SASAKI-EINSTEIN METRICS ON A CLASS OF 7-MANIFOLDS

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Abstract. In this note we give an explicit construction of Sasaki-Einstein metrics on a class of simply connected 7-manifolds with the rational cohomology of the 2-fold connected sum of $S^2 \times S^5$. The homotopy types are distinguished by torsion in H^4.

Introduction

Sasaki-Einstein metrics on 7-manifolds continue to play an important role in M-theory as well as black hole physics [Spa11, GLPS17, MT18]. An important reason for this is that Sasaki-Einstein manifolds admit supersymmetry and are used in the AdS-CFT correspondence. For such reasons it seems important to have a large list of explicit examples of Sasaki-Einstein 7-manifolds that can be used as possible models. Since these SE metrics are toric, general existence of such metrics in well known [FOW09, Leg16]. What is new here is an explicit construction of such metrics, their relation with Bott manifolds (orbifolds), and the topological description of the 7-manifolds. We give an explicit construction of toric Sasaki-Einstein (SE) 7-manifolds which can be represented as S^1 orbibundles over 2-twist stage 3 Bott orbifolds. All of these are obtained by adding orbifold structures to certain stage 3 Bott manifolds which were studied in [BCTF18]. The 7-manifolds all have the rational cohomology of the 2-fold connected sum $2(S^2 \times S^5)$ and are generalizations of the SE 7-manifolds given by Theorem 1.2 of [BTF15].

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1. Stage Three Bott Towers and Orbifolds

Following [GK94] and [BCTF18] we consider *Bott towers* which in arbitrary dimension is represented by a lower triangular unipotent matrix A over $\mathbb{Z}$. Here we deal only with stage 3 Bott towers, so the matrix A in [GK94, BCTF18] takes the form

$$A = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \tag{1}$$

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with $a, b, c \in \mathbb{Z}$. The Bott manifold can be realized as the quotient of $S^3 \times S^3 \times S^3$ by the $\mathbb{T}^3$ action

$$(z_j^0, z_j^\infty)_{j=1}^3 \mapsto (t_j z_j^0, \prod_{i=1}^3 t_i^{A_j^i} z_j^\infty). \tag{2}$$

The quotient M_3 which is called a *Bott manifold* can be represented as a sequence, called a *Bott tower*

$$M_3 \xrightarrow{\pi_3} M_2 \xrightarrow{\pi_2} M_1 \xrightarrow{\pi_1} M_0 = pt, \tag{3}$$

where the jth S^3 is written as $|z_j^0|^2 + |z_j^\infty|^2 = 1$. Then M_k is the compact complex manifold arising as the total space of the $\mathbb{CP}^1$ bundle $\pi_k \colon \mathbb{P}(\mathbb{1} \oplus \mathcal{L}_k) \to M_{k-1}$. At each stage we have *zero* and *infinity sections* $\sigma_k^0 \colon M_{k-1} \to M_k$ and $\sigma_k^\infty \colon M_{k-1} \to M_k$ which respectively identify M_{k-1} with $\mathbb{P}(\mathbb{1} \oplus 0)$ and $\mathbb{P}(0 \oplus \mathcal{L}_k)$. We consider these to be part of the structure of the Bott tower $(M_k, \pi_k, \sigma_k^0, \sigma_k^\infty)_{k=1}^n$. Here in our case $M_3 = M_3(a, b, c)$ is a stage 3 Bott manifold, $M_2(a)$ is a Hirzebruch surface $\mathcal{H}_a = \mathbb{P}(\mathbb{1} \oplus \mathcal{O}(a)) \longrightarrow \mathbb{CP}^1$, and $M_1 = \mathbb{CP}^1$. $M_3(a, b, c)$ can be viewed as the total space of $\mathbb{CP}^1$ bundle over the Hirzebruch surface $\mathcal{H}_a$, and also a bundle of Hirzebruch surfaces over $\mathbb{CP}^1$ with fiber $\mathcal{H}_c$. Bott towers form the object set $\mathcal{BT}_0$ of a groupoid whose morphisms $\mathcal{BT}_1$ are biholomorphisms [BCTF18], and elements of the quotient space $\mathcal{BT}_0/\mathcal{T}_1$ are identfied with biholomorphism classes of Bott manifolds. Since Bott manifolds are toric, they are described by a fan, and it follows from the Bott tower description (3) that the fan of the Bott tower $M_3(a, b, c)$ is described by the primitive collections (cf. [CLS11])

$$\{v_1, u_1\}, \quad \{v_2, u_2\}, \quad \{v_3, u_3\} \tag{4}$$

with normal vectors $u_1 = -v_1 - av_2 - bv_3, u_2 = -v_2 - cv_3, u_3 = -v_3$, and thus has the combinatorial type of a cube. The symmetry group of a cube is the Coxeter group $BC_3 \cong \mathrm{Sym}_3 \ltimes \mathbb{Z}_2^3$ where Sym_3 is the symmetric group on 3 letters. However, not all elements of BC_3 are induced by equivalences. We refer to [BCTF18] and references therein for details.

The structure of Bott towers implies the existence of 3 pairs of $\mathbb{T}^3$ invariant divisors $\{D_{v_j}, D_{u_j}\}$ which are the zero and infinity sections of $M_j \xrightarrow{\pi_j} M_{j-1}$ with $j = 1, 2, 3$. Thus, elements of the subgroup $\mathbb{Z}_2^3$ are induced by the *fiber inversion* maps that interchange the zero and infinity sections, so these equivalences always exist. However, elements of Sym_3 are induced by equivalences only in special cases (see Lemma 1.11 and Example 1.6 of [BCTF18] for details).

Recall (Definition 1.6 of [BCTF18]) that the *holomorphic twist* of an n dimensional Bott tower is the number $t \in \{0, \ldots, n-1\}$ of holomorphically nontrivial $\mathbb{CP}^1$ bundles in the tower. So for $n = 3$ there are only 3 possibilities $t = 0, 1, 2$. Of course, $t = 0$ is the well understood product $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$, whereas $t = 1$ leads to the Koiso-Sakane case, so here we restrict our attention to $t = 2$. For the $t = 2$ case we have a nontrivial holomorphic bundle over a Hirzebruch surface $\mathcal{H}_a$ with $a \neq 0$ whose fiber is $\mathbb{CP}^1$. The case of interest to us can be obtained as an $S^3_{\mathbf{w}}$-join with the $Y^{p,q}$ structures of [GMSW04] on $S^2 \times S^3$, that is, $(S^2 \times S^3) \star_{l_1, l_2} S^3_{\mathbf{w}}$. So the stage 3 Bott manifold has an additional orbifold structure which we now describe.

1.1. **Invariant Divisors.** Here we consider the $\mathbb{T}^3$ invariant divisors D_{v_i}, D_{u_i} defined by the normal vectors v_i, u_i with equivalences

$$D_{v_1} \sim D_{u_1}, \qquad D_{v_2} \sim aD_{u_1} + D_{u_2}, \qquad D_{v_3} \sim bD_{u_1} + cD_{u_2} + D_{u_3}.$$

We have 4 sets of distinguished invariant bases of the Chow group $A_2(M_3)$ of invariant divisor classes

$$\{[D_{u_1}], [D_{u_2}], [D_{u_3}]\}, \tag{5}$$
$$\{[D_{u_1}], [D_{u_2}], [D_{v_3}]\}, \tag{6}$$
$$\{[D_{u_1}], [D_{v_2}], [D_{u_3}]\}, \tag{7}$$
$$\{[D_{u_1}], [D_{v_2}], [D_{v_3}]\}. \tag{8}$$

This gives rise to the 4 sets of dual bases of cohomology classes in $H^2(M_3, \mathbb{Z})$, viz

$$\{x_1, x_2, x_3\}, \tag{9}$$
$$\{x_1, x_2, y_3\}, \tag{10}$$
$$\{x_1, y_2, x_3\}, \tag{11}$$
$$\{x_1, y_2, y_3\}, \tag{12}$$

where x_i is dual to $[D_{u_i}]$ and y_i is dual to $[D_{v_i}]$. Both the ample and Kähler cones can be easily worked out, see Example 3.3 of [BCTF18].

1.2. **Bott Orbifolds and log pairs.** We are interested in these Bott manifolds, but with an additional special orbifold structure along the $\mathbb{T}^3$-invariant divisors D_{v_i}, D_{u_i}. The orbifold structure on $M_3(a, b, c)$ that we are interested in is given by the log pair $(M_3(a, b, c), \Delta_{\mathbf{m}})$ where $\Delta_{\mathbf{m}}$ is the branch divisor

(13)
$$\Big(1-\frac{1}{m_1^0}\Big)D_{v_1}+\Big(1-\frac{1}{m_1^\infty}\Big)D_{u_1}+\Big(1-\frac{1}{m_2^0}\Big)D_{v_2}+\Big(1-\frac{1}{m_2^\infty}\Big)D_{u_2}+\Big(1-\frac{1}{m_3^0}\Big)D_{v_3}+\Big(1-\frac{1}{m_3^\infty}\Big)D_{u_3}$$

where $m_j^0, m_j^\infty \in \mathbb{Z}^+$ are the ramification indices. We define $\Delta_{\mathbf{m}_3} = \Delta_{\mathbf{m}}$ and

$$\begin{aligned}\Delta_{\mathbf{m}_2} &= \Big(1 - \frac{1}{m_1^0}\Big)D_{v_1} + \Big(1 - \frac{1}{m_1^\infty}\Big)D_{u_1} + \Big(1 - \frac{1}{m_2^0}\Big)D_{v_2} + \Big(1 - \frac{1}{m_2^\infty}\Big)D_{u_2}\\ \Delta_{\mathbf{m}_1} &= \Big(1 - \frac{1}{m_1^0}\Big)D_{v_1} + \Big(1 - \frac{1}{m_1^\infty}\Big)D_{u_1}.\end{aligned}$$

Clearly we have

$$\Delta_{\mathbf{m}_1} \subset \Delta_{\mathbf{m}_2} \subset \Delta_{\mathbf{m}_3}.$$

From [GK94] or [BCTF18] one easily sees that

Lemma 1.1. *We have the sequence of Bott towers of log pairs*

$$\big(M_3(a, b, c), \Delta_{\mathbf{m}_3}\big) \xrightarrow{\pi_3} \big(M_2(a), \Delta_{\mathbf{m}_2}\big) \xrightarrow{\pi_2} \big(M_1, \Delta_{\mathbf{m}_1}\big) \xrightarrow{\pi_1} (\{pt\}, \emptyset), \tag{14}$$

where $M_2(a) = \mathcal{H}_a$ is a Hirzebruch surface, $M_1 = \mathbb{CP}^1$, and π_i is the natural projection.

The invariant branch divisors are related to the section maps by

$$D_{v_3} = \sigma_2^0(\mathcal{H}_a),\ D_{u_3} = \sigma_2^\infty(\mathcal{H}_a),\ D_{v_2} = \pi_3^{-1}(\sigma_1^0(\mathbb{CP}^1)),\ D_{u_2} = \pi_3^{-1}(\sigma_1^\infty(\mathbb{CP}^1))$$

and

$$D_{v_1} = (\pi_2 \circ \pi_3)^{-1}(\sigma_1^0(\{pt\})),\ D_{u_1} = (\pi_2 \circ \pi_3)^{-1}(\sigma_1^\infty(\{pt\})).$$

We denote by $\mathfrak{BO}_A$ the set of all such Bott orbifolds.

1.3. **The Orbifold First Chern Class.** We can now compute the orbifold canonical divisor K^{orb} and dually the orbifold first Chern class. We compute the orbifold first Chern class c_1^{orb} in the $\{x_j\}$ basis for any n-dimensional Bott manifold $M_n(A)$;

$$\begin{aligned} c_1^{orb}(M_{\mathbf{n}}(A),\Delta_{\mathbf{m}}) &= c_1(M_{\mathbf{n}}(A)) - \sum_{i=1}^n\Big(\big(1-\frac{1}{m_i^0}\big)y_i + \big(1-\frac{1}{m_i^\infty}\big)x_i\Big) \\ &= \sum_{i=1}^n(x_i+y_i) - \sum_{i=1}^n\Big(\big(1-\frac{1}{m_i^0}\big)y_i + \big(1-\frac{1}{m_i^\infty}\big)x_i\Big) \\ &= \sum_{j=1}^n\big(\frac{1}{m_j^0}y_j + \frac{1}{m_j^\infty}x_j\big) = \sum_{i=1}^n\Big(\big(\frac{1}{m_i^0}+\frac{1}{m_i^\infty}\big)x_i + \sum_{j=1}^{i-1}\frac{A_i^j}{m_i^0}x_j\Big) \\ &= \sum_{j=1}^{n-1}\Big(\frac{1}{m_j^0}+\frac{1}{m_j^\infty}+\sum_{i=j+1}^n\frac{A_i^j}{m_i^0}\Big)x_j + \Big(\frac{1}{m_n^0}+\frac{1}{m_n^\infty}\Big)x_n. \end{aligned} \tag{15}$$

In dimension n there are 2^{n-1} invariant bases in which to compute c_1^{orb}. This becomes much more manageable for $n=3$. We have $c_1^{orb}(M_3(a,b,c),\Delta_{\mathbf{m}})$ in the four bases (9)-(12), respectively

$$(\frac{1}{m_1^0}+\frac{1}{m_1^\infty}+\frac{a}{m_2^0}+\frac{b}{m_3^0})x_1 + (\frac{1}{m_2^0}+\frac{1}{m_2^\infty}+\frac{c}{m_3^0})x_2 + (\frac{1}{m_3^0}+\frac{1}{m_3^\infty})x_3, \tag{16}$$

$$(\frac{1}{m_1^0}+\frac{1}{m_1^\infty}+\frac{a}{m_2^0}-\frac{b}{m_3^\infty})x_1 + (\frac{1}{m_2^0}+\frac{1}{m_2^\infty}-\frac{c}{m_3^\infty})x_2 + (\frac{1}{m_3^0}+\frac{1}{m_3^\infty})y_3, \tag{17}$$

$$(\frac{1}{m_1^0}+\frac{1}{m_1^\infty}-\frac{a}{m_2^\infty}+\frac{b-ac}{m_3^0})x_1 + (\frac{1}{m_2^0}+\frac{1}{m_2^\infty}+\frac{c}{m_3^0})y_2 + (\frac{1}{m_3^0}+\frac{1}{m_3^\infty})x_3, \tag{18}$$

$$(\frac{1}{m_1^0}+\frac{1}{m_1^\infty}-\frac{a}{m_2^\infty}-\frac{b-ac}{m_3^\infty})x_1 + (\frac{1}{m_2^0}+\frac{1}{m_2^\infty}-\frac{c}{m_3^\infty})y_2 + (\frac{1}{m_3^0}+\frac{1}{m_3^\infty})y_3. \tag{19}$$

Note also that as a cohomology class $c_1^{orb}(M_3(a,b,c),\Delta_{\mathbf{m}})$ as an element of $H^{1,1}(M_n(a,b,c),\mathbb{R})$ makes perfect sense for all $m_j^0, m_j^\infty \in \mathbb{R}^+$. We shall make use of this fact shortly.

Equations (16)-(19) implies

Lemma 1.2. *Let $(M_3(a,b,c),\Delta_{\mathbf{m}})$ be a Bott orbifold. Then the following are equivalent:*

1. *$(M_3(a,b,c),\Delta_{\mathbf{m}})$ is log Fano,*
2. *$c_1^{orb}(M_3(a,b,c),\Delta_{\mathbf{m}})$ lies in the Kähler cone $\mathcal{K}(M_3(a,b,c))$,*

(3) *the inequalities*

$$\begin{aligned}
&\frac{1}{m_1^0}+\frac{1}{m_1^\infty}+\frac{a}{m_2^0}+\frac{b}{m_3^0}>0, \quad \frac{1}{m_2^0}+\frac{1}{m_2^\infty}+\frac{c}{m_3^0}>0\\
&\frac{1}{m_1^0}+\frac{1}{m_1^\infty}+\frac{a}{m_2^0}-\frac{b}{m_3^\infty}>0, \quad \frac{1}{m_2^0}+\frac{1}{m_2^\infty}-\frac{c}{m_3^\infty}>0\\
&\frac{1}{m_1^0}+\frac{1}{m_1^\infty}-\frac{a}{m_2^\infty}+\frac{(b-ac)}{m_3^0}>0, \quad \frac{1}{m_2^0}+\frac{1}{m_2^\infty}+\frac{c}{m_3^0}>0\\
&\frac{1}{m_1^0}+\frac{1}{m_1^\infty}-\frac{a}{m_2^\infty}-\frac{(b-ac)}{m_3^\infty}>0, \quad \frac{1}{m_2^0}+\frac{1}{m_2^\infty}-\frac{c}{m_3^\infty}>0
\end{aligned}$$

hold.

2. M^7 as the join $Y^{p,q}\star_{l_1,l_2}S^3_{\mathbf{w}}$

Not every S^1 orbibundle over a Bott orbifold can be realized as a join; however, the 2 twist stage 3 Bott orbifolds that we study here can be realized as Kähler quotients of the join $Y^{p,q}\star_{l_1,l_2}S^3_{\mathbf{w}}$ where $Y^{p,q}$ are the well known SE structures on $S^2\times S^3$ discovered by the physicists [GMSW04]. In fact, since $Y^{p,q}$ is itself a join of two S^3's, it is an iterated join of three S^3's which in the terminology of [BHLTF18] is completely cone decomposable. Now from Example 6.8 of [BTF16] we have

$$Y^{p,q}=S^3\star_{l,p}S^3_{\frac{p+q}{l},\frac{p-q}{l}} \tag{20}$$

where $l=\gcd(p+q,p-q)$ which equals 2 if p,q are both odd, and equals 1 if p,q have opposite parity. Note that we choose the standard SE structure on the lefthand S^3 factor, whereas it is the weighted Sasakian structure with weights $(\frac{p+q}{l},\frac{p-q}{l})$ on the righthand S^3.

If we assume that $p>q\geq 1$ are such that $\sqrt{4p^2-3q^2}\in\mathbb{N}$, then the Sasaki-Einstein structure on $Y^{p.q}$ is quasi-regular [GMSW04]. Indeed, the η-Einstein structure corresponds to a ray in the so-called $\mathbf{w}$ subcone is determined by co-prime solutions (v_2^0,v_2^∞) of

$$\int_{-1}^1((v_2^0-v_2^\infty)-(v_2^0+v_2^\infty)\mathfrak{z})(((p+q)v_2^\infty+(p-q)v_2^0)+((p+q)v_2^\infty-(p-q)v_2^0)\mathfrak{z})\,d\mathfrak{z}=0 \tag{21}$$

(from e.g. (68) in [BTF16]). Following Theorem 3.8 in [BTF16], for any choice of quasi-regular ray determined by the co-prime pair (v_2^0,v_2^∞), the quotient Hirzebruch orbifold is $(\mathcal{H}_a,\Delta_{\mathbf{m}_2})$ where $\Delta_{\mathbf{m}_2}=(1-\frac{1}{m_2^0})D_{v_2}+(1-\frac{1}{m_2^\infty})D_{u_2}$ is the branch divisor with

$$\begin{aligned}
\mathbf{m}_2 &= (m_2^0,m_2^\infty)=m_2(v_2^0,v_2^\infty),\\
m_2 &= \frac{p}{\gcd(p,|\frac{p+q}{l}v_2^\infty-\frac{p-q}{l}v_2^0|)},\\
a &= \frac{(p+q)m_2^\infty-(p-q)m_2^0}{p}.
\end{aligned} \tag{22}$$

The join $M_{l_1,l_2,\mathbf{w}} = Y^{p,q} \star_{l_1,l_2} S^3_{\mathbf{w}}$, where $Y^{p,q}$ has a quasi-regular Sasaki structure as above, can then be obtained as the quotient of the following $\mathbb{T}^2$ action on $S^3 \times S^3 \times S^3$:

(23)
$$(x, u; u_1, u_2; z_1, z_2) \mapsto (x, e^{ipl_2\theta}u; e^{i(l_2m_2v_2^0\phi-(p+q)\theta)}u_1, e^{i(l_2m_2v_2^\infty\phi-(p-q)\theta)}u_2; e^{-il_1w_1\phi}z_1, e^{-il_1w_2\phi}z_2).$$

First we notice that without loss of generality we can assume that $\gcd(l_1, m_2) = 1$, for otherwise we can redefine ϕ. So our gcd conditions are $\gcd(l_1, l_2m_2) = 1$, $\gcd(w_1, w_2) = 1 = \gcd(v_2^0, v_2^\infty)$, and $\gcd(p, q) = 1$. Note that $p+q$ and $p-q$ can have a common factor if and only if both p and q are odd, in which case the common factor is 2. However, $M_{l_1,l_2,\mathbf{w}}$ may not be a smooth manifold. we have

Lemma 2.1. *The join $M_{l_1,l_2,\mathbf{w}} = Y^{p,q} \star_{l_1,l_2} S^3_{\mathbf{w}}$, with Sasakian structure on $Y^{p,q}$ given by the Reeb field $\xi_{\mathbf{v}_2}$ with $\mathbf{v}_2 = (v_2^0, v_2^\infty)$, is a smooth manifold if and only if* $\gcd(l_2m_2v_2^i, l_1w_j) = 1$ *for $i = 0, \infty$ and $j = 1, 2$, where $m_2 = \frac{p}{\gcd(p, |\frac{p+q}{l}v_2^\infty - \frac{p-q}{l}v_2^0|)}$.*

Proof. From Proposition 7.6.7 of [BG08] $M_{l_1,l_2,\mathbf{w}}$ is smooth if and only if $\gcd(l_1\Upsilon_2, l_2\Upsilon_1) = 1$ where Υ_1 is the order of $Y^{p,q}$ and Υ_2 is the order of $S^3_{\mathbf{w}}$. The latter is $\Upsilon_2 = w_1w_2$ with w_1, w_2 coprime. The order Υ_1 with quasi-regular Reeb field $\xi_{\mathbf{v}_2}$ is $\Upsilon_1 = m_2v_2^0v_2^\infty$. □

The analysis in Section 3 of [BTF16] holds equally well when the manifold M in the join $M \star_{l_1,l_2} S^3_{\mathbf{w}}$ has any quasi-regular Sasakian structure. The major difference is having more complicated computations. For example we need the Fano index of of $Y^{p,q}$. As described above, the quotient of any quasi-regular Sasakian structure in the $\mathbf{w}$ subcone of $Y^{p,q}$ is a Hirzebruch orbifold of the form $(\mathcal{H}_a, \Delta_{\mathbf{m}_2})$. We have

Lemma 2.2. *Let $\xi_{\mathbf{v}_2}$ be a quasiregular Reeb vector field with quotient Hirzebruch orbifold $(\mathcal{H}_a, \Delta_{\mathbf{m}_2})$ with $a > 0$. Then its Fano index $\mathcal{I}_{\mathbf{v}_2}$ is given by*

$$\mathcal{I}_{\mathbf{v}_2} = \gcd(2m_2v_2^0 + a)v_2^\infty, v_2^0 + v_2^\infty)$$

where $\mathbf{m}_2 = m_2(v_2^0, v_2^\infty)$ and v_2^0, v_2^∞ are coprime.

Proof. Recall (Definition 4.4.24 of [BG08]) that the Fano index $\mathcal{I}$ of an orbifold $\mathcal{Z}$ is the largest positive integer k such that $p^*c_1^{orb}/k$ is an element of $H^2_{orb}(\mathcal{Z}, \mathbb{Z}) = H^2(B\mathcal{Z}, \mathbb{Z})$. Now the classifying map $p : B\mathcal{Z} \longrightarrow \mathcal{Z}$ is an $m_2v_2^0v_2^\infty$-fold cover, and c_1^{orb} is

$$c_1^{orb} = (2 + \frac{a}{m_2v_2^0})x_1 + \frac{v_2^0 + v_2^\infty}{m_2v_2^0v_2^\infty}x_2 = \frac{1}{m_2v_2^0v_2^\infty}\Big((2m_2v_2^0 + a)v_2^\infty x_1 + (v_2^0 + v_2^\infty)x_2\Big). \tag{24}$$

So $p^*c_1^{orb} = (2m_2v_2^0 + a)v_2^\infty p^*x_1 + (v_2^0 + v_2^\infty)p^*x_2$ from which the result follows. □

From now on we assume that $p > q \geq 1$ are such that $\sqrt{4p^2 - 3q^2} \in \mathbb{N}$ and $Y^{p,q}$ has the quasi-regular Sasaki-Einstein structure. Thus co-prime (v_2^0, v_2^∞) are chosen such that (21) is satisfied. We then present a version of Theorem 3.8 of [BTF16] that allows the join $M_{l_1,l_2,\mathbf{w}} = Y^{p,q} \star_{l_1,l_2} S^3_{\mathbf{w}}$

Theorem 2.3. *Consider the join $M_{l_1,l_2,\mathbf{w}} = Y^{p,q} \star_{l_1,l_2} S^3_{\mathbf{w}}$ where $Y^{p,q}$ has a quasi-regular Sasaki-Einstein structure determined by the co-prime pair (v_2^0, v_2^∞) satisfying Equation (21). Let $\mathcal{S}_{\mathbf{v}_3} = (\xi_{\mathbf{v}_3}, \eta_{\mathbf{v}_3}, \Phi_{\mathbf{v}_3}, g_{\mathbf{v}_3})$ be a quasi-regular Sasakian structure that lies in the $\mathbf{w}$ subcone of the Sasaki cone with $\mathbf{v}_3 = (v_3^0, v_3^\infty)$ where v_3^0, v_3^∞ are coprime. Then the* *quotient of $M_{l_1,l_2,\mathbf{w}}$ by the S^1 action generated by $\xi_{\mathbf{v}_3}$ is the Bott orbifold given by the log pair $(M_3(a,b,c),\Delta_{\mathbf{m}})$, where a is determined by (22),*

$$\begin{aligned} b &= n\hat{b} = n\tfrac{(2m_2v_2^0+a)v_2^\infty}{\mathcal{I}_{\mathbf{v}_2}} \\ c &= n\hat{c} = n\tfrac{(v_2^0+v_2^\infty)}{\mathcal{I}_{\mathbf{v}_2}}, \end{aligned}$$

$n = l_1\frac{w_1v_3^\infty - w_2v_3^0}{\gcd(|w_1v_3^\infty - w_2v_3^0|,l_2)}$, *$\mathcal{I}_{\mathbf{v}_2}$ is given by Lemma 2.2, and $\Delta_{\mathbf{m}}$ is the branch divisor*

$$\Big(1-\frac{1}{m_1^0}\Big)D_{v_1}+\Big(1-\frac{1}{m_1^\infty}\Big)D_{u_1}+\Big(1-\frac{1}{m_2^0}\Big)D_{v_2}+\Big(1-\frac{1}{m_2^\infty}\Big)D_{u_2}+\Big(1-\frac{1}{m_3^0}\Big)D_{v_3}+\Big(1-\frac{1}{m_3^\infty}\Big)D_{u_3}$$

with $(m_1^0,m_1^\infty)=(1,1)$, (m_2^0,m_2^∞) given by (22), and

$$(m_3^0,m_3^\infty) = m_3(v_3^0,v_3^\infty) = \frac{l_2}{\gcd(|w_1v_3^\infty - w_2v_3^0|,l_2)}(v_3^0,v_3^\infty).$$

Proof. We can follow the proof of Theorem 3.8 of [BTF16] with the caveat that N is a Hirzebruch orbifold $(\mathcal{H}_a,\Delta_{\mathbf{m}_2})$. As in Equation (3) of [BTF16] we have the commutative diagram

$$S^3\times S^3\times S^3 \longrightarrow Y^{p,q}\times S^3_{\mathbf{w}};\quad Y^{p,q}\times S^3_{\mathbf{w}} \xrightarrow{\pi_L} Y^{p,q}\star_{l_1,l_2,\mathbf{w}} S^3_{\mathbf{w}};\quad Y^{p,q}\times S^3_{\mathbf{w}} \xrightarrow{\pi_2} (\mathcal{H}_a,\Delta_{\mathbf{m}_2})\times\mathbb{CP}^1[\mathbf{w}];\quad Y^{p,q}\star_{l_1,l_2,\mathbf{w}} S^3_{\mathbf{w}} \xrightarrow{\pi_1} (\mathcal{H}_a,\Delta_{\mathbf{m}_2})\times\mathbb{CP}^1[\mathbf{w}] \tag{25}$$

where the πs are the obvious projections, and the orbifold $(\mathcal{H}_a,\Delta_{\mathbf{m}_2})$ is the quotient by the locally free S^1 action on $Y^{p,q}$ generated by the quasi-regular Reeb vector field $\xi_{\mathbf{m}_2}$ where $\mathbf{m}_2 = m_2(v_2^0,v_2^\infty)$ and a is given in Equations (22). The holomorphic line bundle $L_n = L^n$ now becomes the holomorphic line orbibundle with L determined by the "primitive" Kähler class in $H^2((\mathcal{H}_a,\Delta_{\mathbf{m}}),\mathbb{Q})$, namely

$$\frac{c_1^{orb}(\mathcal{H}_a,\Delta_{\mathbf{m}_2})}{\mathcal{I}_{\mathbf{v}_2}} = \frac{1}{m_2v_2^0v_2^\infty}\frac{\big((2m_2v_2^0+a)v_2^\infty x_1 + (v_2^0+v_2^\infty)x_2\big)}{\mathcal{I}_{\mathbf{v}_2}}. \tag{26}$$

We make note of the fact that $Y^{p,q}\longrightarrow(\mathcal{H}_a,\Delta_{\mathbf{m}_2})$ is an $m_2v_2^0v_2^\infty$-fold covering map.

Now as in Section 3.5 of [BTF16] we want to describe the base orbifold $B_{\mathbf{l},\mathbf{v},\mathbf{w}}$ of the S^1 orbibundle generated by the locally free action of the quasi-regular Reeb vector field $\xi_{\mathbf{m}_3}$ of the SE structure on the join $Y^{p,q}\star_\ell S^3_{\mathbf{w}}$. We see that the analysis of Section 3.5 of [BTF16] goes through verbatim through Remark 3.7 with $M=Y^{p,q}$ and $N=(\mathcal{H}_a,\Delta_{\mathbf{m}_2})$. In particular, from Lemma 3.6 of [BTF16] we obtain the base orbifold

$B_{\ell,\mathbf{v},\mathbf{w}} \approx (B_{\ell,1,\mathbf{w}'},\Delta)$ with

$$\Delta=\Bigl(1-\frac{1}{m_3^0}\Bigr)D_{v_3}+\Bigl(1-\frac{1}{m_3^\infty}\Bigr)D_{u_3},\quad \mathbf{w}'=(v_3^\infty w_1,v_3^0w_2),$$

and from Theorem 3.8 of [BTF16] we have

$$\mathbf{m}_3=m_3(v_3^0,v_3^\infty),\ s=\gcd(|w_1v_3^\infty-w_2v_3^0|,l_2),\ l_2=sm_3,\ n=\frac{l_1}{s}(w_1v_3^\infty-w_2v_3^0). \tag{27}$$

Then from the proof of Theorem 3.8 we see that the quotient is the total space of the projective orbibundle $\mathbb{P}(\mathbb{1}\oplus L^n)$ over $(\mathcal{H}_a,\Delta_{\mathbf{m}_2})$ whose invariant divisors are generally branch divisors of an orbifold. But then using Lemma 1.1 this is precisely a stage 3 Bott orbifold $(M_3(a,b,c),\Delta_{\mathbf{m}})$ for some b,c and where a is given in Equations (22). The fact that $Y^{p,q}$ as a join has the form of Equation (20) with the standard regular Sasakian structure on the first S^3 implies that the ramification indices $\mathbf{m}_1=(1,1)$.

It remains to check the equations for b and c. For this we make use of an orbifold version of Proposition 1.5 in [BCTF18]. Explicitly, we have

Lemma 2.4. *The pullback of $c_1(L^n)$ of the orbifold line bundle L^n to $M_3(a,b,c)$ is bx_1+cx_2 where n is given in Equations* (27)*.*

We know that L^n is the nth tensor product of the line orbibundle L which is determined by the Kähler class

$$\frac{c_1^{orb}(\mathcal{H}_a,\Delta_{\mathbf{m}_2})}{\mathcal{I}_{\mathbf{v}_2}}=\frac{\Bigl((2m_2v_2^0+a)v_2^\infty x_1+(v_2^0+v_2^\infty)x_2\Bigr)}{m_2v_2^0v_2^\infty\mathcal{I}_{\mathbf{v}_2}}. \tag{28}$$

Now the projection $p:(M_3(a,b,c),\Delta)\longrightarrow(\mathcal{H}_a,\Delta_{\mathbf{m}_2})$ is a $m_2v_2^0v_2^\infty$-fold covering map. So pulling back we have $c_1(L^n)=nc_1(L)=np^*\Bigl(\frac{c_1^{orb}(\mathcal{H}_a,\Delta_{\mathbf{m}_2})}{\mathcal{I}_{\mathbf{v}_2}}\Bigr)$. The equations for b and c then follow by equating coefficients in this and in Lemma 2.4. □

Remark 2.5. The Poincaré dual to $c_1^{orb}(M_3(a,b,c),\Delta_{\mathbf{m}})$ is a $\mathbb{Q}$-divisor on $M_3(a,b,c)$ which is ample when $c_1^{orb}(M_3(a,b,c),\Delta_{\mathbf{m}})$ is positive. Such a class gives a polarization to the orbifold $(M_3(a,b,c),\Delta_{\mathbf{m}})$, and a $\mathbb{T}^3$ invariant orbifold 2-form representing $c_1^{orb}(M_3(a,b,c),\Delta_{\mathbf{m}})$ gives an orbifold Kähler metric $g_{a,b,c,\mathbf{m}}$ on $M_3(a,b,c)$.

Remark 2.6. Note that the real cohomology class $c_1^{orb}(M_3(a,b,c),\Delta_{\mathbf{m}})$ makes perfect sense for $m_j^0,m_j^\infty\in\mathbb{R}^+$, and we denote the set of all such classes by $\mathfrak{BL}_A$. In this case $(M_3(a,b,c),\Delta_{\mathbf{m}})$ can be understood as having Kähler metrics with conical singularities along the corresponding $\mathbb{R}$-divisors D_{v_i} and D_{u_i} with cone angle $\frac{2\pi}{m_i^0}$ and $\frac{2\pi}{m_i^\infty}$, respectively [Don12, CDS15]. By this we mean that there is a Kähler metric ω which is smooth on $M_n(a,b,c)\setminus\Delta_{\mathbf{m}}$ which extends to $M_n(a,b,c)$ as a closed positive $(1,1)$ current satisfying certain uniformity requirements. See Definition 1.3 of [CDS15] for the precise statement. For arbitrary $\mathbf{m}$ we say that $c_1^{orb}(M_3(a,b,c),\Delta_{\mathbf{m}})$ represents a cone singularity along the divisor $\Delta_{\mathbf{m}}$. The rational entries in the interval $(0,1)$ are related to the so-called 'ramifolds' [RT11]. As in this reference we shall also assume hereafter that $m_j^0,m_j^\infty\in(0,\infty)$. We denote by $\mathfrak{LF}_A$ the subset of $\mathfrak{BL}_A$ whose Kähler metrics are log Fano and have cone singularities along the divisor $\Delta_{\mathbf{m}}$. Then the natural map from $\mathfrak{LF}_A$ to the

Kähler cone $\mathcal{K}(M_3(a,b,c))$ is surjective, and the conclusion of Lemma 1.2 holds for all $(M_3(a,b,c),\Delta_{\mathbf{m}})\in\mathfrak{LF}_A$.

Remark 2.7. We consider the action of the affine monoid $\mathfrak{M}(\mathbb{R})$ on $(\mathbb{R}^+)^6$ defined by the affine linear map

$$(m_j^0,m_j^\infty)\mapsto(\lambda_j^0m_j^0+a_j^0,\lambda_j^\infty m_j^\infty+a_j^\infty)=(\tilde{m}_j^0,\tilde{m}_j^\infty) \tag{29}$$

with $1\leq\lambda_j<\infty$ and $0\leq a_j<\infty$. Restricting to the positive integers gives an action of the submonoid $\mathfrak{M}(\mathbb{Z})$ on $(\mathbb{Z}^+)^6$. One easily checks that this induces an action of $\mathfrak{M}(\mathbb{R})$ on $\mathfrak{BL}$ that leaves the subset $\mathfrak{LF}_A$ invariant for all $\lambda_j^0,\lambda_j^\infty\in[1,\infty)$ and $a_i^0,a_j^\infty\in[0,\infty)$ sending $c_1^{orb}(M_3(a,b,c),\Delta_{\mathbf{m}})$ to $c_1^{orb}(M_3(a,b,c),\Delta_{\tilde{\mathbf{m}}})$.

3. The Topology of $M^7=Y^{p,q}\star_{l_1,l_2}S^3_{\mathbf{w}}$

It is important to remember that generally the topology of a join depends on the choice of Sasakian structure (through its Reeb vector field) of each factor. We assume that (p,q) are relatively prime with $1\leq q<p$ and that l_1,l_2,w_1,w_2 are chosen such that M^7 is smooth. We show first that our Sasaki 7-manifolds M^7 have the rational cohomology of the 2-fold connected sum $(S^2\times S^5)\#(S^2\times S^5)$. The integer cohomology groups are only distinguished by torsion in H^4. Moreover, the torsion depends on the choice of quasi-regular Sasakian structures on $Y^{p,q}$ and S^3. For the most generality we choose arbitrary quasi-regular Sasakian structure in the so-called $\mathbf{w}$ subcone of the Sasaki cones for both $Y^{p,q}$ and S^3 (of course, the $\mathbf{w}$ cone of S^3 is its entire Sasaki cone).

First we note that any quotient of a quasi-regular Reeb vector field $\xi_{\mathbf{m}}$ in the $\mathbf{w}$ cone of $Y^{p,q}$ has the form of a Hirzebruch orbifold $(\mathcal{H}_a,\Delta_{\mathbf{m}})$. Moreover, $Y^{p,q}$ is itself the join $S^3\star_{l_1,p}S^3_{\mathbf{w}}$ with $\mathbf{w}=(\frac{p+q}{l_1},\frac{p-q}{l_1})$. Here $l_1=2$ if p,q are both odd, and $l_1=1$ is p,q have opposite parities. In any case the relation with the ramification indices is $\mathbf{m}=m(v^0,v^\infty)$ with v^0,v^∞ coprime and $m=p$.

The purpose of this section is to prove

Theorem 3.1. *Let $Y^{p,q}$ have a quasi-regular Sasakian structure with Reeb vector field $\xi_{\mathbf{m}}$. Then the 7-manifolds $M^7=Y^{p,q}\star_{l_1,l_2}S^3_{\mathbf{w}}$ have the rational cohomology of the connected sum $(S^2\times S^5)\#(S^2\times S^5)$. Furthermore, the only torsion that occurs is $H^4(M^7,\mathbb{Z})\approx\mathbb{Z}_{v^0v^\infty m^2l_2^2}\oplus\mathbb{Z}_{w_1w_2l_1^2}$.*

We begin with some lemmas.

Lemma 3.2. *The 7-manifolds $M^7=Y^{p,q}\star_{l_1,l_2}S^3_{\mathbf{w}}$ satisfy the following conditions:*

1. $H_1(M^7,\mathbb{Z})=\pi_1(M^7)=0$,
2. $\pi_2(M^7)=\mathbb{Z}^2$,
3. $H^2(M^7,\mathbb{Z})=H_2(M^7,\mathbb{Z})=\mathbb{Z}^2$,
4. $H^3(M^7,\mathbb{Z})$ *is torsion free.*
5. $b_3(M^7)=b_4(M^7)$ *is even.*

Proof. From the long exact homotopy sequence for the fibration

$$\mathbb{T}^2\longrightarrow S^3\times S^3\times S^3\longrightarrow M^7 \tag{30}$$

we conclude that M^7 is simply connected and that $\pi_2(M^7)=\mathbb{Z}^2$. Thus, by Hurewicz $H_2(M^7,\mathbb{Z})=\mathbb{Z}^2$ which implies (4) by universal coefficients, and then by item (1)

$H^2(M^7,\mathbb{Z})=\mathbb{Z}^2$. Item (5) follows from Poincaré duality and the fact that M^7 admits a Sasakian structure.

□

Actually we have

Lemma 3.3. *$H^3(M^7,\mathbb{Z})=0$.*

Proof. First by (4) of Lemma 3.2 $H^3(M^7,\mathbb{Z})$ is torsion free, so it suffices to work with $\mathbb{Q}$ coefficients. Since M^7 is simply connected and is an S^1 orbibundle over a stage 3 Bott orbifold $(M_3(a,b,c),\Delta_{\mathbf{m}})$ we can apply the Leray-Serre Theorem with $\mathbb{Q}$ coefficients. The differential $d_2:E_2^{0,1}\longrightarrow E_2^{2,0}=H^2(M^7,\mathbb{Q})$ sends the class α of the fiber S^1 to the Kähler class $c_1x_1+c_2x_2+c_3x_3$ where $c_i\in\mathbb{Q}^+$. So by naturality we have $d_2(\alpha\otimes x_i)=(c_1x_1+c_2x_2+c_3x_3)x_i$. Suppose there would exist a class $w=w_1x_1+w_2x_2+w_3x_3\in E_2^{2,0}$ such that $d_2(\alpha\otimes w)=0$. Then the 3-class $\alpha\otimes w$ would survive to the limit giving a nonzero element in $H^3(M^7,\mathbb{Q})$ by the Leray-Serre Theorem. Now the cohomology ring of $M_3(a,b,c)$ is [CMS10]

$$\mathbb{Z}[x_1,x_2,x_3]/\big(x_1^2,x_2(ax_1+x_2),x_3(bx_1+cx_2+x_3)\big).$$

So computing the d_2 differential we have

$$\begin{aligned}0&=d_2(\alpha\otimes w)=d_2(\alpha)\otimes w=(c_1x_1+c_2x_2+c_3x_3)(w_1x_1+w_2x_2+w_3x_3)\\&=(c_1w_2+c_2w_1-c_2w_2a)x_1x_2+(c_1w_3+c_3w_1-c_3w_3b)x_1x_3+(c_2w_3+c_3w_2-c_3w_3c)x_2x_3\end{aligned}$$

which gives

$$\begin{pmatrix}c_2&c_1-c_2a&0\\c_3&0&c_1-c_3b\\0&c_3&c_2-c_3c\end{pmatrix}\begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix}=0. \tag{31}$$

Since the coefficients c_1,c_2,c_3 are all positive, the rank of the matrix

$$C=\begin{pmatrix}c_2&c_1-c_2a&0\\c_3&0&c_1-c_3b\\0&c_3&c_2-c_3c\end{pmatrix}$$

is either 3 or 2 which can be seen by putting C in Jordan canonical form. If the rank of C were 2 there would be exactly one solution to Equation (31) which is an element of $E_2^{1,2}$ and since $E_2^{3,0}=H^3(M_3(a,b,c),\mathbb{Q})=0$, there would be precisely one generator in $H^3(M^7,\mathbb{Q})$ which contradicts the fact that $b_3(M^7)$ is even. □

Proof of Theorem 3.1: It follows from Lemmas 3.2 and 3.3 and the fact that the homology (cohomology) of a connected sum is the direct sum of the homology (cohomology) of the two factors that M^7 has the rational homology (cohomology) of the connected sum $(S^2\times S^5)\#(S^2\times S^5)$. By Poincaré duality and Lemma 3.2 it follows that $H^5(M^7,\mathbb{Z})$ and $H^6(M^7,\mathbb{Z})$ have no torsion. So it suffices to compute the torsion in $H^4(M^7,\mathbb{Z})$. First

Lemma 3.4. *We have*

$$H^r_{orb}((\mathcal{H}_a,\Delta_{\mathbf{m}}),\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } r=0 \\ \mathbb{Z}^2 & \text{if } r=2 \\ \mathbb{Z}\oplus\mathbb{Z}_{m^0}\oplus\mathbb{Z}_{m^\infty} & \text{if } r=4 \\ \mathbb{Z}_{m^0}\oplus\mathbb{Z}_{m^\infty} & \text{if } r=6,8,\cdots \\ 0 & \text{if } r \text{ is odd.} \end{cases}$$

Proof. The Leray sheaf of the map

$$p:\mathsf{B}(\mathcal{H}_a,\Delta_{\mathbf{m}})\longrightarrow\mathcal{H}_a$$

is the derived functor sheaf $R^sp\mathbb{Z}$, that is, the sheaf associated to the presheaf $U\mapsto H^s(p^{-1}(U),\mathbb{Z})$. For $s>0$ the stalks of $R^sp\mathbb{Z}$ at points of U vanish if U lies in the regular locas of $(\mathcal{H}_a,\Delta_{\mathbf{m}})$ which is the complement of the union of the zero e_0 and infinity e_∞ sections of the natural projection $\mathcal{H}_a\longrightarrow\mathbb{CP}^1$. However, at points of e_0 and e_∞ the fibers of p are (up to homotopy) the Eilenberg-MacLane spaces $K(\mathbb{Z}_{m^0},1)$ and $K(\mathbb{Z}_{m^\infty},1)$, respectively. So at points of $e_0(e_\infty)$ the stalks are the group cohomology $H^s(\mathbb{Z}_{m^0},\mathbb{Z})\big(H^s(\mathbb{Z}_{m^\infty},\mathbb{Z})\big)$. This is $\mathbb{Z}$ for $s=0$ and $\mathbb{Z}_{m^0}(\mathbb{Z}_{m^\infty})$ at points of $e_0(e_\infty)$ when $s>0$ is even; it vanishes when s is odd. The E_2 term of the Leray spectral sequence of the map p is

$$E_2^{r,s}=H^r(\mathcal{H}_a,R^sp\mathbb{Z})$$

and by Leray's theorem this converges to the orbifold cohomology $H^{r+s}_{orb}((\mathcal{H}_a,\Delta_{\mathbf{m}}),\mathbb{Z})$. Now $E_2^{r,0}=H^r(\mathcal{H}_a,\mathbb{Z})$ and $E_2^{r,s}=0$ for r or s odd. For $r=0$ since $R^sp\mathbb{Z}$ has its support in the orbifold singular locus $e_0\cup e_\infty$, the only continuous section of $R^sp\mathbb{Z}$ is the 0 section which implies that $E_2^{0,s}=0$ for all s. Now we have $E_2^{2r,2s}=0$ for $r>1$ and

$$E_2^{2,2s}=H^2(\mathcal{H}_a,R^{2s}p)=H^2(e_0,\mathbb{Z}_{m^0})\oplus H^2(e_\infty,\mathbb{Z}_{m^\infty})=\mathbb{Z}_{m^0}\oplus\mathbb{Z}_{m^\infty}.$$

One easily sees this spectral sequence collapses whose limit is the orbifold cohomology $H^r_{orb}((\mathcal{H}_a,\Delta_{\mathbf{m}}),\mathbb{Z})$ which implies the result. □

To continue the proof of Theorem 3.1, as in [BTF16, BTF15], we use the commutative diagram of fibrations

$$\begin{array}{ccccc} Y^{p,q}\times S^3_{\mathbf{w}} & \longrightarrow & M_{l_1,l_2,\mathbf{w}} & \longrightarrow & \mathsf{B}S^1 \\ \Big\downarrow{=} & & \Big\downarrow & & \Big\downarrow{\psi} \\ Y^{p,q}\times S^3_{\mathbf{w}} & \longrightarrow & \mathsf{B}(\mathcal{H}_a,\Delta_{\mathbf{m}})\times\mathsf{B}\mathbb{CP}^1[\mathbf{w}] & \longrightarrow & \mathsf{B}S^1\times\mathsf{B}S^1. \end{array} \tag{32}$$

Here $\mathsf{B}G$ is the classifying space of a group G or Haefliger's classifying space [Hae84] of an orbifold if G is an orbifold. The lower exact fibration is a product of fibrations. We denote the orientation classes of $Y^{p,q}\times S^3=S^2\times S^3\times S^3$ by α,β,γ, respectively. As in 3.2.2 of [BTF15] we have $d_4(\gamma)=w_1w_2s_2^2$. For the fibration

$$Y^{p,q}\longrightarrow\mathsf{B}(\mathcal{H},\Delta_{\mathbf{m}})\longrightarrow\mathsf{B}S^1$$

we have $d_4(\beta)=m^2v^0v^\infty s_1^2$. The map ψ in Diagram (32) is induced by the map $e^{i\theta}\mapsto(e^{il_2\theta},e^{-il_1\theta})$, so $\psi^*s_1=l_2s$ and $\psi^*s_2=-l_1s$ the result follows by the commutativity of Diagram (32). □

Remark 3.5. Although the case $(p,q)=(1,0)$ does not fit directly into the $Y^{p,q}$ scheme of [GMSW04], it can, nevertheless, be identified with the homogeneous Sasaki-Einstein structure on $S^2\times S^3$. Then if we take $(v^0,v^\infty)=(w_1,w_2)=(l_1,l_2)=(1,1)$ and $m=1$ we obtain the homogeneous Sasaki-Einstein structure on an S^1 bundle over $\mathbb{CP}^1\times\mathbb{CP}^1\times\mathbb{CP}^1$ which is a 7-manifold with the integral cohomology of the 2-fold connected sum $2(S^2\times S^5)$. See Sections 11.1.1 and 11.4.2 of [BG08] for details. The general $S^3_{\mathbf{w}}$ join with $Y^{1,0}$ was treated in Section 3.2.2 of [BTF15].

4. A Generalized Orbifold Calabi construction

We now discuss a family of explicit examples or orbifold Kähler-Einstein metrics that we may view as arising from a special case of the generalized Calabi construction as presented in [ACGTF04, Section 2.5] and further discussed in [ACGTF11, Section 2.3] and [BCTF18, Section 5.1] - here generalized to allowing certain mild orbifold singularities.

The base of the construction will in this case be a Kähler-Einstein Hirzebruch orbifold $(\mathcal{H}_a,\Delta_{\mathbf{m}_2})$. This is the Kähler quotient of the quasi-regular Sasaki-Einstein $Y^{p,q}$ examples produced in [GMSW04]). Now $Y^{p,q}$ may be viewed as the total space of a S^1 principal orbi-bundle, P, over $(\mathcal{H}_a,\Delta_{\mathbf{m}_2})$ defined by the class $\frac{c_1^{orb}(\mathcal{H}_a,\Delta_{\mathbf{m}_2})}{\mathcal{I}_{\mathbf{v}_2}}=\frac{(2m_2v_2^0+a)v_2^\infty x_1+(v_2^0+v_2^\infty)x_2}{m_2v_2^0v_2^\infty\mathcal{I}_{\mathbf{v}_2}}$, where the notation is as in Lemma 2.2. In particular, the index of $(\mathcal{H}_a,\Delta_{\mathbf{m}_2})$ is $\mathcal{I}_{\mathbf{v}_2}=\gcd(2m_2v_2^0+a)v_2^\infty,v_2^0+v_2^\infty)$, where $(m_2^0,m_2^\infty)=m_2(v_2^0,v_2^\infty)$ and v_2^0,v_2^∞ are coprime. Let g_{base} denote the Kähler-Einstein orbifold metric whose Kähler form, ω_{base}, satisfies that

$$\left[\frac{\omega_{base}}{2\pi}\right]=\frac{(2m_2v_2^0+a)v_2^\infty x_1+(v_2^0+v_2^\infty)x_2}{m_2v_2^0v_2^\infty\mathcal{I}_{\mathbf{v}_2}}.$$

As we saw in Example 6.8 of [BTF16], the metric g_{base} is explicit and "admissible" in the sense of [ACGTF08]. Note that the Ricci form of g_{base} is given by $\rho_{base}=\mathcal{I}_{\mathbf{v}_2}\omega_{base}$.

We consider the generalized Calabi construction of orbifold Kähler metrics on the bunde $P\times_{S^1}\mathbb{CP}^1_{m_3^0,m_3^\infty}\to(\mathcal{H}_a,\Delta_{\mathbf{m}_2})$. This may also be viewed as an admissible construction - extended to mild orbifold cases.

Definition 4.1. Generalized orbifold Calabi data for our purposes.

(1) *A log pair* $(\mathcal{H}_a,\Delta_{\mathbf{m}_2})$ *with Kähler-Einstein structure* (ω_{base},g_{base}) *such that* $\left[\frac{\omega_{base}}{2\pi}\right]=\frac{(2m_2v_2^0+a)v_2^\infty x_1+(v_2^0+v_2^\infty)x_2}{m_2v_2^0v_2^\infty\mathcal{I}_{\mathbf{v}_2}}$.

(2) *The weighted projective line* $(\mathbb{CP}^1_{m_3^0,m_3^\infty}=\mathbb{CP}^1_{v_3^0,v_3^\infty}/\mathbb{Z}_{m_3},g_{\mathbf{m}_3},\omega_{\mathbf{m}_3})$ *with rational Delzant polytope* $[-1,1]\subseteq\mathbb{R}^*$ *and momentum map* $\mathfrak{z}\colon\mathbb{CP}^1_{m_3^0,m_3^\infty}\to[-1,1]$. *Here* $(m_3^0,m_3^\infty)=m_3(v_3^0,v_3^\infty)$ *and* v_3^0,v_3^∞ *are coprime.*

(3) *A principal* S^1 *orbi-bundle,* $P_n\to(\mathcal{H}_a,\Delta_{\mathbf{m}_2})$, *with a principal connection of curvature* $n\,\omega_{base}\in\Omega^{1,1}((\mathcal{H}_a,\Delta_{\mathbf{m}_2}),\mathbb{R})$, *where* S^1 *acts on* $\mathbb{CP}^1_{m_3^0,m_3^\infty}$, $n\in\mathbb{Z}\setminus\{0\}$, *and* $\gcd(n,m_3)=1$. *Note that* $n\in span_{\mathbb{Z}}\{v_3^0,v_3^\infty\}$ *(since* v_3^0,v_3^∞ *are coprime), so* $m_3n\in span_{\mathbb{Z}}\{m_3^0,m_3^\infty\}$.

(4) *A constant* $0<|r_3|<1$ *of same sign as* n
[ensuring that the $(1,1)$-form $(1/r_3+\mathfrak{z})n\,\omega_{base}$ is positive for $\mathfrak{z}\in[-1,1]$].

From this data we may define the orbifold

$$M_3 = P_n \times_{S^1} \mathbb{CP}^1_{m_3^0, m_3^\infty} = \mathring{M}_3 \times_{\mathbb{C}^*} \mathbb{CP}^1_{m_3^0, m_3^\infty} \to (\mathcal{H}_a, \Delta_{\mathbf{m}_2}),$$

where $\mathring{M}_3 = P_n \times_{S^1} (\mathfrak{z}^{-1}(-1,1))$. Since the curvature 2-form of P_n has type $(1,1)$, $\mathring{M}_3$ is a holomorphic principal $\mathbb{C}^*$ bundle with connection $\theta \in \Omega^1(\mathring{M}_3, \mathbb{R})$ and M_3 is a complex orbifold.

On $\mathring{M}_3$ we define Kähler structures of the form

$$\begin{aligned} g &= (1/r_3 + \mathfrak{z}) n\, g_{base} + \frac{1}{\Theta(\mathfrak{z})} d\mathfrak{z}^2 + \Theta(\mathfrak{z})\theta^2 \\ \omega &= (1/r_3 + \mathfrak{z}) n\, \omega_{base} + d\mathfrak{z} \wedge \theta \\ d\theta &= n\, \omega_{base}, \end{aligned} \tag{33}$$

where $\frac{1}{\Theta(\mathfrak{z})} = \frac{d^2U}{d\mathfrak{z}^2}$ and U is the symplectic potential [Gui94] of the chosen toric Kähler structure $g_{\mathbf{m}_3}$ on $\mathbb{CP}^1_{m_3^0, m_3^\infty}$.

The *generalized Calabi construction* arises from seeing (33) as a blueprint for the construction of various orbifold Kähler metrics on M_3 by choosing various smooth functions $\Theta(\mathfrak{z})$ on $(-1,1)$ satisfying that

- [boundary values] the following endpoint conditions are satisfied

$$\Theta(\pm 1) = 0 \qquad \text{and} \qquad \Theta'(-1) = 2/m_3^\infty \qquad \text{and} \qquad \Theta'(1) = -2/m_3^0; \tag{34}$$

- [positivity] the function $\Theta(\mathfrak{z})$ is positive for $\mathfrak{z} \in (-1,1)$.

Then (33) extends to an orbifold Kähler metric on $(M(a,b,c), \Delta_{\mathbf{m}})$, where $b = n\frac{(2m_2 v_2^0 + a)v_2^\infty}{\mathcal{I}_{\mathbf{v}_2}}$, $c = n\frac{v_2^0 + v_2^\infty}{\mathcal{I}_{\mathbf{v}_2}}$, and $\mathbf{m} = (1, 1, m_2^0, m_2^\infty, m_3^0, m_3^\infty)$. Metrics constructed this way are called *compatible Kähler metrics* with *compatible Kähler classes* parametrized by r_3. From [ACG06] (and specifically to our notation from [BTF16, Proposition 5.4]) we have that the compatible metric defined by $\Theta(\mathfrak{z})$ is Kähler-Einstein exactly when

$$2r_3 \mathcal{I}_{\mathbf{v}_2}/n = (1 + r_3)/m_3^\infty + (1 - r_3)/m_3^0 \tag{35}$$

and

$$\int_{-1}^{1} ((1-\mathfrak{z})/m_3^\infty - (1+\mathfrak{z})/m_3^0)(1 + r_3\mathfrak{z})^2 \, d\mathfrak{z} = 0. \tag{36}$$

5. Explicit Sasaki-Einstein metrics

We now look more closely for explicit Sasaki-Einstein examples arising from the join from Theorem 2.3. The arguments in Sections 6.1 and 6.2 of [BTF16] (see specifically page 1053) carry through so that we have an adapted version of Theorem 1.4 of [BTF16]:

Theorem 5.1. *Consider the join $M^7 = Y^{p,q} \star_{l_1,l_2} S^3_{\mathbf{w}}$ where $Y^{p,q}$ has a quasi-regular Sasaki-Einstein structure with quotient Hirzebruch orbifold $(\mathcal{H}_a, \Delta_{\mathbf{v}_2})$ with $a > 0$ and l_1, l_2 are given by*

$$l_1 = \frac{\mathcal{I}_{\mathbf{v}_2}}{\gcd(|\mathbf{w}|, \mathcal{I}_{\mathbf{v}_2})}, \qquad l_2 = \frac{|\mathbf{w}|}{\gcd(|\mathbf{w}|, \mathcal{I}_{\mathbf{v}_2})}.$$

Then for each vector $\mathbf{w} = (w_1, w_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ *with relatively prime components satisfying* $w_1 > w_2$ *there exist a Reeb vector field* $\xi_{\mathbf{v}_3}$ *in the 2-dimensional* $\mathbf{w}$*-Sasaki cone on* M^7 *such that the corresponding Sasakian structure is Sasaki-Einstein.*

Specifically, using equation (59) from [BTF16], we know that if the ray defined by co-prime (v_3^0, v_3^∞) is quasi-regular, then we ought to look at the Kähler class determined by $r_3 = \frac{w_1 v_3^\infty - w_2 v_3^0}{w_1 v_3^\infty + w_2 v_3^0}$. Now with this r_3, and the above choice of (l_1, l_2), (35) is automatically solved and (36) becomes (similarly to (68) in [BTF16])

$$\int_{-1}^{1} \left((v_3^0 - v_3^\infty) - (v_3^0 + v_3^\infty)\mathfrak{z}\right)\left((w_1 v_3^\infty + w_2 v_3^0) + (w_1 v_3^\infty - w_2 v_3^0)\mathfrak{z}\right)^2 d\mathfrak{z} = 0. \tag{37}$$

This equation defines a priori a quasi-regular Sasaki η-Einstein ray (and thus, up to transverse homothety, a Sasaki-Einstein structure), but, by the same arguments as in Section 6.1 of [BTF16], any solution $(v_3^0, v_3^\infty) \in \mathbb{R}^+ \times \mathbb{R}^+$ of (37) defines a Sasaki η-Einstein ray in the $\mathbf{w}$-cone, which is irregular unless $v_3^\infty / v_3^0 \in \mathbb{Q}$. This gives

Corollary 5.2. *For any pair of relatively prime positive integers* (w_1, w_2) *satisfying* $w_1 > w_2$*, we obtain an SE metric by solving the cubic equation*

$$3w_2 k^3 + (2w_2 - w_1)k^2 - (2w_1 - w_2)k - 3w_1 = 0. \tag{38}$$

The ray of the Reeb vector field of the SE metric is then given by

$$\frac{v_3^\infty}{v_3^0} = \frac{3 + 2k + k^2}{(1 + 2k + 3k^2)} = k\frac{w_2}{w_1}. \tag{39}$$

The SE metric is quasi-regular when k *is a rational root of* (38) *and irregular when it is an irrational root.*

It follows from the analysis in Section 6.2 of [BTF16] that the positive real root k lies in the open interval $(1, \infty)$.

5.1. **Quasi-regular examples.** Changing our point of view we see that for any $k \in (1, +\infty) \cap \mathbb{Q}$ we can choose (w_1, w_2) such that

$$w_2 / w_1 = \frac{3 + 2k + k^2}{k(1 + 2k + 3k^2)} \tag{40}$$

and then the Sasaki-Einstein metric from Theorem 5.1 is quasi-regular with ray $\mathbf{v}_3$ defined by Equation (39). We now give some examples.

Example 5.3. Here we give an example that builds on a bouquet of Sasaki cones from Example[1] 6.8 of [BTF16]. Corollary 5.5 of [BP14] describes the well known $Y^{p,q}$ structures [GMSW04] on $S^2 \times S^3$ as a $|\phi(p)|$-bouquet of Sasaki cones where $|\phi(p)|$ denotes the order of the Euler phi function $\phi(p)$. Let us consider the example when $p = 13$. Since 13 is prime the bouquet consists of $p - 1 = 12$ Sasaki cones labeled by the 12 positive integers $1 \leq q < 13$ and as such contains 12 Sasaki-Einstein metrics. However, in order to construct SE metrics on our 7-manifolds, we need the SE structure on $Y^{p,q}$ to be quasi-regular. It is easy to check that for the bouquet $\bigcup\{Y^{13,q}\}_{q=1}^{12}$ the only values of q where the SE metric is quasi-regular is for $q = 7, 8$, all the other SE metrics in the bouquet are irregular. Let us look at these two cases a bit closer.

[1]Example 6.8 in this reference has a small typo, namely v_2^0 and v_2^∞ got switched.

In the case $Y^{13,8}$ we have $a=70$, $v_2^0=7$, $v_2^\infty=5$, $m_2=13$, so $m_2^0=91$ and $m_2^\infty=65$. This gives us that $\mathcal{I}_{\mathbf{v}_2}=12$. If we put $k=2$ in the quasi-regular SE prescription above given by Equation (38), we see that $(w_1,w_2)=(34,11)$, so $l_1=4$ and $l_2=15$, and $(v_3^0,v_3^\infty)=(17,11)$. Now we calculate that $s=\gcd(|w_1v_3^\infty-w_2v_3^0|,l_2)=1$ so $n=l_1\frac{w_1v_3^\infty-w_2v_3^0}{s}=748$, and $(m_3^0,m_3^\infty)=15(17,11)$. Then the quotient is the Bott orbifold given by the log pair $(M(a,b,c),\Delta_{\mathbf{m}})$, where

$$\begin{aligned} a &= 70,\\ b &= n\hat{b}=n\frac{(2m_2v_2^0+a)v_2^\infty}{\mathcal{I}_{\mathbf{v}_2}}=78540,\\ c &= n\hat{c}=n\frac{v_2^0+v_2^\infty}{\mathcal{I}_{\mathbf{v}_2}}=748,\\ \mathbf{m} &= (1,1,91,65,255,165). \end{aligned}$$

In the case where $p=7$ in $Y^{p,q}$ we have $a=36$, $v_2^0=4$, $v_2^\infty=3$, $m_2=13$, so $m_2^0=52$ and $m_2^\infty=39$. This gives us that $\mathcal{I}_{\mathbf{v}_2}=7$. Again we put $k=2$ in Equation (38) which gives $(w_1,w_2)=(34,11)$ and $(v_3^0,v_3^\infty)=(17,11)$ respectively. Now $l_1=7$ and $l_2=45$, and $s=\gcd(|w_1v_3^\infty-w_2v_3^0|,l_2)=1$ so $n=l_1\frac{w_1v_3^\infty-w_2v_3^0}{s}=1309$, and $(m_3^0,m_3^\infty)=45(17,11)$. Then the quotient is the Bott orbifold given by the log pair $(M(a,b,c),\Delta_{\mathbf{m}})$, where

$$\begin{aligned} a &= 36,\\ b &= n\hat{b}=n\frac{(2m_2v_2^0+a)v_2^\infty}{\mathcal{I}_{\mathbf{v}_2}}=78540,\\ c &= n\hat{c}=n\frac{v_2^0+v_2^\infty}{\mathcal{I}_{\mathbf{v}_2}}=1309,\\ \mathbf{m} &= (1,1,52,39,765,495). \end{aligned}$$

One can easily check from the torsion in Theorem 3.1 that the two SE 7-manifolds $Y^{13,8}\star_{4,15}S^3_{(34,11)}$ and $Y^{13,7}\star_{7,45}S^3_{(34,11)}$ are not homotopy equivalent. For both of these 7-manifolds the Reeb field that gives the SE metric is quasi-regular. Moreover, they are both induced from the same Reeb ray, namely $\{\xi_{a(17,11)}\}_{a>0}$ of the same $S^3_{(34,11)}$, and b is the same for the quotient orbifolds.

For each choice of the rational number $k>1$ we obtain a pair of quasi-regular SE 7-manifolds induced by the $S^3_{\mathbf{w}}$ join and its Reeb field $\xi_{\mathbf{v}_3}$ where $\mathbf{w}$ and $\mathbf{v}_3$ are determined by Equations (38) and (39).

Example 5.4. Here we give a 1-parameter family of smooth quasi-regular examples. First, let $k_2\in\mathbb{Z}^{\geq 0}$ be given (using the subscript "2" to indicate that this is a choice at the second stage). Then from [GMSW04]) we get a quasi-regular Sasaki-Einstein $Y^{p,q}$ example by choosing

$$p=12k_2^2+18k_2+7 \qquad \text{and} \qquad q=12k_2^2+16k_2+5.$$

It is not hard to check that $\gcd(p,q)=1$ and $\gcd(p+q,p-q)=2$. Accordingly we recognize from (20) that

$$Y^{p,q}=S^3\star_{l,p}S^3_{\frac{p+q}{l},\frac{p-q}{l}}=S^3\star_{2,(12k_2^2+18k_2+7)}S^3_{(2+3k_2)(3+4k_2),(1+k_2)}$$

and that the quotient Hirzebruch orbifold of the quasi-regular Sasaki-Einstein metric is $(\mathcal{H}_a, \Delta_{\mathbf{m}_2})$ where, using (21) and (22),

$$\begin{aligned} \mathbf{m}_2 &= (m_2^0, m_2^\infty) = (12k_2^2 + 18k_2 + 7)((3+4k_2), 2(1+k_2)), \\ m_2 &= 12k_2^2 + 18k_2 + 7, \\ a &= 6(1+k_2)(1+2k_2)(3+4k_2). \end{aligned}$$

We can calculate from Lemma 2.2 that $\mathcal{I}_{\mathbf{v}_2} = 5 + 6k_2$.

Now choosing $k = 3$ in (38) and (39), we have, from Theorem 5.1, a quasi-regular Sasaki-Einstein structure on $M^7 = Y^{p,q} \star_{l_1,l_2} S^3_{\mathbf{w}}$ with quotient log pair $(M_3(a,b,c), \Delta_{\mathbf{m}})$ given by Theorem 2.3. Indeed, we have

$$\begin{aligned} a &= 6(1+k_2)(1+2k_2)(3+4k_2) \\ b &= 4(1+k_2)(2+3k_2)(3+4k_2)n \\ c &= n \\ \mathbf{m} &= (m_1^0, m_1^\infty, m_2^0, m_2^\infty, m_3^0, m_3^\infty) \\ (m_1^0, m_1^\infty) &= (1,1) \\ (m_2^0, m_2^\infty) &= (12k_2^2 + 18k_2 + 7)((3+4k_2), 2(1+k_2)) \\ (m_3^0, m_3^\infty) &= m_3(v_3^0, v_3^\infty), \end{aligned}$$

where

$$\begin{aligned} n &= l_1 \tfrac{102}{\gcd(102, l_2)} \\ m_3 &= \tfrac{l_2}{\gcd(102, l_2)} \\ l_1 &= \tfrac{\mathcal{I}_{\mathbf{v}_2}}{\gcd(20, 5+6k_2)} \\ l_2 &= \tfrac{20}{\gcd(20, 5+6k_2)} \\ (w_1, w_2) &= (17, 3) \\ (v_3^0, v_3^\infty) &= (17, 9). \end{aligned}$$

Using Lemma 2.1, we know that the corresponding Sasaki structure is a smooth manifold if and only if

$$\begin{aligned} \gcd(l_2(12k_2^2 + 18k_2 + 7)(3+4k_2), 17l_1) &= 1 \\ \gcd(l_2((12k_2^2 + 18k_2 + 7)(3+4k_2), 3l_1) &= 1 \\ \gcd(l_2(12k_2^2 + 18k_2 + 7))2((1+k_2)), 17l_1) &= 1 \\ \gcd(l_2(12k_2^2 + 18k_2 + 7)2(1+k_2), 3l_1) &= 1 \end{aligned}$$

In order to get smooth examples, let us now assume $k_2 = 255\,t + 10$ with $t \in \mathbb{Z}^+$. Then we have

$$\begin{aligned} \mathcal{I}_{\mathbf{v}_2} &= 5(306\,t+13) \\ l_1 &= 306\,t+13 \\ l_2 &= 4 \\ n &= 51(306\,t+13) \\ m_3 &= 2 \end{aligned}$$

and then the corresponding Sasaki structure is a smooth manifold if and only if

$$\begin{aligned} \gcd(4(1387+65790t+780300t^2)(1020\,t+43), 17(306\,t+13)) &= 1 \\ \gcd(4(1387+65790t+780300t^2)(1020\,t+43), 3(306\,t+13)) &= 1 \\ \gcd(8(1387+65790t+780300t^2))(255\,t+11), 17(306\,t+13)) &= 1 \\ \gcd(8(1387+65790t+780300t^2))(255\,t+11), 3(306\,t+13)) &= 1. \end{aligned}$$

if and only if

$$\begin{aligned} \gcd((1387+65790t+780300t^2)(1020\,t+43), (306\,t+13)) &= 1 \\ \gcd((1387+65790t+780300t^2))(255\,t+11), (306\,t+13)) &= 1. \end{aligned}$$

Since $\forall t \in \mathbb{Z}^+$,

$$\begin{aligned} 6(255\,t+11) - 5(306\,t+13) &= 1 \\ 10(306\,t+13) - 3(1020\,t+43) &= 1 \\ 3\,(780300t^2+65790t+1387) - (7650t+320)(306t+13) &= 1, \end{aligned}$$

we have that this is always satisfied. Note that with $k_2 = 255\,t + 10$ we get

$$\begin{aligned} a &= 6(255\,t+11)(510\,t+21)(1020\,t+43) \\ b &= 204(255\,t+11)(765\,t+32)(1020\,t+43)(306\,t+13) \\ c &= 51(306\,t+13) \\ \mathbf{m} &= (m_1^0, m_1^\infty, m_2^0, m_2^\infty, m_3^0, m_3^\infty) \\ (m_1^0, m_1^\infty) &= (1,1) \\ (m_2^0, m_2^\infty) &= (1387+65790t+780300t^2)(1020\,t+43, 2(255\,t+11)) \\ (m_3^0, m_3^\infty) &= 2(17,9). \end{aligned}$$

Finally, the p and q in $Y^{p,q}$ are here given by

$$p = 780300t^2 + 65790t + 1387$$

and

$$q = 15(170t + 7)(306t + 13),$$

so the smooth Sasaki Einstein structures live on

$$Y^{780300t^2+65790t+1387,15(170t+7)(306t+13)} \star_{306\,t+13,4} S^3_{17,3}.$$

References

[ACG06] Vestislav Apostolov, David M. J. Calderbank, and Paul Gauduchon, *Hamiltonian 2-forms in Kähler geometry. I. General theory*, J. Differential Geom. **73** (2006), no. 3, 359–412. MR 2228318 (2007b:53149)

[ACGTF04] V. Apostolov, D. M. J. Calderbank, P. Gauduchon, and C. W. Tønnesen-Friedman, *Hamiltonian 2-forms in Kähler geometry. II. Global classification*, J. Differential Geom. **68** (2004), no. 2, 277–345. MR 2144249

[ACGTF08] Vestislav Apostolov, David M. J. Calderbank, Paul Gauduchon, and Christina W. Tønnesen-Friedman, *Hamiltonian 2-forms in Kähler geometry. III. Extremal metrics and stability*, Invent. Math. **173** (2008), no. 3, 547–601. MR MR2425136 (2009m:32043)

[ACGTF11] ______, *Extremal Kähler metrics on projective bundles over a curve*, Adv. Math. **227** (2011), 2385–2424.

[BCTF18] Charles P. Boyer, David M.J. Calderbank, and Christina W. Tønnesen-Friedman, *The Kähler geometry of Bott manifolds*, Preprint: arXiv:1801.09641 (2018).

[BG08] Charles P. Boyer and Krzysztof Galicki, *Sasakian geometry*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2008. MR MR2382957 (2009c:53058)

[BHLTF18] Charles P. Boyer, Hongnian Huang, Eveline Legendre, and Christina W. Tønnesen-Friedman, *Reducibility in Sasakian geometry*, Trans. Amer. Math. Soc. **370** (2018), no. 10, 6825–6869. MR 3841834

[BP14] Charles P. Boyer and Justin Pati, *On the equivalence problem for toric contact structures on S^3-bundles over S^2*, Pacific J. Math. **267** (2014), no. 2, 277–324. MR 3207586

[BTF15] Charles P. Boyer and Christina W. Tønnesen-Friedman, *On the topology of some Sasaki-Einstein manifolds*, NYJM **21** (2015), 57–72.

[BTF16] ______, *The Sasaki join, Hamiltonian 2-forms, and constant scalar curvature*, J. Geom. Anal. **26** (2016), no. 2, 1023–1060. MR 3472828

[CDS15] Xiuxiong Chen, Simon Donaldson, and Song Sun, *Kähler-Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities*, J. Amer. Math. Soc. **28** (2015), no. 1, 183–197. MR 3264766

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011. MR 2810322 (2012g:14094)

[CMS10] Suyoung Choi, Mikiya Masuda, and Dong Youp Suh, *Topological classification of generalized Bott towers*, Trans. Amer. Math. Soc. **362** (2010), no. 2, 1097–1112. MR 2551516 (2011a:57050)

[Don12] S. K. Donaldson, *Kähler metrics with cone singularities along a divisor*, Essays in mathematics and its applications, Springer, Heidelberg, 2012, pp. 49–79. MR 2975584

[FOW09] Akito Futaki, Hajime Ono, and Guofang Wang, *Transverse Kähler geometry of Sasaki manifolds and toric Sasaki-Einstein manifolds*, J. Differential Geom. **83** (2009), no. 3, 585–635. MR MR2581358

[GK94] Michael Grossberg and Yael Karshon, *Bott towers, complete integrability, and the extended character of representations*, Duke Math. J. **76** (1994), no. 1, 23–58. MR 1301185 (96i:22030)

[GLPS17] Jakob C. Geipel, Olaf Lechtenfeld, Alexander D. Popov, and Richard J. Szabo, *Sasakian quiver gauge theories and instantons on cones over round and squashed seven-spheres*, arXiv:1706.07383 [hep-th] (2017).

[GMSW04] J. P. Gauntlett, D. Martelli, J. Sparks, and D. Waldram, *Sasaki-Einstein metrics on* $S^2 \times S^3$, Adv. Theor. Math. Phys. **8** (2004), no. 4, 711–734. MR 2141499

[Gui94] V. Guillemin, *Kaehler structures on toric varieties*, J. Differential Geom. **40** (1994), no. 2, 285–309. MR 1293656 (95h:32029)

[Hae84] A. Haefliger, *Groupoïdes d'holonomie et classifiants*, Astérisque (1984), no. 116, 70–97, Transversal structure of foliations (Toulouse, 1982). MR 86c:57026a

[Leg16] Eveline Legendre, *Toric Kähler-Einstein metrics and convex compact polytopes*, J. Geom. Anal. **26** (2016), no. 1, 399–427. MR 3441521

[MT18] Ruben Monten and Chiara Toldo, *Black holes with halos*, Classical Quantum Gravity **35** (2018), no. 3, 035001, 28. MR 3755957

[RT11] Julius Ross and Richard Thomas, *Weighted projective embeddings, stability of orbifolds, and constant scalar curvature Kähler metrics*, J. Differential Geom. **88** (2011), no. 1, 109–159. MR 2819757

[Spa11] James Sparks, *Sasaki-Einstein manifolds*, Surveys in differential geometry. Volume XVI. Geometry of special holonomy and related topics, Surv. Differ. Geom., vol. 16, Int. Press, Somerville, MA, 2011, pp. 265–324. MR 2893680 (2012k:53082)

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