (i.e. d_a , s_a , g_a etc.) is fixed. Indeed, we have

$$\Omega_{\mathbf{r}} = \sum_{a \in \mathcal{A}} \frac{[\omega_{N_a}]}{r_a} + \Xi,$$

where Ξ is a certain fixed cohomology class on M . In the event that $d_0 = d_\infty = 0$, then we have that Ξ is the Poincaré dual of $2\pi(e_0 + e_\infty)$. For a more thorough description of Ξ , please consult Section 1.3 of [ACGTF08].

Define a function $F(\mathfrak{z})$ by the formula $\Theta(\mathfrak{z}) = F(\mathfrak{z})/p_c(\mathfrak{z})$, where $p_c(\mathfrak{z}) = \prod_{a \in \hat{\mathcal{A}}} (1 + r_a \mathfrak{z})^{d_a}$. Since $p_c(\mathfrak{z})$ is positive for $-1 < \mathfrak{z} < 1$, conditions (52) imply the following conditions on $F(\mathfrak{z})$, which are only necessary for compactification of the metric g :

(53)

$$(i) F(\mathfrak{z}) > 0, \quad -1 < \mathfrak{z} < 1, \quad (ii) F(\pm 1) = 0, \quad (iii) F'(\pm 1) = \mp 2p_c(\pm 1).$$

A.2. Extremal Admissible Metrics. From Proposition 1 in [ACGTF08] we know that an admissible metric g given by (51) is extremal precisely when

$$F''(\mathfrak{z}) = \left(\prod_{a \in \hat{\mathcal{A}}} (1 + r_a \mathfrak{z})^{d_a - 1} \right) P(\mathfrak{z}),$$

where $P(\mathfrak{z})$ is a polynomial of degree at most $|\hat{\mathcal{A}}| + 1$ (at most $|\hat{\mathcal{A}}|$ if g has constant scalar curvature) satisfying that

$$P(-1/r_a) = 2d_a s_a r_a \prod_{j \in \hat{\mathcal{A}} \setminus \{a\}} \left(1 - \frac{r_j}{r_a} \right).$$

For fixed admissible data and with a choice of allowable parameters r_a , $a \in \mathcal{A}$ (that is, a choice of an admissible Kähler class $\Omega_{\mathbf{r}}$), it is shown in [ACGTF08] that the system above has a unique polynomial solution, $F_{extr}(\mathfrak{z})$, satisfying conditions (ii) and (iii) from (53). Further, $\Theta(\mathfrak{z})$ given by $F_{extr}(\mathfrak{z})/p_c(\mathfrak{z})$ satisfies (ii) and (iii) of (52). An admissible extremal Kähler metric, with Kähler class $\Omega_{\mathbf{r}}$, then results exactly when also (i) of (53) is satisfied.

A root counting argument due to Guan [Gua95] and Hwang [Hwa94], and further explored by Hwang-Singer [HS02]), was adopted in [ACGTF08] to prove Proposition 4.2.

When not all the CSC Kähler metrics $\pm g_a$ have non-negative scalar curvature, the root counting argument breaks down and in general there are many well known such examples where $F_{extr}(\mathfrak{z})$ will not satisfy (i) of (53) for all admissible Kähler classes (see e.g. Remark 7 in [ACGTF08]). While it is also known that if r_a is sufficiently small for all $a \in \mathcal{A}$, then (i) of (53) is satisfied by $F_{extr}(\mathfrak{z})$; however, for our purpose this is not useful. For the most part, in this paper, the Kähler class is unspecified. Thus, we are looking for cases where we can say that every Kähler class has an extremal Kähler metric. Within the current framework, this means that we are looking for cases where we can say that every admissible Kähler class has an extremal Kähler metric and that every Kähler class is admissible.

When N is a compact Riemann surface of genus at least two, Theorem 6 in [ACGTF08], which we apply to obtain Proposition 5.7, provides some positive results that seems to indicate that larger d_0 and/or d_∞ could be of value.

Now assume that $N = \mathbb{CP}^1 \times \Sigma_g$, where Σ_g is a compact Riemann surface of genus g at least two. We set $\mathcal{A} = \{1, 2\}$. Note that all Kähler classes are admissible in this case.

From Section 3.1 in [ACGTF08] we know that

$$F_{extr}(\mathfrak{z}) = (1 - \mathfrak{z}^2) (p_c(\mathfrak{z}) + (1 + \mathfrak{z})^{d_0+1}(1 - \mathfrak{z})^{d_\infty+1}q(\mathfrak{z})) .$$

where $q(\mathfrak{z}) = c\mathfrak{z} + e$ and c, e are determined by the conditions above. Working this out explicitly and in general is straightforward (albeit messy), but it gets hopelessly complicated to sort out when the admissible data is such that $F_{extr}(\mathfrak{z})$ satisfies (i) of (53).

However, to illustrate that further existence results are out there, let us for simplicity assume that $s_1 = 2$, $0 < r_1 < 1$ (so $\omega_{N_1}/(2\pi)$ is a Kähler form with primitive Kähler class on \mathbb{CP}^1) and $s_2 = -2$, $0 < r_2 < 1$ (meaning $\omega_{N_2}/(2\pi)$ is a Kähler form whose class is $(g-1)$ times a primitive Kähler class on Σ_g). We will also assume that $d_0 = d_\infty = 1$. Now one can calculate that

$$F_{extr}(\mathfrak{z}) = \frac{(1 - \mathfrak{z}^2)^2 h(\mathfrak{z})}{3(3r_1^2 r_2^2 - 7r_1^2 + 8r_1 r_2 - 7r_2^2 + 35)}$$

with

$$h(\mathfrak{z}) = a_0 + a_1(\mathfrak{z} + 1) + a_2(\mathfrak{z} + 1)^2 + a_3(1 + \mathfrak{z})^3,$$

where

$$\begin{aligned}
a_0 &= 3(1-r_1)(1-r_2)(3r_1^2r_2^2 - 7r_1^2 + 8r_1r_2 - 7r_2^2 + 35) \\
a_1 &= (1-r_2)(12r_1^3r_2^2 + 15r_1^3r_2 + 7r_1^3 + 105r_1 + 49r_2^2 + 105r_2) \\
&\quad - (1-r_2)(21r_1^2r_2^2 + 13r_1^2r_2 + 56r_1^2 + 48r_1r_2^2 + 91r_1r_2) \\
a_2 &= 2(3r_1^3r_2^3 + 3r_1^3r_2^2 + 8r_1^3r_2 + 2r_1^2r_2^2 + 14r_1^2 + 49r_1r_2 + 7r_2^3 + 14r_2^2) \\
&\quad - 2(7r_1^3 + 3r_1^2r_2^3 + 30r_1^2r_2 + 22r_1r_2^3 + 30r_1r_2^2) \\
a_3 &= 10r_1r_2(2-r_1+r_2)(r_1+r_2)
\end{aligned}$$

Since all the coefficients, a_i , for the powers of $(1+\mathfrak{z})$ are positive (bear in mind that $0 < r_i < 1$), we see that $h(\mathfrak{z}) > 0$ for $\mathfrak{z} > -1$ and hence $F_{extr}(\mathfrak{z}) > 0$ for $-1 < \mathfrak{z} < 1$. In conclusion, we have the following statement

Proposition A.2. *Let P be the admissible manifold $P = \mathbb{P}(E_0 \oplus E_\infty) \rightarrow \mathbb{C}\mathbb{P}^1 \times \Sigma_g$, where Σ_g is a compact Riemann surface of genus $g > 1$. Let Ω_1 denote the standard Fubini-Study metric on $\mathbb{C}\mathbb{P}^1$ (with volume one) and let Ω_2 denote the standard Kähler area form on Σ_g (with scalar curvature $8\pi(1-g)$). Assume $E_0 = L_0 \times \mathbb{C}^2$, $E_\infty = L_\infty \times \mathbb{C}^2$, $c_1(L_0) = [-\omega_0]$, and $c_1(L_\infty) = [-\omega_\infty]$, where $\omega_0 = 2\Omega_1 + g\Omega_2$ and $\omega_\infty = \Omega_1 + \Omega_2$. Then every Kähler class on P admits an extremal (admissible) metric.*

A.3. Regular transverse Kähler class: $d = 1$ admissible case.

At this point we will not attempt to determine the Kähler class of the regular quotient of a fiber join in general. However, in the special case of an admissible fiber join with $d = 1$, we can extract this information from (9). We shall see that in this case -with one additional assumption added- the Kähler class is indeed admissible (up to scale).

So assume we have a $d = 1$ fiber join as defined in Theorem 3.4 and the complex manifold arising as the quotient of the regular Reeb vector field ξ_1 is equal to $\mathbb{P}(L_1^* \oplus L_2^*) \rightarrow N$ where the base N is a local product of Kähler manifolds (N_a, Ω_a) , $a \in \mathcal{A} \subset \mathbb{N}$, where \mathcal{A} is a finite index set. For convenience set $L_1^* = L_0$ and $L_2^* = L_\infty$, where L_0 and L_∞ are some holomorphic line bundles, and set $\omega_1 = \omega_0$ and $\omega_2 = \omega_\infty$. Then $c_1(L_0) = -[\omega_0]$ and $c_1(L_\infty) = -[\omega_\infty]$. Note that for $j = 1, 2$, the pullback of ω_j to the Sasaki manifold M_j is $d\eta_j/(2\pi)$ (the factor of 2π is occasionally ignored or neglected). As part of the admissible assumption we have that $c_1(L_\infty) - c_1(L_0) = \sum_{a \in \mathcal{A}} [\epsilon_a \Omega_a]$, where $\epsilon_a = \pm 1$ or we may write $c_1(L_\infty) - c_1(L_0) = \frac{1}{2\pi}[\omega_N]$, where $\omega_N = \sum_{a \in \mathcal{A}} \omega_{N_a}$ and $\omega_{N_a} = 2\pi\epsilon_a\Omega_a$.

This means that $2\pi([\omega_0] - [\omega_\infty]) = [\omega_N]$. We shall make one additional assumption, namely that

$$(54) \quad 2\pi([\omega_0] + [\omega_\infty]) = \sum_{a \in \mathcal{A}} \frac{1}{r_a} [\omega_{N_a}],$$

for some r_a with same sign as ω_{N_a} and $0 < |r_a| < 1$.

Equation (9) with $\mathbf{a} = (1, 1)$ for $d = 1$ now reads as

$$d\eta_{\mathbb{1}} = r_1^2 d\eta_1 + 2(r_1 dr_1 \wedge (\eta_1 + d\theta_1)) + r_2^2 d\eta_2 + 2(r_2 dr_2 \wedge (\eta_2 + d\theta_2))$$

Consider now the transverse Kähler structure on $\mathbb{P}(L_0 \oplus L_\infty) \rightarrow N$ (same as $\mathbb{P}(\mathbb{1} \oplus L_0^* \otimes L_\infty) \rightarrow N$). We follow the ideas on page 557 in [ACGTF08] with no blow-downs: Since (r_j, θ_j) are polar coordinates of L_j^* we can say that $z_0 := \frac{1}{2}r_1^2$ and $z_\infty := \frac{1}{2}r_2^2$ are the moment maps of the natural S^1 action on L_0 and L_∞ , respectively. On $2 = z_0 + z_\infty$, the function $z := z_0 - 1 = 1 - z_\infty$ descends to a fiberwise moment map (with range $[-1, 1]$) for the induced S^1 action on $\mathbb{P}(\mathbb{1} \oplus L_0^* \otimes L_\infty) \rightarrow N$. This allows us to rewrite (9) as

$$d\eta_{\mathbb{1}} = 2(d\eta_1 + d\eta_2) + 2d(z\theta),$$

where $\theta = (\eta_1 + d\theta_1) - (\eta_2 + d\theta_2)$ is the connection form on $L_0^* \otimes L_\infty$. According to pages 556-557 in [ACGTF08] we have that this descends to a Kähler form on $\mathbb{P}(\mathbb{1} \oplus L_0 \otimes L_\infty^*) \rightarrow N$ with Kähler class

$$2(2\pi([\omega_1] + [\omega_2]) + \Xi),$$

where Ξ is the fixed class described above. Due to our extra assumption we then have that - up to scale - the Kähler class is

$$\sum_{a \in \mathcal{A}} \frac{1}{r_a} [\omega_{N_a}] + \Xi,$$

with r_a as given in (54), which is precisely an admissible Kähler class.

Remark A.3. If colinearity holds on top of the above assumptions, we have according to Proposition 3.15 that the fiber join is just a regular join. Here $\omega_i = b_i \omega_N$. Connecting with the notation in [BTF16] we have $l_1(w_1, w_2) = (b_1, b_2)$, $l_2 = 1$, and therefore in Theorem 3.8 of [BTF16], the regular quotient has $n = b_1 - b_2$. In the above setting \mathcal{A} is a singleton, $[\omega_{N_a}] = 2\pi n [\omega_N]$, and $2\pi([\omega_1] + [\omega_2]) + \Xi = 2\pi(b_1 + b_2) [\omega_N] + \Xi = \frac{1}{r_a} [\omega_{N_a}] + \Xi$ with $r_a = \frac{b_1 - b_2}{b_1 + b_2}$, as it should be according to (44) and (59) in [BTF16].

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