# CANONICAL SASAKIAN METRICS

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ABSTRACT. Let $M$ be a closed manifold of Sasaki type. A polarization of $M$ is defined by a Reeb vector field, and for any such polarization, we consider the set of all Sasakian metrics compatible with it. On this space we study the functional given by the square of the $L^2$-norm of the scalar curvature. We prove that its critical points, or canonical representatives of the polarization, are Sasakian metrics that are transversally extremal. We define a Sasaki-Futaki invariant of the polarization, and show that it obstructs the existence of constant scalar curvature representatives. For a fixed CR structure of Sasaki type, we define the Sasaki cone of structures compatible with this underlying CR structure, and prove that the set of polarizations in it that admit a canonical representative is open. We use our results to describe fully the case of the sphere with its standard CR structure, showing that each element of its Sasaki cone can be represented by a canonical metric; we compute their Sasaki-Futaki invariant, and use it to describe the canonical metrics that have constant scalar curvature, and to prove that just the standard polarization can be represented by a Sasaki-Einstein metric.

## 1. INTRODUCTION

With the knowledge that the set of Kähler metrics representing a given Kähler class is an affine space modeled after the smooth functions, Calabi used [11, 9] a natural Riemannian functional on this space with the hope of using it to find canonical representatives of the given class. In effect, his functional, or Calabi energy, is simply the squared $L^2$-norm of the scalar curvature, and the critical point minimizing it would fix the affine parameter alluded to above, yielding the desired representative of the class. Calabi named these critical points extremal Kähler metrics. It was then determined that if the Futaki character [17] of the class vanishes, a plausible extremal Kähler representative must

*Key words and phrases.* Sasakian structure, canonical Sasakian metric, extremal metric, Sasaki-Futaki invariant.

During the preparation of this work, the first two authors were partially supported by NSF grant DMS-0504367.

be a metric of constant scalar curvature, and if under that condition we look at the case where the Kähler class in question is a multiple of the first Chern class, the extremal representative must then be Kähler-Einstein.

One of the most important problems in Kähler geometry today involves the subtle questions regarding the existence of extremal Kähler metrics representing a given class. Over the years, starting with the formulation of the famous Calabi Conjecture and its proof by Yau in 1978, various tools have been used or developed to attack this problem. The continuity method, Tian's $\alpha$-invariant, the Calabi-Lichnerowicz-Matsushima obstruction, the Futaki invariant and its generalizations, the Mabuchi K-energy, and more recently the various notions of stability proposed and studied by Tian, Donaldson and others. Substantial progress has been made, but the general existence problem remains open.

Sasakian geometry sits naturally in between two Kähler geometries. On the one hand, Sasakian manifolds are the bases of metric cones which are Kähler. On the other hand, any Sasakian manifold is contact, and the one dimensional foliation associated to the characteristic Reeb vector field is transversally Kähler. In many interesting situations, the orbits of the Reeb vector field are all closed, in which case the Sasakian structure is called quasi-regular. Compact quasi-regular Sasakian manifolds have the structure of an orbifold circle bundle over a compact Kähler orbifold, which must be algebraic and which has at most cyclic quotient singularities. Since much of the study of compact Kähler manifolds and extremal metrics can be extended to the orbifold case, extension often done in a fairly straightforward way, it is not surprising that we can then "translate" statements involving compact Kähler orbifolds to conclude parallel statements regarding quasi-regular Sasakian structures. This is an approach that has been spectacularly successful in constructing new quasi-regular Sasaki-Einstein metrics on various contact manifolds of odd dimension greater than 3 (cf. [5, 3, 4, 20, 21], and references therein).

For some time now, it has been believed that the only interesting (canonical) metrics in Sasakian geometry occur precisely in this orbibundle setting. In 1994, Cheeger and Tian conjectured that any compact Sasaki-Einstein manifold must be quasi-regular [12]. Their conjecture was phrased in terms of the properties of the Calabi-Yau cone rather than its Sasaki-Einstein base[1], and until recently, compact

[1]More precisely Cheeger and Tian used the term *standard cone*, and the conjecture states that all Calabi-Yau cones are standard [12]

Sasakian manifolds with non-closed leaves were certainly known, but there was no evidence to suspect that we could get such structures with Einstein metrics as well. Hence, it was reasonable to believe that all Sasaki-Einstein metrics could be understood well by simply studying the existence of Kähler-Einstein metrics on compact cyclic orbifolds.

As it turns out, the conjecture of Cheeger and Tian mentioned above is not true, and the first examples of irregular Sasaki-Einstein manifolds, that is to say, Sasaki-Einstein manifolds that are not quasi-regular, came first from the physics surrounding the famous CFT/AdS Duality Conjecture [18, 19, 25, 26, 27, 28, 14]. It now appears that there are irregular structures of this type on many compact manifolds in any odd dimension greater than 3. These Sasaki-Einstein metrics represent canonical points in the space of metrics adapted to the underlying geometric setting. However, although their Calabi-Yau cones are smooth outside the tip of the cone, their space of leaves is not even Hausdorff, and so the whole "orbibundle over a Kähler-Einstein base" approach proves itself insufficient in the study of the problem.

The discovery of these new metrics make a strong case in favor of a variational formulation of the study of these Sasakian metrics, in a way analogous to the notion of the Calabi energy and extremality. With the proper set-up, the quasi-regularity property should no longer be a key factor, and all Sasaki-Einstein metrics should indeed appear as minima of a suitable Riemannian functional. This would put on equal footing the analysis of all Sasakian structures, the quasi-regular or the irregular ones. Thus, we should be able to study the existence and uniqueness of these canonical Sasakian metrics in ways parallel to those used in Kähler geometry.

Until now, this approach for finding canonical Sasakian structures has not been pursued, perhaps due to the lack of evidence that the orbibundle approach would be insufficient. We propose here to look at the squared $L^2$-norm of the scalar curvature functional, defined over a suitable space of Sasakian metrics that are determined by fixing the Reeb vector field, which we think of as polarizing the Sasakian manifold. Its critical points are, by definition, canonical Sasakian metrics representing the said polarization.

Recently, Martelli, Sparks and Yau presented a similar point of view [28], opting to look at the Sasaki-Einstein metrics as minima of the Hilbert action instead. Our point of view has important advantages, several of which are elucidated in the present article. In particular, we see that for certain manifolds of Sasaki type, the *optimal* Sasaki metric on it cannot possibly have constant scalar curvature, showing the need to enlarge the plausible set of metrics to be considered, if at least, from a mathematical point of view.

In general, the minimization of the $L^2$-norm of the scalar curvature over metrics of fixed volume is intimately related to the search for Einstein metrics. We use this functional here over a smaller space of metrics, thus laying the foundation for the study of canonical Sasakian metrics in a way that parallels what is done in Kähler geometry for the extremal metric problem. This point of view eliminates the need to make a distinction between the quasi-regular and the irregular case, discussing them both on an equal footing. For we introduce the notion of a polarized Sasakian manifold, polarized by a Reeb vector field, and analyze the variational problem for the $L^2$-norm of the scalar curvature over the space of Sasakian metrics representing the said polarization. Given a CR structure of Sasaki type, we define the cone of Sasakian polarizations compatible with this underlying CR structure, and discuss the variational problem for this functional as we vary the polarization on this cone also. The quasi-regularity or not of the resulting critical Sasakian structures is just a property of the characteristic foliation defined by the Reeb vector field. This foliation must clearly sit well with the Sasakian metrics under consideration in our approach, but it stands in its own right. A canonical Sasakian metric, a critical point of the said functional, interacts with the underlying characteristic foliation, but neither one of them determines the other.

We organize the paper as follows. In §2 we recall and review the necessary definitions of Sasakian manifolds and associated structures. In §3 we define the notion of a polarized Sasakian manifold, and describe the space of Sasakian metrics that represent a given polarization, a space consisting of metrics with the same transversal holomorphic structure. We then analyze the variational problem for the $L^2$-norm of the scalar curvature with it as its domain of definition, and show that the resulting critical points are Sasakian metrics for which the basic vector field $\partial^{\#}_g s^T_g = \partial^{\#}_g s_g$ is transversally holomorphic, that is to say, metrics that are transversally extremal. In §4 we study various transformation groups of Sasakian structures and their Lie algebras, proving the Sasakian version of the Lichnerowicz-Matsushima theorem. In §5 we define and study the Sasaki-Futaki invariant, and prove that a canonical Sasakian metric is of constant scalar curvature if, and only if, this invariant vanishes for the polarization under consideration. In §6 we define and study the Sasaki cone, and end up in §7 by proving that the polarizations in the Sasaki cone that admit canonical representatives form an open set, proving that the openness theorem for the extremal cone in Kähler geometry [22] holds in the Sasakian context also. We illustrate the power of this result by providing a detailed analysis of the Sasaki cone for the standard CR-structure on the unit sphere $\mathbb{S}^{2n+1}$, and use it to show that all of its elements admit canonical representatives. We describe explicitly those that are of constant scalar curvature, and show that the standard metric is the only one of these that is Sasaki-Einstein.

## 2. Sasakian manifolds

We recall that an almost contact structure on a differentiable manifold $M$ is given by a triple $(\xi, \eta, \Phi)$, where $\xi$ is a vector field, $\eta$ is a one form, and $\Phi$ is a tensor of type $(1,1)$, subject to the relations

$$\eta(\xi) = 1\,, \quad \Phi^2 = -\mathbb{1} + \xi \otimes \eta\,.$$

The vector field $\xi$ defines the *characteristic foliation* $\mathcal{F}_\xi$ with one-dimensional leaves, and the kernel of $\eta$ defines the codimension one sub-bundle $\mathcal{D}$. This yields a canonical splitting

$$TM = \mathcal{D} \oplus L_\xi\,, \tag{1}$$

where $L_\xi$ is the trivial line bundle generated by $\xi$. The sub-bundle $\mathcal{D}$ inherits an almost complex structure $J$ by restriction of $\Phi$. Clearly, the dimension of $M$ must be an odd integer $2n+1$. We refer to $(M, \xi, \eta, \Phi)$ as an almost contact manifold. If we disregard the tensor $\Phi$ and characteristic foliation, that is to say, if we just look at the sub-bundle $\mathcal{D}$ forgetting altogether its almost complex structure, we then refer to the contact structure $(M, \mathcal{D})$, or simply $\mathcal{D}$ when $M$ is understood. Here, and further below, the reader can no doubt observe that the historical development of the terminology is somewhat unfortunate, and for instance, it is an almost contact structure the one that gives rise to a contact one, rather than the other way around.

A Riemannian metric $g$ on $M$ is said to be compatible with the almost contact structure $(\xi, \eta, \Phi)$ if for any pair of vector fields $X, Y$, we have that

$$g(\Phi(X), \Phi(Y)) = g(X,Y) - \eta(X)\eta(Y)\,.$$

Any such $g$ induces an almost Hermitian metric on the sub-bundle $\mathcal{D}$. We say that $(\xi, \eta, \Phi, g)$ is an almost contact metric structure.

In the presence of a compatible Riemannian metric $g$ on $(M, \xi, \eta, \Phi)$, the canonical decomposition (1) is orthogonal. Furthermore, requiring that the orbits of the field $\xi$ be geodesics is equivalent to requiring that $\pounds_\xi \eta = 0$, a condition that in view of the relation $\xi \lrcorner \eta = 1$, can be re-expressed as $\xi \lrcorner d\eta = 0$.

An almost contact metric structure $(\xi, \eta, \Phi, g)$ is said to be a contact metric structure if for all pair of vector fields $X$, $Y$, we have that

$$g(\Phi X, Y) = d\eta(X, Y) \,. \tag{2}$$

We then say that $(M, \xi, \eta, \Phi, g)$ is a contact metric manifold. Notice that in such a case, the volume element defined by $g$ is given by

$$d\mu_g = \frac{1}{n!}\eta \wedge (d\eta)^n \,. \tag{3}$$

It is convenient to reinterpret the latter structure in terms of the cone construction. Indeed, on $C(M) = M \times \mathbb{R}^+$, we introduce the metric

$$g_C = dr^2 + r^2 g \,.$$

The radial vector field $r\partial_r$ satisfies

$$\pounds_{r\partial_r} g_C = 2g_C \,,$$

and we may define an almost complex structure $I$ on $C(M)$ by

$$I(Y) = \Phi(Y) + \eta(Y) r\partial_r \,, \quad I(r\partial_r) = -\xi \,.$$

The almost contact manifold $(M, \xi, \eta, \Phi)$ is said to be *normal* if the pair $(C(M), I)$ is a complex manifold. In that case, the induced almost complex structure $J$ on $\mathcal{D}$ is integrable.

**Definition 2.1.** *An contact metric structure* $(\xi, \eta, \Phi, g)$ *on a manifold* $M$ *is said to be a* **Sasakian structure** *if* $(\xi, \eta, \Phi)$ *is normal. A smooth manifold provided with one such structure is said to be a* **Sasakian manifold***, or a* **manifold of Sasaki type**.

For a Sasakian structure $(\xi, \eta, \Phi, g)$, the integrability of the almost complex structure $I$ on the cone $C(M)$ implies that the Reeb vector field $\xi$ leaves both, $\eta$ and $\Phi$, invariant [3]. We obtain a codimension one integrable strictly pseudo-convex CR structure $(\mathcal{D}, J)$, where $\mathcal{D} = \ker \eta$ is the contact bundle and $J = \Phi|_{\mathcal{D}}$, and the restriction of $g$ to $\mathcal{D}$ defines a positive definite symmetric form on $(\mathcal{D}, J)$ that we shall refer to as the transverse Kähler metric $g^T$.

By (2), the Kähler form of the transverse Kähler metric is given by the form $d\eta$. Therefore, the Sasakian metric $g$ is determined fully in terms of $(\xi, \eta, \Phi)$ by the expression

$$g = d\eta \circ (\mathbb{1} \otimes \Phi) + \eta \otimes \eta \,, \tag{4}$$

where the fact that $d\eta$ is non-degenerate over $\mathcal{D}$ is already built in. Since $\xi$ leaves invariant $\eta$ and $\Phi$, it is a Killing field, its orbits are geodesics, and the decomposition (1) is orthogonal. Despite its dependence on the other elements of the structure, we insist on explicitly referring to $g$ as part of the Sasakian structure $(\xi, \eta, \Phi, g)$.

The discussion may be turned around to produce an alternative definition of the notion of Sasakian structure. For the contact metric structure $(\xi, \eta, \Phi, g)$ defines a Sasakian structure if $(\mathcal{D}, J)$ is a complex sub-bundle of $TM$, and $\xi$ generates a group of isometries. This alternative approach appears often in the literature.

If we look at the Sasakian structure $(\xi, \eta, \Phi, g)$ from the point of view of CR geometry, its underlying strictly pseudo-convex CR structure $(\mathcal{D}, J)$, with associated contact bundle $\mathcal{D}$, has Levi form $d\eta$. (In the sequel, when referring to any CR structure, we shall always mean one that is integrable and of codimension one.)

**Definition 2.2.** *Let $(\mathcal{D}, J)$ be a strictly pseudo-convex* CR *structure on $M$. We say that $(\mathcal{D}, J)$ is of* **Sasaki type** *if there exists a Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ such that $\mathcal{D} = \ker\, \eta$ and $\Phi|_{\mathcal{D}} = J$.*

If $(\mathcal{D}, J)$ is a CR structure of Sasaki type, the Sasakian structures $\mathcal{S} = (\xi, \eta, \Phi, g)$ with $\mathcal{D} = \ker\, \eta$ and $\Phi|_{\mathcal{D}} = J$ will be said to be Sasakian structures *with underlying* CR *structure* $(\mathcal{D}, J)$.

The following well known result will be needed later. Let us observe that since the fibers of the Riemannian foliation defined by a Sasaki structure $(\xi, \eta, \Phi, g)$ are geodesics, their second fundamental forms are trivial.

**Proposition 2.3.** *Let $(M, \xi, \eta, \Phi, g)$ be a Sasakian manifold. Then we have that*

a) $Ric_g(X, \xi) = 2n\eta(X)$ *for any vector field $X$.*
b) $Ric_g(X, Y) = Ric_T(X, Y) - 2g(X, Y)$ *for any pair of sections $X, Y$ of $\mathcal{D}$.*
c) $s_g = s_T - |A|^2 = s_T - 2n$, *where $A$ is the O'Neill tensor of the "corresponding" Riemannian submersion.*

*In these statements, the subscript $T$ denotes the corresponding intrinsic geometric quantities of the transversal metric $g^T$.*

*Proof.* The Riemannian submersion of the Sasakian structure has totally geodesic fibers. The vector field $\xi$ spans the only vertical direction, and we have that $A_X Y = -(d\eta(X, Y))\xi$, and that $A_X \xi = -\Phi(X)$ for arbitrary horizontal vector fields $X, Y$. The first two results follow easily from O'Neill's formulae [32]. The computation of the $L^2$-norm of $A$ is a simple consequence of the $J$-invariance of the induced metric on $\mathcal{D}$. $\square$

## 3. Canonical representatives of polarized Sasakian manifolds

Let us consider a Sasakian structure $(\xi, \eta, \Phi, g)$ on $M$ that shall remain fixed throughout this section. There are two naturally defined sets of deformations of this structure, those where the Reeb vector field remains fixed while the underlying CR structure changes by a diffeomorphism, and those where the underlying CR structure stays put while the Reeb vector field varies. We study the first type of deformations in this section. They turn out to be the Sasakian analogues of the set of Kähler metrics representing a given polarization on a manifold of Kähler type. They have different but isomorphic underlying CR structure, and they all share the same transverse holomorphic structure. The other set of deformations that fix the underlying CR structure will lead to the Sasakian analogue of the Kähler cone of a manifold of Kähler type, and they will be analyzed in §6 below.

We begin by recalling that a function $\varphi \in C^\infty(M)$ is said to be basic if it is annihilated by the vector field $\xi$, that is to say, if $\xi(\varphi) = 0$. We denote by $C^\infty_B(M)$ the space of all real valued basic functions on $M$. We observe that the notion of basic can be extended to covariant tensors of any order in the obvious manner. In particular, when looking at the transversal Kähler metric of $(\xi, \eta, \Phi, g)$, its Kähler form is basic, and so must be all of its curvature tensors as well. This observation will play a crucial rôle in the sequel.

We consider the set [3]

$$\mathcal{S}(\xi) = \{\text{Sasakian structure } (\tilde{\xi}, \tilde{\eta}, \tilde{\Phi}, \tilde{g}) \mid \tilde{\xi} = \xi\}\,, \tag{5}$$

and provide it with the $C^\infty$ compact-open topology as sections of vector bundles. For any element $(\tilde{\xi}, \tilde{\eta}, \tilde{\Phi}, \tilde{g})$ in this set, the 1-form $\zeta = \tilde{\eta} - \eta$ is basic, and so $[d\tilde{\eta}]_B = [d\eta]_B$. Here, $[\,\cdot\,]_B$ stands for a cohomology class in the basic cohomology ring, ring that is defined by the restriction $d_B$ of the exterior derivative $d$ to the subcomplex of basic forms in the de Rham complex of $M$. Thus, all of the Sasakian structures in $\mathcal{S}(\xi)$ correspond to the same basic cohomology class. We call $\mathcal{S}(\xi)$ the *space of Sasakian structures compatible with* $\xi$, and say that the Reeb vector field $\xi$ *polarizes* the Sasakian manifold $M$.

Given the Reeb vector field $\xi$, we have its characteristic foliation $\mathcal{F}_\xi$, so we let $\nu(\mathcal{F}_\xi)$ be the vector bundle whose fiber at a point $p \in M$ is the quotient space $T_pM/L_\xi$, and let $\pi_\nu : TM \to \nu(\mathcal{F}_\xi)$ be the natural projection. The background structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ induces a complex structure $\bar{J}$ on $\nu(\mathcal{F}_\xi)$. This is defined by $\bar{J}\bar{X} := \overline{\Phi(X)}$, where $X$ is any vector field in $M$ such that $\pi(X) = \bar{X}$. Furthermore, the underlying CR structure $(\mathcal{D}, J)$ of $\mathcal{S}$ is isomorphic to $(\nu(\mathcal{F}_\xi), \bar{J})$ as a complex vector bundle. For this reason, we refer to $(\nu(\mathcal{F}_\xi), \bar{J})$ as the complex normal bundle of the Reeb vector field $\xi$, although its identification with $(\mathcal{D}, J)$ is not canonical. We shall say that $(M, \xi, \nu(\mathcal{F}_\xi), \bar{J})$, or simply $(M, \xi, \bar{J})$, is a polarized Sasakian manifold.

We define $\mathcal{S}(\xi, \bar{J})$ to be the subset of all structures $(\tilde{\xi}, \tilde{\eta}, \tilde{\Phi}, \tilde{g})$ in $\mathcal{S}(\xi)$ such that the diagram

$$\begin{array}{ccc} TM & \stackrel{\tilde{\Phi}}{\rightarrow} & TM \\ \downarrow\pi_\nu & & \downarrow\pi_\nu \\ \nu(\mathcal{F}_\xi) & \stackrel{\bar{J}}{\rightarrow} & \nu(\mathcal{F}_\xi), \end{array} \tag{6}$$

commutes. This set consists of elements of $\mathcal{S}(\xi)$ with the same transverse holomorphic structure $\bar{J}$, or more precisely, the same complex normal bundle $(\nu(\mathcal{F}_\xi), \bar{J})$. We have [3]

**Lemma 3.1.** *The space $\mathcal{S}(\xi, \bar{J})$ of all Sasakian structures with Reeb vector field $\xi$ and transverse holomorphic structure $\bar{J}$ is an affine space modeled on $(C^\infty_B(M)/\mathbb{R}) \times (C^\infty_B(M)/\mathbb{R}) \times H^1(M, \mathbb{Z})$. Indeed, if $(\xi, \eta, \Phi, g)$ is a given Sasakian structure in $\mathcal{S}(\xi, \bar{J})$, any other Sasakian structure $(\xi, \tilde{\eta}, \tilde{\Phi}, \tilde{g})$ in it is determined by real valued basic functions $\varphi$ and $\psi$ and integral closed 1-form $\alpha$, such that*

$$\begin{array}{rcl} \tilde{\eta} & = & \eta + d^c\varphi + d\psi + i(\alpha)\,, \\ \tilde{\Phi} & = & \Phi - (\xi \otimes (\tilde{\eta} - \eta)) \circ \Phi\,, \\ \tilde{g} & = & d\tilde{\eta} \circ (\mathbb{1} \otimes \tilde{\Phi}) + \tilde{\eta} \otimes \tilde{\eta}\,, \end{array}$$

*where $d^c = \frac{i}{2}(\overline{\partial} - \partial)$, and $i : H^1(M, \mathbb{Z}) \mapsto H^1(M, \mathbb{R}) = H^1(\mathfrak{F}_\xi)$ is the homomorphism induced by inclusion. In particular, $d\tilde{\eta} = d\eta + i\partial\overline{\partial}\varphi$.*

The complex structure defining the operators $\partial$ and $\overline{\partial}$ in this Lemma is $\bar{J}$, as basic covariant tensors on $M$ define multilinear maps on $\nu(\mathcal{F}_\xi)$. We think of these as tensors on a transversal Kähler manifold that does not necessarily exist. Be as it may, the cohomology class of the transverse Kähler metrics arising from elements in $\mathcal{S}(\xi, \bar{J})$ is fixed, and it is natural to ask if there is a way of fixing the affine parameters $\varphi$ and $\psi$ also, which would yield then a canonical representative of this set. We proceed to discuss and answer this problem.

We start by introducing a Riemannian functional whose critical point will fix a canonical choice of metric for structures in $\mathcal{S}(\xi, \bar{J})$. This, in effect, will fix the desire preferred representative that we seek.

We denote by $\mathfrak{M}(\xi, \bar{J})$ the set of all compatible Sasakian metrics arising from structures in $\mathcal{S}(\xi, \bar{J})$, and define the functional

$$\begin{array}{rcl} \mathfrak{M}(\xi, \bar{J}) & \stackrel{E}{\to} & \mathbb{R}\,, \\ g & \mapsto & \displaystyle\int_M s_g^2 d\mu_g\,, \end{array} \tag{7}$$

the squared $L^2$-norm of the scalar curvature of $g$.

The variation of a metric in $\mathfrak{M}(\xi, \bar{J})$ depends upon the two *affine parameters* of freedom $\varphi$ and $\psi$ of Lemma 3.1. However, the transversal Kähler metric varies as a function of $\varphi$ only, and does so within a fixed basic cohomology class. The critical point of (7), should it exist, will allow us to fix the parameter $\varphi$ since, not surprisingly, it shall be determined by the condition that $d\tilde{\eta} = d\eta + i\partial\overline{\partial}\varphi$ be an extremal Kähler metric [9] on $\mathcal{D}$. The remaining gauge function parameter $\psi$ represents nothing more than a change of coordinates in the representation of the form $\tilde{\eta}$ of the Sasakian structure in question. Thus, the finding of a critical point of $E$ produces a canonical representative of $\mathcal{S}(\xi, \bar{J})$.

3.1. **Variational formulae.** In order to derive the Euler-Lagrange equation of (7), we describe the infinitesimal variations of the volume form, Ricci tensor, and scalar curvature, as a metric in $\mathfrak{M}(\xi, \bar{J})$ is deformed within that space.

We begin by recalling that the Ricci form $\rho$ of a Sasaki structure $(\xi, \eta, \Phi, g)$ is defined on $\mathcal{D}$ by the expression

$$\rho_g(X, Y) = Ric_g(JX, Y)\,.$$

This is extended trivially on the characteristic foliation $L_\xi$, and by Proposition 2.3, we easily see that

$$\rho_g = \rho_g^T - 2d\eta\,, \tag{8}$$

where $\rho^T$ denotes the form arising from the Ricci tensor of the transversal metric, a basic two form that we think of as a $\bar{J}$-invariant two tensor in $\nu(\mathcal{F}_\xi)$. Thus, $\rho_g$ induces a well defined bilinear map on $\nu(\mathcal{F}_\xi)$ that is $\bar{J}$-invariant.

Though the notation suggests so, it is not the case that the trace of the form (8) yields the scalar curvature of $g$. This form only encodes information concerning second covariant derivatives of $g$ along directions in $\nu(\mathcal{F}_\xi)$.

**Proposition 3.2.** *Let $(\xi, \eta_t, \Phi_t, g_t)$ be a path in $\mathcal{S}(\xi, \bar{J})$ that starts at $(\xi, \eta, \Phi, g)$ when $t = 0$, and is such that $d\eta_t = d\eta + ti\partial\overline{\partial}\varphi$ for certain basic function $\varphi$, and for $t$ sufficiently small. Then we have the* *expansions*

$$\begin{array}{rcl} d\mu_t & = & (1-\frac{t}{2}\Delta_B\varphi))d\mu + O(t^2)\,, \\ \rho_t & = & \rho - ti\partial\overline{\partial}\left(\frac{1}{2}\Delta_B\varphi+\varphi\right)+O(t^2)\,, \\ s_t & = & s^T - 2n - t\left(\frac{1}{2}\Delta_B^2\varphi + 2(\rho^T, i\partial\overline{\partial}\varphi)\right)+O(t^2)\,, \end{array}$$

*for the volume form, Ricci form, and scalar curvature of $g_t$, respectively. Here, the geometric terms without sub-index are those corresponding to the starting metric $g$, and $\Delta_B$ is the Laplacian acting on basic functions.*

*Proof.* By Lemma 3.1, there exists a function $\psi$ such that

$$\eta_t = \eta + t(d^c\varphi + d\psi) + O(t^2)\,.$$

Since $\varphi$ and $\psi$ are both basic, we have that $d\mu_t = \frac{1}{n!}\eta_t \wedge (d\eta_t)^n = \eta \wedge (d\eta_t)^n$, and we obtain

$$d\mu_t = \frac{1}{n!}\eta \wedge (d\eta + ti\partial\overline{\partial}\varphi)^n = d\mu + \frac{t}{(n-1)!}\eta \wedge (d\eta)^{n-1} \wedge i\partial\overline{\partial}\varphi + O(t^2)\,.$$

Now, $\omega_B = d\eta$, the Kähler form of the induced metric on $\mathcal{D}$, is basic. We then have that $*_B(\omega_B)^{n-1}/(n-1)! = \omega_B$, and conclude that

$$d\mu_t = \left(1 - \frac{t}{2}\Delta_B\varphi\right) d\mu + O(t^2)\,.$$

By (8), we may compute the variation of $\rho$ by computing the variation of $\rho^T$. This is well known to be [34]

$$\rho_t^T = \rho^T + \frac{t}{2} i\partial\overline{\partial}(\Delta_B\varphi) + O(t^2)\,.$$

Since $d\eta_t = d\eta + ti\partial\overline{\partial}\varphi$, we obtain

$$\rho_t = \rho + ti\partial\overline{\partial}\left(\frac{1}{2}\Delta_B\varphi + \varphi\right) + O(t^2)\,,$$

as stated.

Finally, by Proposition 2.3 once again, we have that the variation of the scalar curvature arises purely from the variation of its transversal part. Since the transversal metric is Kähler, we obtain

$$s_t = s - t\left(\frac{1}{2}\Delta_B^2\varphi + 2(\rho^T, i\partial\overline{\partial}\varphi)\right) + O(t^2)\,.$$

as desired. □

**Remark 3.3**. The forms $\rho^T$ and $i\partial\overline{\partial}\varphi$ are basic. Hence, the metric pairing of these forms that appears in the Proposition above involves only the transversal metric. On the other hand, in view of the analogous variational formulae in the Kähler case [34], we might think that the expression for $\rho_t$ above is a bit strange. We see that this is not so if we just keep in mind that the Sasakian $\rho_g$ encodes $\nu(\mathcal{F}_\xi)$-covariant derivatives information only.

3.2. **Euler-Lagrange equations.** Associated to any Sasakian structure $(\xi, \eta, \Phi, g)$ in $\mathcal{S}(\xi, \bar{J})$, we introduce a basic differential operator $L_g^B$, of order 4, whose kernel consists of basic functions with transverse holomorphic gradient.

Given a *basic* function $\varphi : M \mapsto \mathbb{C}$, we consider the vector field $\partial^{\#}\varphi$ defined by the identity

$$g(\partial^{\#}\varphi, \cdot\,) = \overline{\partial}\varphi\,. \tag{9}$$

Thus, we obtain the (1,0) component of the gradient of $\varphi$, a vector field that, generally speaking, is not transversally holomorphic. In order to ensure that, we would need to impose the condition $\bar{\partial}\partial^{\#}\varphi = 0$, that is equivalent to the fourth-order equation

$$(\bar{\partial}\partial^{\#})^{*}\bar{\partial}\partial^{\#}\varphi = 0\,, \tag{10}$$

because $\langle \varphi, (\bar{\partial}\partial^{\#})^{*}\bar{\partial}\partial^{\#}\varphi\rangle_{L^2} = \|\bar{\partial}\partial^{\#}\varphi\|^2_{L^2}$.

We have that

$$L_g^B\varphi := (\bar{\partial}\partial^{\#})^{*}\bar{\partial}\partial^{\#}\varphi = \frac{1}{4}(\Delta_B^2\varphi + 4(\rho^T, i\partial\overline{\partial}\varphi) + 2(\partial s^T)\lrcorner\partial^{\#}\varphi)\,. \tag{11}$$

The functions on $M$ that are transversally constant are always in the kernel of $L_g^B$. The only functions of this type that are basic are the constants. Thus, the kernel of $L_g^B$ has dimension at least 1.

**Proposition 3.4.** *The first derivative of $E$ at $g \in \mathfrak{M}(\xi, \bar{J})$ in the direction of the deformation defined by $(\varphi, \psi)$ is given by*

$$\frac{d}{dt}E(g_t)\mid_{t=0} = -4\int_M (s^T - 2n)\left((\bar{\partial}\partial^{\#})^{*}\bar{\partial}\partial^{\#}\varphi\right)\, d\mu\,.$$

*Proof.* This result follows readily from the fact that $s = s^T - 2n$, the variational formulae of Proposition 3.2, and identity (11). □

As a corollary to Proposition 3.4, we have:

**Theorem 3.5.** *A Sasakian metric $g \in \mathfrak{M}(\xi, \bar{J})$ is a critical point of the energy functional $E$ of* (7) *if the basic vector field $\partial_g^{\#}s_g^T = \partial_g^{\#}s_g$ is transversally holomorphic.*

We are thus led to our fundamental definition:

**Definition 3.6.** *We say that $\mathcal{S} = (\xi, \eta, \Phi, g)$ is a* **canonical representative** *of $\mathcal{S}(\xi, \bar{J})$ if the metric $g$ satisfies the condition of Theorem* 3.5*, that is to say, if, and only if, $g$ is transversally extremal.*

## 4. Transformation groups of Sasakian structures

In this section we discuss some important transformation groups associated to Sasakian structures, and their corresponding Lie algebras.

Let us begin with a CR structure $(\mathcal{D}, J)$ of Sasaki type on $M$. If $\eta$ is a contact form, we have the group $\mathfrak{Con}(M, \mathcal{D})$ of contact diffeomorphims, that is to say, the subgroup of the diffeomorphisms group $\mathfrak{Diff}(M)$ consisting of those elements that leave the contact subbundle $\mathcal{D}$ invariant:

$$\mathfrak{Con}(M, \mathcal{D}) = \{\phi \in \mathfrak{Diff}(M) \mid \phi^* \eta = f_\phi \eta\,,\ f_\phi \in C^\infty(M)^*\}\,. \tag{12}$$

Here, $C^\infty(M)^*$ denotes the subset of nowhere vanishing functions in $C^\infty(M)$. The dimension of the group so defined is infinite.

We may also consider the subgroup $\mathfrak{Con}(M, \eta)$ of strict contact transformations, whose elements are those $\phi \in \mathfrak{Con}(M, \mathcal{D})$ such that $f_\phi = 1$:

$$\mathfrak{Con}(M, \eta) = \{\phi \in \mathfrak{Diff}(M) \mid \phi^* \eta = \eta\}\,.$$

This subgroup is also infinite dimensional.

The Lie algebras of these two groups are quite important. The first of these is the Lie algebra of *infinitesimal contact transformations*,

$$\mathfrak{con}(M, \mathcal{D}) = \{X \in \mathfrak{X}(M) \mid \pounds_X \eta = a(X)\eta\,,\ a(X) \in C^\infty(M)\}\,, \tag{13}$$

while the second is the subalgebra of *infinitesimal strict contact transformations*

$$\mathfrak{con}(M, \eta) = \{X \in \mathfrak{X}(M) \mid \pounds_X \eta = 0\}\,. \tag{14}$$

If we now look at the pair $(\mathcal{D}, J)$, we have the group of CR automorphisms of $(\mathcal{D}, J)$, defined by

$$\mathfrak{CR}(M, \mathcal{D}, J) = \{\phi \in \mathfrak{Con}(M, \mathcal{D}) \mid \phi_* J = J \phi_*\}\,. \tag{15}$$

This is a Lie group [13]. Its Lie algebra $\mathfrak{cr}(M, \mathcal{D}, J)$ can be characterized as

$$\mathfrak{cr}(M, \mathcal{D}, J) = \{X \in \mathfrak{con}(M, \mathcal{D}) \mid \pounds_X J = 0\}\,. \tag{16}$$

Notice that in this defining expression, $\pounds_X J$ makes sense even though $J$ is not a tensor field on $M$, the reason being that the vector field $X$ leaves $\mathcal{D}$ invariant.

Let us now consider a Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with underlying CR structure $(\mathcal{D}, J)$. We are interested in the subgroup of $\mathfrak{Diff}(M)$ that leaves the tensor field $\Phi$ invariant. So we define

$$\mathfrak{S}_\Phi = \{\phi \in \mathfrak{Diff}(M) :\ \phi_* \circ \Phi = \Phi \circ \phi_*\}\,. \tag{17}$$

We also have

$$\mathfrak{Fol}(M, \mathcal{F}_\xi) = \{\phi \in \mathfrak{Diff}(M) :\ \phi_* \mathcal{F}_\xi \subset \mathcal{F}_\xi\}\,,$$

the subgroup of $\mathfrak{Diff}(M)$ that preserves the characteristic foliation of the Sasakian structure $\mathcal{S}$.

In order to simplify the notation, we will often drop $M$ from the notation when referring to these various groups and algebras.

**Lemma 4.1.** *Let* $(\mathcal{D}, J)$ *be a strictly pseudo-convex* CR *structure of Sasaki type on* $M$*, and fix a Sasakian structure* $\mathcal{S} = (\xi, \eta, \Phi, g)$ *with underlying CR structure* $(\mathcal{D}, J)$*. Then*

$$\mathfrak{S}_\Phi = \mathfrak{CR}(\mathcal{D}, J) \cap \mathfrak{Fol}(\mathcal{F}_\xi)\,.$$

*Proof.* If $\phi \in \mathfrak{S}_\Phi$, the identity $\eta \circ \Phi = 0$ implies that $\phi$ preserves $\mathcal{D}$. If $X$ is a section of $\mathcal{D}$, then we have that $\phi_* J(X) = \phi_* \Phi(X) = \Phi(\phi_* X) = J(\phi_* X)$, which implies that $\phi \in \mathfrak{CR}(\mathcal{D}, J)$. But we have $\phi_* \Phi(\xi) = 0 = \Phi(\phi_* \xi)$ also, which implies that $\phi \in \mathfrak{Fol}(\mathcal{F}_\xi)$.

Conversely, suppose that $\phi \in \mathfrak{CR}(\mathcal{D}, J) \cap \mathfrak{Fol}(\mathcal{F}_\xi)$. Then $\phi$ leaves all three $\mathcal{D}$, $\mathcal{F}_\xi$, and $J$ invariant, and therefore, it preserves the splitting (1). Since the relation between $J$ and $\Phi$ is given by

$$\Phi(X) = \begin{cases} J(X) \text{ if } X \text{ is a section of } \mathcal{D}\,, \\ 0 \text{ if } X = \xi\,, \end{cases} \tag{18}$$

in order to conclude that $\phi \in \mathfrak{S}_\Phi$, it suffices to show that $\phi_* \Phi(\xi) = \Phi(\phi_* \xi)$. But this is clear as both sides vanish. □

Since $\mathfrak{S}_\Phi$ is closed in the Lie group $\mathfrak{CR}(\mathcal{D}, J)$, it is itself a Lie group. Generally speaking, the inclusion $\mathfrak{S}_\Phi \subset \mathfrak{CR}(\mathcal{D}, J)$ is strict, and the group $\mathfrak{Fol}(M, \mathcal{F}_\xi)$ is infinite dimensional.

The automorphism group $\mathfrak{Aut}(\mathcal{S})$ of the Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ is defined to be the subgroup of $\mathfrak{Diff}(M)$ that leaves all the tensor fields in $(\xi, \eta, \Phi, g)$ invariant. It is a Lie group, and one has natural group inclusions

$$\mathfrak{Aut}(\mathcal{S}) \subset \mathfrak{S}_\Phi \subset \mathfrak{CR}(\mathcal{D}, J) \tag{19}$$

whenever the CR structure $(\mathcal{D}, J)$ is of Sasaki type, and $\mathcal{S}$ has it as its underlying CR structure.

The Lie algebras of $\mathfrak{S}_\Phi$ and $\mathfrak{Fol}(\mathcal{F}_\xi)$ are given by

$$\mathfrak{s}_\Phi = \{X \in \mathfrak{X}(M) \mid \pounds_X \Phi = 0\}\,, \tag{20}$$

and

$$\mathfrak{fol}(\mathcal{F}_\xi) = \{X \in \mathfrak{X}(M) \mid [X, \xi] \text{ is tangent to the leaves of } \mathcal{F}_\xi\}\,, \tag{21}$$

respectively. The latter is just the Lie algebra of *foliate* vector fields of the foliation $\mathcal{F}_\xi$. On the other hand, we may restate the defining condition for $\mathfrak{s}_\Phi$ as

$$X \in \mathfrak{s}_\Phi \Longleftrightarrow [X, \Phi(Y)] = \Phi([X, Y]) \text{ for all } Y\,,$$

and we can see easily that

$$\mathfrak{s}_\Phi = \mathfrak{cr}(\mathcal{D}, J) \cap \mathfrak{fol}(\mathcal{F}_\xi)\,. \tag{22}$$

We can characterize now CR structures of Sasaki type in terms of their relations to the Lie algebras above.

**Lemma 4.2.** *Let $(\mathcal{D}, J)$ be a strictly pseudoconvex* CR *structure on $M$, and let $\eta$ be a compatible contact form representing $\mathcal{D}$, with Reeb vector field $\xi$. Define the tensor field $\Phi$ by Equation* (18)*. Then $(\mathcal{D}, J)$ is of Sasaki type if, and only if, $\xi \in \mathfrak{cr}(\mathcal{D}, J)$.*

*Proof.* Given a strictly pseudoconvex CR structure $(\mathcal{D}, J)$, with contact 1-form $\eta$ that represents $\mathcal{D}$, and Reeb vector field $\xi$, we consider the $(1,1)$ tensor field $\Phi$ defined by (18). By Proposition 3.5 of [2], $(\xi, \eta, \Phi)$ defines a Sasakian structure if, and only if, the CR structure is integrable and $\pounds_\xi \Phi = 0$. Since $\xi$ is a foliate vector field, the condition that $\xi \in \mathfrak{cr}(\mathcal{D}, J)$ ensures that $\xi \in \mathfrak{s}_\Phi$, and the result follows. □

For any strictly pseudoconvex CR structure with contact 1-form $\eta$, the Reeb vector field $\xi$ belongs to $\mathfrak{con}(M, \eta)$. Therefore, if $\xi \in \mathfrak{s}_\Phi$, the structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ is Sasakian, and $\xi \in \mathfrak{aut}(\mathcal{S})$.

The study of the group $\mathfrak{CR}(\mathcal{D}, J)$ has a long and ample history [36, 23, 33]. We state the most general result given in [33]. This holds for a general CR manifold $M$, so we emphasize its consequence in the case where $M$ is closed.

**Theorem 4.3.** *Let $M$ be a $2n+1$ dimensional manifold with a strictly pseudoconvex* CR *structure $(\mathcal{D}, J)$, and* CR *automorphism group $G$. If $G$ does not act properly on $M$, then:*

1. *If $M$ is a non-compact manifold, then it is* CR *diffeomorphic to the Heisenberg group with its standard* CR *structure.*
2. *If $M$ is a compact manifold, then it is* CR *diffeomorphic to the sphere $\mathbb{S}^{2n+1}$ with its standard* CR *structure.*

*In particular, if $M$ is a closed manifold not* CR *diffeomorphic to the sphere, the automorphisms group of its* CR *structure is compact.*

We recall that the inclusion $\mathfrak{Aut}(\mathcal{S}) \subset \mathfrak{CR}(\mathcal{D}, J)$ (see (19)), generally speaking, is proper. However, we do have the following.

**Proposition 4.4.** *Let $(\mathcal{D}, J)$ be a strictly pseudoconvex* CR *structure on a closed manifold $M$, and suppose that $(\mathcal{D}, J)$ is of Sasaki type. Then there exists a Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ with underlying* CR *structure $(\mathcal{D}, J)$, whose automorphism group $\mathfrak{Aut}(\mathcal{S})$ is a maximal compact subgroup of $\mathfrak{CR}(M, \mathcal{D}, J)$. In fact, except for the case* *when* $(M,\mathcal{D},J)$ *is* CR *diffeomorphic to the sphere* $\mathbb{S}^{2n+1}$ *with its standard* CR *structure, the automorphisms group* $\mathfrak{Aut}(\mathcal{S})$ *of* $\mathcal{S}$ *is equal to* $\mathfrak{CR}(M,\mathcal{D},J)$.

*Proof.* Let $G$ be a maximal compact subgroup of $\mathfrak{CR}(\mathcal{D},J)$. By Theorem 4.3, we have that $G=\mathfrak{CR}(\mathcal{D},J)$ except when $(M,\mathcal{D},J)$ is CR diffeomorphic to the sphere $\mathbb{S}^{2n+1}$. Let $\tilde{\mathcal{S}}=(\tilde{\xi},\tilde{\eta},\tilde{\Phi},\tilde{g})$ be a Sasakian structure with underlying CR structure $(\mathcal{D},J)$.

If $\phi\in G$, then $\phi^*\tilde{\eta}=f\tilde{\eta}$ for some nowhere vanishing real-valued function $f$. By averaging $\tilde{\eta}$ over $G$, we obtain a $G$-invariant contact form $\eta$ with associated contact structure $\mathcal{D}$. Let $\xi$ be its Reeb vector field. As $\xi$ is uniquely determined by $\eta$, we conclude that $\phi_*\xi=\xi$. We then define a $(1,1)$-tensor $\Phi$ by the expression in (18). The conditions $\phi_*J=J\phi_*$ and $\phi_*\xi=\xi$ imply that $\phi_*\Phi=\Phi\phi_*$. The triple $(\xi,\eta,\Phi)$ defines a Sasakian structure $\mathcal{S}$, and $\phi\in\mathfrak{Aut}(\mathcal{S})$. □

We now look at the case where the manifold $M$ is polarized by $(\xi,\bar{J})$. Then, $(\nu(\mathcal{F}_\xi),\bar{J})$ is a complex vector bundle, and any $\phi\in\mathfrak{Fol}(M,\mathcal{F}_\xi)$ induces a map $\bar{\phi}_*:\nu(\mathcal{F}_\xi)\rightarrow\nu(\mathcal{F}_\xi)$. We define the group of *transversely holomorphic transformations* $\mathfrak{H}^T(\xi,\bar{J})$ by

$$\mathfrak{H}^T(\xi,\bar{J})=\{\phi\in\mathfrak{Fol}(M,\mathcal{F}_\xi)\mid\bar{\phi}_*\circ\bar{J}=\bar{J}\circ\bar{\phi}_*\}. \tag{23}$$

Since a 1-parameter subgroup of any smooth section of $L_\xi$ induces the identity on $\nu(\mathcal{F}_\xi)$, this group is infinite dimensional. We are mainly interested in the infinitesimal version. Given a choice of $\mathcal{S}=(\xi,\eta,\Phi,g)$ in $\mathcal{S}(\xi,\bar{J})$, $(\nu(\mathcal{F}_\xi),\bar{J})$ is identified with the underlying CR structure $(\mathcal{D},J)$. In this case, we may use the decomposition (1) to write any vector field as

$$X=X_{\mathcal{D}}+c(X)\xi\,, \tag{24}$$

which defines the component function $X\mapsto c(X):=\eta(X)$, and the class $\bar{X}$ on $\nu(\mathcal{F}_\xi)$ defined by the vector field $X$ is represented by $X_{\mathcal{D}}$.

The Lie bracket operation induces a bilinear mapping on $\nu(\mathcal{F}_\xi)$ by

$$[\bar{X},\bar{Y}]:=\overline{[X,Y]}\,.$$

This operation allows us to generalize the notion of transversally holomorphic vector field already encountered in §3.2.

**Definition 4.5.** *Let* $(M,\xi,\bar{J})$ *be a polarized Sasakian manifold. We say that a vector field* $X$ *is* **transversally holomorphic** *if given any section* $\bar{Y}$ *of* $\nu(\mathcal{F}_\xi)$*, we have that*

$$[\bar{X},\bar{J}\bar{Y}]=\bar{J}\,\overline{[X,Y]}\,.$$

*The set of all such vector fields will be denoted by* $\mathfrak{h}^T(\xi,\bar{J})$.

It is now desirable to express the defining condition for $X$ to be in $\mathfrak{h}^T(\xi, \bar{J})$ in terms intrinsic to $X$ itself. The reader may consult [7] for relevant discussions.

**Lemma 4.6.** *Let $(M, \xi, \bar{J})$ be a polarized Sasakian manifold. If $X \in \mathfrak{h}^T(\xi, \bar{J})$ is a transversally holomorphic vector field, for any Sasakian structure $(\xi, \eta, \Phi, g) \in \mathcal{S}(\xi, \bar{J})$ we have that*

$$(\pounds_X \Phi)(Y) = \eta([X, \Phi(Y)])\xi \, .$$

*Proof.* We have that

$$(\pounds_X \Phi)(Y) = [X, \Phi(Y)] - \Phi([X, Y]) \, ,$$

which implies that $c((\pounds_X \Phi)(Y)) = \eta([X, \Phi(Y)])$. Thus,

$$(\pounds_X \Phi)(Y) = ((\pounds_X \Phi)(Y))_{\mathcal{D}} + \eta([X, \Phi(Y)])\xi \, .$$

The result follows after simple considerations. □

The set $\mathfrak{h}^T(\xi, \bar{J})$ is a Lie algebra contained in $\mathfrak{fol}(M, \mathcal{F}_\xi)$. If we represent $(\nu(\mathcal{F}_\xi), \bar{J})$ as $(\mathcal{D}, J)$ for a choice of $\mathcal{S} = (\xi, \eta, \Phi, g)$ in $\mathcal{S}(\xi, \bar{J})$ with underlying CR structure $(\mathcal{D}, J)$, by the decomposition (24) we see that for a transversally holomorphic vector field $X$ we have that

$$[X_{\mathcal{D}}, J(Y_{\mathcal{D}})]_{\mathcal{D}} = J([X_{\mathcal{D}}, Y_{\mathcal{D}}]_{\mathcal{D}}) \, ,$$

for any vector field $Y$. Thus, $X_{\mathcal{D}}$ preserves the transverse complex structure $J$. This characterization can be reformulated by saying that if $X \in \mathfrak{h}^T(\xi, \bar{J})$, the vector field of type $(1,0)$ given by

$$\Xi_X = \frac{1}{2}(X_{\mathcal{D}} - iJ(X_{\mathcal{D}})) \tag{25}$$

is in the kernel of the transverse Cauchy-Riemann equations. Thus, the mapping into the space of sections of $\nu(\mathcal{F}_\xi)$ given by

$$\begin{array}{ccc} \mathfrak{h}^T(\xi, \bar{J}) & \rightarrow & \Gamma(\nu(\mathcal{F}_\xi)) \\ X & \mapsto & \bar{X} \end{array} \tag{26}$$

has an image that can be identified with the space of sections of $(\mathcal{D}, J)$ satisfying the Cauchy Riemann equations, there finite dimensional. We denote this image by $\mathfrak{h}^T(\xi, \mathcal{F}_\xi)/L_\xi$.

Their one dimensional foliations made Sasakian manifolds a bit special. In particular, they carry no non-trivial parallel vector field (see [3]). For by a result of Tachibana, any harmonic one form must annihilate the Reeb vector field $\xi$, and so any parallel vector fields $X$ must be orthogonal to $\xi$, that is to say, it must be a section of $\mathcal{D}$. But then, since the metric is covariantly constant and $\xi$ is Killing, we must have that $0 = \nabla_Y g(\xi, X) = g(\Phi(Y), X)$ for all $Y$, which forces $X$ to be identically zero.

**Remark 4.7**. Notice that for any Sasakian structure with underlying CR structure $(\mathcal{D}, J)$, the Lie algebra $\mathfrak{cr}(\mathcal{D}, \eta)$ is reductive. For either $\mathfrak{CR}(\mathcal{D}, J)$ is compact, or $\mathfrak{cr}(\mathcal{D}, \eta) = \mathfrak{su}(n+1, 1)$. In particular, let $(N, \omega)$ be a Kähler manifold, and consider the circle bundle $\pi : M \to N$ with Euler class $[\omega]$. Then $M$ has a natural Sasakian structure $(\xi, \eta, \Phi, g)$ such that $\pi^*\omega = d\eta$. Then the horizontal lifts of non-trivial parallel vector fields in $(N, \omega)$ are holomorphic but not parallel on $M$. On the other hand, if we take a holomorphic field that lies in the non-reductive part of the algebra of holomorphic vector fields of $N$, its horizontal lift is a transversally holomorphic vector field that does not lie in the reductive component of the algebra $\mathfrak{h}^T$, and thus, it cannot possibly leave the contact subbundle $\mathcal{D}$ invariant. □

If $X \in \mathfrak{h}^T(\xi, \bar{J})$, given any real valued function $f$, $X + f\xi \in \mathfrak{h}^T(\xi, \bar{J})$ also, and so, $\mathfrak{h}^T(\xi, \bar{J})$ cannot have finite dimension. The remark above alludes to the special structure that $\mathfrak{h}^T(\xi, \bar{J})/L_\xi$ has, and in fact, we are now ready to extend to the Sasakian context a result of Calabi [10] on the structure of the algebra of holomorphic vector fields of a Kähler manifold that carries an extremal metric. Calabi's theorem is, in turn, an extension of work of Lichnerowicz [24] on constant scalar curvature metrics, and the latter is itself an extension of a result of Matsushima [29] in the Kähler-Einstein case. We also point the reader to the theorem for harmonic Kähler foliations in [31], which is relevant in this context.

Consider a Sasakian structure $(\xi, \eta, \Phi, g)$ in $\mathcal{S}(\xi, \bar{J})$. Let $\mathcal{H}^B_g$ be the space of basic functions in the kernel of the operator $L^B_g$ in (11), and consider the mapping

$$\partial^\#_g : \mathcal{H}^B_g \to \mathfrak{h}^T(\xi, \bar{J})/L_\xi \,, \tag{27}$$

where $\partial^\#_g$ is the operator defined in (9). We use the Sasakian metric $g$ to identify the quotient space in the right side above with the holomorphic vector fields that are sections of $(\mathcal{D}, J)$, which we shall refer to from here on as $\mathfrak{h}(\xi, \mathcal{D}, J)$. The notation for this Lie algebra is a bit non-standard in that $\mathfrak{h}(\xi, \mathcal{D}, J)$ depends on $\xi$ or rather on the foliation $\mathcal{F}_\xi$ and are holomorphic sections of $\mathcal{D}$ with respect to $J$. It should be noted, however, that while elements in $\mathfrak{h}(\xi, \mathcal{D}, J)$ leave both $\mathcal{F}_\xi$ and $\bar{J}$ invariant, they do not necessarily leave $\mathcal{D}$ invariant.

We also define the operator $\bar{L}^B_g$ on $\mathcal{H}^B_g$ by $\bar{L}^B_g\varphi = \overline{L^B_g\bar{\varphi}}$. It follows that

$$(\bar{L}^B_g - L^B_g)\varphi = \partial^\#_g s_g \lrcorner \partial\varphi - \partial^\#_g \varphi \lrcorner \partial s_g \,, \tag{28}$$

where $s_g$ is the scalar curvature of $g$. The fact that $s_g$ is a basic function implies that $\partial^{\#}_g s_g$ is a $(1,0)$ section of $(\mathcal{D}, J)$. The identity above implies that $L^B$ and $\bar{L}^B$ coincide if $s_g$ is constant. If the metric $g$ is canonical, then we have that $\partial^{\#}_g s_g \in \mathfrak{h}(\xi, \mathcal{D}, J)$, and the operators $L^B$ and $\bar{L}^B$ commute.

The image $\mathfrak{h}_0 \cong \mathcal{H}^B_g/\mathbb{C}$ of the mapping (27) is an ideal in $\mathfrak{h}(\xi, \mathcal{D}, J)$, and can be identified with the space of holomorphic fields that have non-empty zero set. The quotient algebra $\mathfrak{h}(\xi, \mathcal{D}, J)/\mathfrak{h}_0$ is Abelian. We also denote by $\mathfrak{aut}(\bar{J}, g_T)$ the Lie subalgebra of $\mathfrak{h}^T(\xi, \nu(\mathcal{F}_\xi))$ that are holomorphic Killing vector fields of the transverse metric $g_T$, that is

$$\mathfrak{aut}(\bar{J}, g_T) = \{\bar{X} \in \mathfrak{h}(\xi, \mathcal{D}, J) \mid \pounds_{\bar{X}} g_T = 0\} . \tag{29}$$

Suppose now that $(\xi, \eta, \Phi, g)$ is a canonical representative of $\mathcal{S}(\xi, \bar{J})$, so that $g$ is Sasaki extremal. Let $\mathfrak{z}_0$ be the image under $\partial^{\#}_g$ of the set of purely imaginary functions in $\mathcal{H}^B_g$. This is just the space of Killing fields for the transversal metric $g^T$ that are of the form $J\nabla_{g^T}\varphi$, $\varphi \in \mathcal{H}^B_g$. Furthermore, by (28) we see that the complexification $\mathfrak{z}_0 \oplus \bar{J}\mathfrak{z}_0$ coincides with the commutator of $\partial^{\#}_g s_g$:

$$\mathfrak{z}_0 \oplus \bar{J}\mathfrak{z}_0 = \{X \in \mathfrak{h}(\xi, \mathcal{D}, J) \, : \, [X, \partial^{\#}_g s_g] = 0\} .$$

**Theorem 4.8.** *Let $(M, \xi, \bar{J})$ be a polarized Sasakian manifold. Suppose that there exists a canonical representative $(\xi, \eta, \Phi, g)$ of $\mathcal{S}(\xi, \bar{J})$. Let $\mathcal{H}^B_g$ be the space of basic functions in the kernel of the operator $L^B_g$ in* (11)*, and let $\mathfrak{h}_0$ be the image of the mapping* (27)*. Then we have the orthogonal decomposition*

$$\mathfrak{h}^T(\xi, \bar{J})/L_\xi \cong \mathfrak{h}(\xi, \mathcal{D}, J) = \mathfrak{a} \oplus \mathfrak{h}_0 ,$$

*where $\mathfrak{a}$ is the algebra of parallel vector fields of the transversal metric $g^T$. Furthermore,*

$$\mathfrak{h}_0 = \mathfrak{z}_0 \oplus \bar{J}\mathfrak{z}_0 \oplus (\oplus_{\lambda>0}\mathfrak{h}^\lambda) ,$$

*where $\mathfrak{z}_0$ is the image of the purely imaginary elements of $\mathcal{H}^B_g$ under $\partial^{\#}_g$, and $\mathfrak{h}^\lambda = \{\bar{X} \in \mathfrak{h}^T(\xi, \bar{J})/L_\xi \, : \, [\bar{X}, \partial^{\#}_g s_g] = \lambda \bar{X}\}$. Moreover, $\mathfrak{z}_0$ is isomorphic to the quotient algebra $\mathfrak{aut}(\xi, \eta, \Phi, g)/\{\xi\}$, so the Lie algebra $\mathfrak{aut}(\bar{J}, g_T)$ of Killing vector fields for the transversal metric $g^T$ is equal to*

$$\mathfrak{aut}(\bar{J}, g_T) = \mathfrak{a} \oplus \mathfrak{z}_0 \cong \mathfrak{a} \oplus \mathfrak{aut}(\xi, \eta, \Phi, g)/\{\xi\} .$$

The presence of the algebra $\mathfrak{a}$ above does not contradict the fact that there are no non-trivial parallel vector fields on a closed Sasakian manifold: a vector field can be parallel with respect to $g^T$ without being parallel with respect to $g$.

*Proof of Theorem* 4.8. We prove the last statement first. In order to see this, we notice that there is an exact sequence [3]

$$0 \to \{\xi\} \to \mathfrak{aut}(\xi,\eta,\Phi,g) \to \mathfrak{aut}(\bar{J},g_T) \xrightarrow{\delta} H^1_B(\mathcal{F}_\xi)\,, \tag{30}$$

where $H^1_B(\mathcal{F}_\xi)$ denotes the basic cohomology associated to the characteristic foliation $\mathcal{F}_\xi$. Using the identification of $\mathfrak{aut}(\bar{J},g_T)$ with elements in $\mathfrak{h}(\xi,\mathcal{D},J)$, let us describe the map $\delta$. Since $\bar{X}\in\mathfrak{aut}(\bar{J},g_T)$ it leaves $d\eta$ invariant, so the 1-form $\bar{X}\lrcorner d\eta$ is closed and basic. It, thus, defines an element in $H^1_B(\mathcal{F}_\xi)$. So we can define $\delta(\bar{X})=[\bar{X}\lrcorner d\eta]_B$. Now the section $\bar{X}\in\mathfrak{aut}(\bar{J},g_T)$ can be extended to an element $X=\bar{X}+a\xi\in\mathfrak{aut}(\xi,\eta,\Phi,g)$ if, and only if, the basic cohomology class $[\bar{X}\lrcorner d\eta]_B$ vanishes, and this determines $a$ up to a constant. By Hodge theory and duality, the image of $\delta$ can be identified with the Lie algebra of parallel vector fields in $\mathfrak{aut}(\bar{J},g_T)$. The splitting then follows as in the Kähler case [10].

For the first part of the theorem we sketch the main points, as the argument is an adaptation to our situation of that in [10]. Given a section $\bar{X}$ in $\mathfrak{h}(\xi,\mathcal{D},J)$, we look at the Hodge decomposition of the $(0,1)$-form that corresponds to it via the metric $g^T$. It is $\overline{\partial}$-closed, and both, its harmonic and $\overline{\partial}$ components, are the dual of holomorphic fields. The vector field dual to the harmonic component is $g^T$-parallel.

Since $g^T$ is an extremal metric, the operators $L^B_g$ and $\bar{L}^B_g$ commute. We then restrict $\bar{L}^B_g$ to the kernel of $L^B_g$, and use the resulting eigenspace decomposition together with the identity (28) to derive the remaining portion of the theorem. □

**Remark 4.9**. This result obstructs the existence of special canonical representatives of a polarized Sasakian manifold in the same way it does so in the Kählerian case. For instance, let $(N,\omega)$ be the one-point or two-points blow-up of $\mathbb{CP}^2$, and consider the circle bundle $\pi: M\to N$ with Euler class $[\omega]$. If $M$ is polarized by its natural Sasakian structure $(\xi,\eta,\Phi,g)$, the one where $\pi^*\omega=d\eta$, then $\mathcal{S}(\xi,\bar{J})$ cannot be represented by a Sasakian structure $(\xi,\tilde{\eta},\tilde{\Phi},\tilde{g})$ with $\tilde{g}$ a metric of constant scalar curvature. The structure of $\mathfrak{h}^T(\xi,\bar{J})/L_\xi$ would obstruct it. □

## 5. A Sasaki-Futaki invariant

Let $(M,\xi,\bar{J})$ be a polarized Sasakian manifold. Given any structure $(\xi,\eta,\Phi,g)\in\mathcal{S}(\xi,\bar{J})$, we denote its underlying CR structure by $(\mathcal{D},J)$. The metric $g$ is an element of $\mathfrak{M}(\xi,\bar{J})$ whose transversal Ricci form $\rho^T$ is basic. We define the Ricci potential $\psi_g$ as the function in the Hodge decomposition of $\rho^T$ given by

$$\rho^T = \rho^T_h + i\partial\overline{\partial}\psi_g \, ,$$

where $\rho^T_h$ is the harmonic representative of the foliated cohomology class represented by $\rho^T$. Notice that if $G^T_g$ is the Green's operator of the transversal metric, we have that

$$\psi_g = -G^T_g(s^T_g) = -G^T_g(s^T_g - 2n) = -G^T_g(s_g) = -G(s_g - s_{g,0}) \, ,$$

where $s_g$ and $G_g$ are the scalar curvature and Green's operator of $g$, and $s_{g,0}$ is the projection of $s_g$ onto the constants. The sequence of equalities above follows by (c) of Proposition 2.3, which implies that $s_g = s^T_g - 2n$ is a basic function. Thus, the Ricci potential $\psi_g$ is itself a basic function.

On $\mathfrak{h}^T(\xi, \bar{J})$, we define the function

$$X \mapsto \int X(\psi_g) d\mu_g \, .$$

Since $\psi_g$ is basic, the integrand in this expression can be fully written in terms of the transversally holomorphic realization $\Xi_X$ (see (25)) of $X$.

**Proposition 5.1.** *The mapping above only depends on the basic cohomology class represented by $d\eta$, and not on the particular transversal Kähler metric induced by $g \in \mathfrak{M}(\xi, \bar{J})$ that is used to represent it.*

*Proof.* We take a path $g_t$ in $\mathfrak{M}(\xi, \bar{J})$ starting at $g$ for which the transversal Kähler form is of the form

$$d\eta_t = d\eta + ti\partial\overline{\partial}\varphi \, ,$$

with the affine parameter $\varphi$ a basic function. From the identity $\Delta_B \psi_g = s^T_{g,0} - s^T$, we see that the variation $\dot{\psi}_g$ of $\psi_g$ satisfies the relation

$$2(i\partial\overline{\partial}\varphi, i\partial\overline{\partial}\psi_g)_g + \Delta_B \dot{\psi}_g = -\dot{s^T} = \frac{1}{2}\Delta_B^2 \varphi + 2(\rho^T, i\partial\overline{\partial}\varphi)_g \, .$$

Hence,

$$\dot{\psi}_g - \frac{1}{v}\int \dot{\psi}_g d\mu_g = \frac{1}{2}\Delta_B \varphi + 2G^T_g(\rho^T_h, i\partial\overline{\partial}\varphi)_g \, ,$$

where $v$ is the volume of $M$ in the metric.

Since $\rho^T_h$ is harmonic, the last summand in the right side can be written as $-2G^T_g(\partial^*(\overline{\partial}^*(\varphi\rho^T_h)))$. For convenience, let us set $\beta = \overline{\partial}^*(\varphi\rho^T_h)$. Hence,

$$\frac{d}{dt}\int X(\psi_t) d\mu_{g_t} = \int X\left(\frac{1}{2}\Delta_B\varphi - 2G^T_g(\partial^*\beta)) - \frac{1}{2}\psi\Delta_B\varphi\right) d\mu_g \, .$$

By the Ricci identity for the transversal metric, we have that

$$\frac{1}{2}(\Delta_B\varphi)_\alpha = -\varphi_{,\gamma\alpha}{}^{\gamma} + \varphi_{,\gamma}(\psi_{,\alpha}{}^{\gamma} + (r_h^T)_\alpha^{\gamma}) = -\varphi_{,\gamma\alpha}{}^{\gamma} + \varphi_{,\gamma}\psi_{,\alpha}{}^{\gamma} + \beta_\alpha\,,$$

and so, after minor simplifications, we conclude that

$$\begin{aligned} \frac{d}{dt}\int X(\psi_t)d\mu_{g_t} \;\; = \;\; & \int \Xi_X^\alpha\left(\varphi_{,\gamma}\psi_\alpha - \varphi_{,\gamma\alpha}\right)^\gamma d\mu_g + \\ & \int \Xi_X^\alpha\left(\beta_\alpha - 2(G_g\partial^*\beta)_{,\alpha}\right)d\mu_g\,, \end{aligned}$$

where $\Xi_X$ is the (1,0)-component of $X_{\mathcal{D}}$ (see (25)).

The first summand on the right above is zero because $\Xi_X$ is holomorphic. This is just a consequence of Stokes' theorem. The second summand is also zero since we have

$$\begin{aligned} \int \Xi_X^\alpha\left(\beta_\alpha - 2(G_g\partial^*\beta)_{,\alpha}\right)d\mu_g \;\; = \;\; & \int (\beta - \Delta_B G_g^T\beta, \Xi_X^\flat)d\mu_g + \\ & \int (2\partial^*\partial G_g^T\beta, \Xi_X^\flat)d\mu_g\,, \end{aligned}$$

and $\beta - \Delta_B G_g^T\eta = 0$ while $\partial\Xi_X^\flat = 0$. Here, of course, $\Xi_X^\flat$ is the $(0,1)$-basic form corresponding the $(1,0)$-vector field $\Xi_X$. $\square$

We may then define the transversal Futaki invariant $\mathfrak{F} = \mathfrak{F}_{(\xi,\bar{J})}$ of the polarized Sasakian manifold $(M,\xi,\bar{J})$ to be the functional

$$\begin{gathered} \mathfrak{F} : \mathfrak{h}^T(\xi,\bar{J}) \longrightarrow \mathbb{C} \\ \mathfrak{F}(X) = \int_M X(\psi_g)d\mu_g = -\int_M X(G_g^T s_g^T)d\mu_g\,, \end{gathered} \tag{31}$$

where $g$ is any metric in $\mathfrak{M}(\xi,\bar{J})$. The Proposition above shows that $\mathfrak{F}$ is well-defined, as this expression depends only on the basic class $[d\eta]$ of a Sasakian structure $(\xi,\eta,\Phi,g)$ in $\mathcal{S}(\xi,\bar{J})$, rather than the specific Sasakian structure chosen to represent it.

It is rather obvious that $\mathfrak{F}(X) = \mathfrak{F}(X_{\mathcal{D}})$, and the usual argument in the Kähler case implies also that $\mathfrak{F}([X,Y]) = 0$ for any pair of vector fields $X$, $Y$ in $\mathfrak{h}^T$.

Our next proposition extends to canonical Sasakian metrics a now well-known result in Kähler geometry originally due to Futaki [17]. The Sasaki version here is analogous to the expanded version of Futaki's result presented by Calabi [10].

**Proposition 5.2.** *Let $(\xi,\eta,\Phi,g)$ be a canonical Sasakian representative of $\mathcal{S}(\xi,J)$. Then, the metric $g$ has constant scalar curvature if, and only if, $\mathfrak{F}(\,\cdot\,) = 0$.*

*Proof.* In one direction the statement is obvious: a constant scalar curvature Sasakian metric has trivial Ricci potential function, and the functional (31) vanishes on any $X \in \mathfrak{h}^T(\xi, \bar{J})$.

In order to prove the converse, we first observe that if $X \in \mathfrak{h}^T(\xi, \bar{J})$ is a transversally holomorphic vector field of the form $X = \partial_g^{\#} f$ for some basic function $f$, then

$$\begin{aligned} \mathfrak{F}(X) = -\int \partial_g^{\#} f (G_g s_g) d\mu_g &= -2\int (\overline{\partial} f, \overline{\partial} G_g^T s_g)_g d\mu_g \\ &= -2\int f(\overline{\partial}_g^* \overline{\partial} G_g^T s_g) d\mu_g \,, \end{aligned}$$

because the scalar curvature $s_g$ is a basic function also. Since $2\overline{\partial}^* \overline{\partial} = \Delta_B$, we conclude that

$$\mathfrak{F}(\partial_g^{\#} f) = -\int f(s_g - s_{g,0}) d\mu_g \,.$$

Now, if the Sasakian metric $g$ is a critical point of the energy function $E$ in (7), then $\partial_g^{\#} s = \partial^{\#} s^T = \partial_g^{\#} s_g$ is a transversally holomorphic vector field, and we conclude that

$$\mathfrak{F}(\partial_g^{\#} s^T) = -\int (s_g - s_{g,0})^2 d\mu_g \,.$$

Thus, if $\mathfrak{F}(\,\cdot\,) = 0$, then $s_g$ must be constant. □

A particular case of constant scalar curvature Sasakian metrics is the case of Sasakian $\eta$-Einstein metrics. These are Sasakian metrics $g$ that satisfy

$$\mathrm{Ric}_g = \lambda g + \nu \eta \otimes \eta \tag{32}$$

for some constants $\lambda$ and $\nu$. The scalar curvature $s_g$ of these metrics is given by $s_g = 2n(1+\lambda)$. We refer the reader to [6], and references therein, for further discussion of these type of metrics.

**Corollary 5.3.** *Let $(\xi, \eta, \Phi, g)$ be a canonical representative of $\mathcal{S}(\xi, \bar{J})$, and suppose that the basic first Chern class $c_1(\mathcal{F}_\xi)$ is a constant multiple, say $a$, of $[d\eta]_B$. Then*

1. *If $a = 0$, then $(\xi, \eta, \Phi, g)$ is a null $\eta$-Einstein Sasakian structure with $\lambda = -2$, whose transverse metric is Calabi-Yau.*
2. *If $a < 0$, then $(\xi, \eta, \Phi, g)$ is a negative $\eta$-Einstein Sasakian structure with $\lambda < -2$, whose transverse metric is Kähler-Einstein with negative scalar curvature.*
3. *If $a > 0$, then $(\xi, \eta, \Phi, g)$ is a positive $\eta$-Einstein Sasakian structure with $\lambda > -2$, whose transverse metric is positive Kähler-Einstein if, and only if, the Futaki-Sasaki invariant $\mathfrak{F}_{\xi, \bar{J}}$*

*vanishes. Moreover, if $\mathfrak{F}_{\xi,\bar{J}}$ vanishes, $g \in \mathfrak{M}(\xi, \bar{J})$ is Sasaki-Einstein if, and only if, $\lambda = 2n$.*

*When an $\eta$-Einstein metric exists, the relation $2\pi a = \lambda + 2$ holds.*

*Proof.* Parts (1) and (2) follow from Proposition 5.2 and Theorem 17 of [6]. For (3) we notice that if $g$ is positive Sasakian $\eta$-Einstein, the result follows immediately from Proposition 5.2. Conversely, if $g$ is a canonical representative $\mathcal{S}(\xi, \bar{J})$ and $\mathfrak{F}_{\xi,\bar{J}}$ vanishes, its scalar curvature is constant. It follows that the scalar curvature of the transversal metric is constant also, and this implies that $\rho_g + 2d\eta = \rho^T$ is transversally harmonic. As the latter form represents $2\pi c_1(\mathcal{F}_\xi)$, which is also represented by a constant multiple of $d\eta$, the uniqueness of the harmonic representative of a class implies that $\rho_g + 2d\eta = \rho^T = 2\pi a d\eta$ for some $a > 0$. It then follows from this that the transverse Ricci tensor $\mathrm{Ric}_T$ satisfies $\mathrm{Ric}_T = 2\pi a g_T$. But then $g$ is a positive Sasakian $\eta$-Einstein metric, and it follows from Equation 32 that $2\pi a = \lambda + 2$. □

This Corollary applies whenever the first Chern class $c_1(\mathcal{D})$ of the contact bundle is a torsion class. We mention also that one can always obtain a Sasaki-Einstein metric from a positive Sasakian $\eta$-Einstein metric by applying a transverse homothety.

## 6. The Sasaki cone

We have the following result for CR structures of Sasaki type on $M$, an immediate consequence of the argument in the proof of Lemma 4.2.

**Proposition 6.1.** *Let $(\mathcal{D}, J)$ be a* CR *structure of Sasaki type on $M$, and let $\mathcal{S} = (\xi, \eta, \Phi, g)$ be a contact metric structure whose underlying* CR *structure is $(\mathcal{D}, J)$. Then $\mathcal{S}$ is a Sasakian structure if and only if $\xi \in \mathfrak{cr}(\mathcal{D}, J)$.*

We fix a strictly pseudoconvex CR structure $(\mathcal{D}, J)$ on $M$, and define the set

$$\mathcal{S}(\mathcal{D}, J) = \left\{ \begin{array}{c} \mathcal{S} = (\xi, \eta, \Phi, g) : \ \mathcal{S} \text{ a Sasakian structure} \\ (\ker \eta, \Phi\, |_{\ker \eta}) = (\mathcal{D}, J) \end{array} \right\}. \tag{33}$$

We think of this as a subspace of sections of a vector bundle, and provide it with the $C^\infty$ compact-open topology. This set is nonempty if, and only if, $(\mathcal{D}, J)$ is of Sasaki type.

**Proposition 6.2.** *Let $(\mathcal{D}, J)$ be a* CR *structure of Sasaki type, and let $\mathcal{S}_0 = (\xi_0, \eta_0, \Phi_0, g_0) \in \mathcal{S}(\mathcal{D}, J)$. If $\mathcal{S} = (\xi, \eta, \Phi, g) \in \mathcal{S}(\mathcal{D}, J)$, we have that $\eta_0(\xi) > 0$, and $\eta = \dfrac{\eta_0}{\eta_0(\xi)}$.*

*Proof.* Using the canonical splitting (1), we write any 1-form $\eta$ as $\eta = f\eta_0 + \alpha$, with $\eta_0$ and $\alpha$ orthogonal to each other. As the kernels of $\eta$ and $\eta_0$ equal $\mathcal{D}$, we must have $\alpha = 0$. Since $\eta$ is a contact form, the function $f$ is nowhere vanishing, and since $\Phi\mid_{\mathcal{D}} = J = \Phi_0\mid_{\mathcal{D}}$, $f$ must be positive. The result follows. □

We thus see that the underlying CR structure fixes both, orientation and co-orientation of the contact structure.

**Definition 6.3.** *Let* $(\mathcal{D}, J)$ *be a strictly pseudoconvex* CR *structure of Sasaki type. We say that a vector field* $X \in \mathfrak{cr}(\mathcal{D}, J)$ *is* **positive** *if* $\eta(X) > 0$ *for any* $\mathcal{S} = (\xi, \eta, \Phi, g) \in \mathcal{S}(\mathcal{D}, J)$. *We denote by* $\mathfrak{cr}^+(\mathcal{D}, J)$ *the subset of all positive elements of* $\mathfrak{cr}(\mathcal{D}, J)$.

We consider the mapping $\iota$ defined by projection,

$$
\begin{array}{ccc} \mathcal{S}(\mathcal{D}, J) & \stackrel{\iota}{\rightarrow} & \mathfrak{cr}^+(\mathcal{D}, J) \\ \mathcal{S} & \mapsto & \xi \end{array} . \tag{34}
$$

By Proposition 6.2, we see that this mapping is injective.

We have the following.

**Lemma 6.4.** *Let* $(\mathcal{D}, J)$ *be a strictly pseudoconvex* CR *structure of Sasaki type. Then*

a) $\mathfrak{cr}^+(\mathcal{D}, J)$ *is naturally identified with* $\mathcal{S}(\mathcal{D}, J)$,
b) $\mathfrak{cr}^+(\mathcal{D}, J)$ *is an open convex cone in* $\mathfrak{cr}(\mathcal{D}, J)$,
c) *The subset* $\mathfrak{cr}^+(\mathcal{D}, J)$ *is invariant under the adjoint action of the Lie group* $\mathfrak{CR}(\mathcal{D}, J)$.

*Proof.* In order to prove (a), we show that the map $\iota$ in (34) is surjective. As in Proposition 6.2, we fix a Sasaki structure $\mathcal{S}_0 = (\xi_0, \eta_0, \Phi_0, g_0)$ in $\mathcal{S}(\mathcal{D}, J)$. For $\xi \in \mathfrak{cr}^+(\mathcal{D}, J)$, we define a 1-form $\eta$ by

$$
\eta = \frac{\eta_0}{\eta_0(\xi)} .
$$

Then, $\eta(\xi) = 1$, and since $\xi \in \mathfrak{cr}^+(\mathcal{D}, J)$, $\xi$ leaves $\mathcal{D}$ invariant. This implies that $\xi \lrcorner d\eta = \pounds_\xi \eta = 0$. Thus, $\xi$ is the Reeb vector field of $\eta$. We then define $\Phi$ by $\Phi = \Phi_0 - \Phi_0(\xi) \otimes \eta$, and a metric $g$ by (4). The structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ belongs to $\mathcal{S}(\mathcal{D}, J)$, and thus, $\iota$ is surjective.

For the proof of (b), we observe that $\mathfrak{cr}^+(\mathcal{D}, J)$ is open and convex, and that if $\xi \in \mathfrak{cr}^+(\mathcal{D}, J)$, then so is $a\xi$ for any positive real number $a$. Indeed, all of these follow by the defining condition of positivity of a vector field $X$ in $\mathfrak{cr}(\mathcal{D}, J)$.

For the final assertion, we observe that for groups of transformations, the adjoint action is that induced by the differential. Thus, given $\phi \in \mathfrak{CR}(\mathcal{D}, J)$ and $\xi \in \mathfrak{cr}^+(\mathcal{D}, J)$, we have that $\eta_0(\phi_* \xi) = (\phi^* \eta_0)(\xi) =$

$f_\phi \eta_0(\xi) > 0$ for some positive function $f_\phi$, which shows that $\phi_*\xi \in \mathfrak{cr}^+(\mathcal{D}, J)$. Thus, $\mathfrak{cr}^+(\mathcal{D}, J)$ is invariant. $\square$

Hereafter, we shall identify the spaces $\mathcal{S}(\mathcal{D}, J)$ and $\mathfrak{cr}^+(\mathcal{D}, J)$. We are interested in the action of the Lie group $\mathfrak{CR}(\mathcal{D}, J)$ on $\mathcal{S}(\mathcal{D}, J) = \mathfrak{cr}^+(\mathcal{D}, J)$.

**Theorem 6.5.** *Let $M$ be a closed manifold of dimension $2n+1$, and let $(\mathcal{D}, J)$ be a* CR *structure of Sasaki type on it. Then the Lie algebra $\mathfrak{cr}(\mathcal{D}, J)$ decomposes as $\mathfrak{cr}(\mathcal{D}, J) = \mathfrak{t}_k + \mathfrak{p}$, where $\mathfrak{t}_k$ is the Lie algebra of a maximal torus $T_k$ of dimension $k$, $1 \leq k \leq n+1$, and $\mathfrak{p}$ is a completely reducible $T_k$-module. Furthermore, every $X \in \mathfrak{cr}^+(\mathcal{D}, J)$ is conjugate to a positive element in the Lie algebra $\mathfrak{t}_k$.*

*Proof.* Let us assume first that $M$ is not the sphere with its standard CR structure. By Proposition 4.4, there is a Sasaki structure $\mathcal{S}_0 \in \mathcal{S}(\mathcal{D}, J)$ such that $\mathfrak{CR}(\mathcal{D}, J) = \mathfrak{Aut}(\mathcal{S}_0)$, which is a compact Lie group. A well known Lie theory result implies that every element in the Lie algebra $\mathfrak{aut}(\mathcal{S}_0)$ is conjugate under the adjoint action of the group $\mathfrak{Aut}(\mathcal{S}_0)$ to one on $\mathfrak{t}_k$, and by (3) of Lemma 6.4, the positivity is preserved under this action. The possible restriction on the dimension of the maximal torus of $\mathfrak{cr}(\mathcal{D}, J)$ is well-known in Sasakian geometry.

In the case where $(\mathcal{D}, J)$ is the standard CR structure on the sphere, we know [36] that $\mathfrak{CR}(\mathcal{D}, J) = \mathbb{SU}(n+1, 1)$, and $\mathfrak{cr}(\mathcal{D}, J) = \mathfrak{su}(n+1, 1)$, which has several maximal Abelian subalgebras. A case by case analysis shows that the only Abelian subgroup where the positivity condition can be satisfied is in that of a maximal torus. (This can be ascertained, for instance, by looking at Theorem 6 of [15].) $\square$

We wish to study further the action of the Lie group $\mathfrak{CR}(\mathcal{D}, J)$ on the space $\mathcal{S}(\mathcal{D}, J)$. The isotropy subgroup of an element $\mathcal{S} \in \mathcal{S}(\mathcal{D}, J)$ is, by definition, $\mathfrak{Aut}(\mathcal{S})$, and this contains the torus $T_k$. More generally, we have

**Lemma 6.6.** *Let $(\mathcal{D}, J)$ be a* CR *structure of Sasaki type on $M$. For each $\mathcal{S} \in \mathcal{S}(\mathcal{D}, J)$, the isotropy subgroup of $\mathfrak{CR}(\mathcal{D}, J)$ at $\mathcal{S}$ is precisely $\mathfrak{Aut}(\mathcal{S})$. Furthermore,*

$$\bigcap_{\mathcal{S} \in \mathcal{S}(\mathcal{D}, J)} \mathfrak{Aut}(\mathcal{S}) = T_k \,.$$

*In particular, $T_k$ is contained in the isotropy subgroup of every $\mathcal{S} \in \mathcal{S}(\mathcal{D}, J)$.*

*Proof.* It suffices to show that for the generic Reeb vector field $\mathfrak{Aut}(\mathcal{S}) = T_k$. So let $\xi \in \mathfrak{cr}^+(\mathcal{D}, J)$ be such that the leaf closure of $\mathcal{F}_\xi$ is a $k$-dimensional torus $T_k$. Since the Reeb field is in the center of

$\mathfrak{Aut}(\mathcal{S})$, continuity implies that all of $T_k$ is in the center of $\mathfrak{Aut}(\mathcal{S})$. But since the center is Abelian and $T_k$ is maximal, the result follows. □

We are interested in the orbit space $\mathcal{S}(\mathcal{D},J)/\mathfrak{CR}(\mathcal{D},J)$. We have

**Definition 6.7.** *Let $(\mathcal{D},J)$ be a* CR *structure of Sasaki type on $M$. We define the* **Sasaki cone** *$\kappa(\mathcal{D},J)$ to be the moduli space of Sasakian structures compatible with $(\mathcal{D},J)$,*

$$\kappa(\mathcal{D},J)=\mathcal{S}(\mathcal{D},J)/\mathfrak{CR}(\mathcal{D},J)\,.$$

Theorem 6.5 together with the mapping (34) says that each orbit can be represented by choosing a positive element in the Lie algebra $\mathfrak{t}_k$ of a maximal torus $T_k$. We denote the subset of positive elements by $\mathfrak{t}_k^+$, so we have an identification $\mathfrak{t}_k^+=\mathfrak{t}_k\cap\mathfrak{cr}^+(\mathcal{D},J)\approx\kappa(\mathcal{D},J)$.

Now the basic Chern class of a Sasakian structure $\mathcal{S}=(\xi,\eta,\Phi,g)$ is represented by the Ricci form $\rho^T/2\pi$ of the transverse metric $g_T$ (up to a factor of $2\pi$). Although the notion of basic changes with the Reeb vector field, the complex vector bundle $\mathcal{D}$ remains fixed. Hence, for any Sasakian structure $\mathcal{S}\in\mathcal{S}(\mathcal{D},J)$, the transverse 2-form $\rho^T/2\pi$ associated to $\mathcal{S}$ represents the first Chern class $c_1(\mathcal{D})$ of the complex vector bundle $\mathcal{D}$.

It is of interest to consider the case where $k=1$, that is to say, the case where the maximal torus of $\mathfrak{CR}(\mathcal{D},J)$ is one dimensional. Since the Reeb vector field is central, the hypothesis that $k=1$ implies that $\dim\mathfrak{aut}(\mathcal{S})=\dim\mathfrak{cr}(\mathcal{D},J)=1$. Hence, we have that $\mathcal{S}(\mathcal{D},J)=\mathfrak{cr}^+(\mathcal{D},J)=\mathfrak{t}_1^+=\mathbb{R}^+$, and $\mathcal{S}(\mathcal{D},J)$ consists of the 1-parameter family of Sasaki structures given by $\mathcal{S}_a=(\xi_a,\eta_a,\Phi_a,g_a)$, where

$$\xi_a=a^{-1}\xi\,,\quad \eta_a=a\eta\,,\quad \Phi_a=\Phi\,,\quad g_a=ag+(a^2-a)\eta\otimes\eta\,, \tag{35}$$

and $a\in\mathbb{R}^+$, the 1-parameter family of transverse homotheties.

In effect, the homotheties described above are the only deformations $(\xi_t,\eta_t,\Phi_t,g_t)$ of a given structure $\mathcal{S}=(\xi,\eta,\Phi,g)$ in the Sasaki cone $\kappa(\mathcal{D},J)$ where the Reeb vector field varies in the form $\xi_t=f_t\xi$, $f_t$ a scalar function. For we then have that the family of tensors $\Phi_t$ is constant, and since $\pounds_{\xi_t}\Phi_t=0$, we see that $f_t$ must be annihilated by any section of the sub-bundle $\mathcal{D}$. But then (2) implies that $df_t=(\xi f_t)\eta$, and we conclude that the function $f_t$ is constant. Thus, in describing fully the tangent space of $\mathcal{S}(\mathcal{D},J)$ at $\mathcal{S}$, it suffices to describe only those deformations $(\xi_t,\eta_t,\Phi_t,g_t)$ where $\dot{\xi}=\partial_t\xi_t\mid_{t=0}$ is $g$-orthogonal to $\xi$. These correspond to deformations where the volume of $M$ in the metric $g_t$ remains constant in $t$, and are parametrized by elements of $\mathfrak{k}(\mathcal{D},J)$ that are $g$-orthogonal to $\xi$.

The terminology we use here is chosen to emphasize the fact that the Sasaki cone is to a CR structure of Sasaki type what the Kähler cone is to a complex manifold of Kähler type. Indeed, for any point $\mathcal{S} = (\xi, \eta, \Phi, g)$ in $\kappa(\mathcal{D}, J)$, the complex normal bundle $(\nu(\mathfrak{F}_\xi), \bar{J})$ is isomorphic to $(\mathcal{D}, J)$, and so is the underlying CR structure of any element of $\mathcal{S}(\xi, \bar{J})$. In this sense, the complex structure $\bar{J}$ is fixed with the fixing of $(\mathcal{D}, J)$, the Reeb vector field $\xi$ polarizes the manifold, and the Sasaki cone $\kappa(\mathcal{D}, J)$ is the set of all possible polarizations.

**Definition 6.8.** *We say that* $(\xi, \eta, \Phi, g) \in \kappa(\mathcal{D}, J)$ *is a* **canonical** *element of the Sasaki cone if the space* $\mathcal{S}(\xi, \bar{J})$ *admits a canonical representative. We denote by* $\mathfrak{e}(\mathcal{D}, J)$ *the set of all canonical elements of the Sasaki cone, and refer to it as the* **canonical Sasaki set** *of the* CR *structure* $(\mathcal{D}, J)$*.*

By the identification of $\kappa(\mathcal{D}, J)$ with $\mathfrak{t}_k^+$, the canonical Sasaki cone singles out the subset of positive Reeb vector fields $\xi$ in $\mathfrak{t}_k^+$ for which the functional (7) admits a critical point.

## 7. Openness of the canonical Sasaki set

Given a canonical Sasakian structure $(\xi, \eta, \Phi, g)$ with underlying CR structure $(\mathcal{D}, J)$, its isometry group will contain the torus $T_k$ of Theorem 6.5. In fact by Lemma 6.6, for a generic element $\xi \in \mathfrak{t}_k^+$, $T_k$ will be exactly the isometry group of $g$. Moreover, Theorem 4.8 says that the isometry group of the transversal metric $g^T$ is a maximal compact subgroup $G$ of the identity component of the automorphism group of the transverse holomorphic structure, and the reductive part of the Lie algebra $\mathfrak{h}^T(\xi, \bar{J})/L_\xi$ consists of the complexification of the Lie algebra of all Killing vector fields for $g^T$ that are Hamiltonian. If $\mathfrak{g}$ is the Lie algebra of $G$ and $\mathfrak{g}_0$ is the ideal of Killing fields of $g^T$ that have zeroes, then $\mathfrak{z}_0 \subset \mathfrak{g}_0$ consists of sections of $(\mathcal{D}, J)$ that are Hamiltonian gradients, and these can be lifted [3] to infinitesimal automorphisms of $(\xi, \eta, \Phi, g)$. Hence, these vector fields are the transversal gradients of functions in $M$ that are $T_k$-invariant, or to put it differently, they are generated by those elements of the Lie algebra $\mathfrak{t}_k$ that correspond to transversal holomorphic gradient sections of $(\mathcal{D}, J)$. Thus, in searching for canonical representatives of elements of the Sasaki cone $\kappa(\mathcal{D}, J)$, it will suffice to consider Sasakian structures that are invariant under $T_k$, and then seek the canonical representatives among them.

We denote by $\mathcal{S}(\xi, \bar{J})^{T_k}$ the collection of all Sasakian structures in $\mathcal{S}(\xi, \bar{J})$ that are $T_k$-invariant, and by $\mathfrak{M}(\xi, \bar{J})^{T_k}$ the space of all $T_k$-invariant metrics in $\mathfrak{M}(\xi, \bar{J})$. The observation made above indicates that, in order to seek canonical representatives of $\mathcal{S}(\xi, \bar{J})$, it would suffices to do so among metrics in $\mathfrak{M}(\xi, \bar{J})^{T_k}$.

Given $(\xi, \eta, \Phi, g) \in \mathcal{S}(\xi, \bar{J})^{T_k}$, we let $G_g$ stand for the Green's operator of $g$ acting on functions. We consider a basis $\{X_1, \dots, X_{k-1}\}$ of $\mathfrak{z}_0 \cap \mathfrak{t}_k$. Then the set of functions

$$
\begin{array}{rcl}
p_0(g) & = & 1 \\
p_j(g) & = & 2iG_g\overline{\partial}^*_g((JX_j + iX_j)\lrcorner d\eta)\, , \quad j = 1, \dots, k-1\, ,
\end{array}
\tag{36}
$$

spans the space of $T_k$-invariant basic real-holomorphy potentials, real-valued functions solutions of equation (10) whose $g^T$-gradients are holomorphic vector fields. Since the argument function on which $G_g$ acts in order to define $p_j(g)$ is basic, we could have used the Green's operator of $g^T$ above instead of $G_g$ itself.

**Definition 7.1.** *We define $\pi_g$ to be the $L^2$-projection onto the space of smooth real holomorphic potential functions in* (36).

By Theorem 3.5, $(\xi, \eta, \Phi, g)$ is a canonical representative of $\mathcal{S}(\xi, \bar{J})$ if and only if $(1 - \pi_g)s_g = 0$, $s_g$ the scalar curvature of $g$.

We denote by $L^2_{B,l,T_k}$ the Hilbert space of of $T_k$-invariant basic real-valued functions of class $L^2_l$. We consider deformations $(\xi_\alpha, \eta_\alpha, \Phi_\alpha, g_\alpha)$ of $(\xi, \eta, \Phi, g)$ in the Sasaki cone $\kappa(\mathcal{D}, J)$ where the Reeb vector field varies as $\xi_\alpha = \xi + \alpha$. We require that $\alpha$ be in a sufficiently small neighborhood of the origin in $\mathfrak{cr}(\mathcal{D}, J)$ so that $\xi_\alpha$ remains positive. For $\varphi$ in a sufficiently small neighborhood of the origin in $L^2_{B,l+4,T_k}$, $l > n$, we then consider the deformations of $(\xi_\alpha, \eta_\alpha, \Phi_\alpha, g_\alpha)$ in $\mathcal{S}(\xi_\alpha, \bar{J})$ to the Sasakian structure defined by

$$
\begin{array}{rcl}
\eta_{\alpha,\varphi} & = & \eta_\alpha + d^c\varphi\, , \\
\Phi_{\alpha,\varphi} & = & \Phi_\alpha - (\xi_\alpha \otimes (\eta_{\alpha,\varphi} - \eta_\alpha)) \circ \Phi_\alpha\, , \\
g_{\alpha,\varphi} & = & d\eta_{\alpha,\varphi} \circ (\mathbb{1} \otimes \Phi_{\alpha,\varphi}) + \eta_{\alpha,\varphi} \otimes \eta_{\alpha,\varphi}\, .
\end{array}
$$

Here, for $(\alpha, \varphi) = (0, 0)$, we have that $(\xi_\alpha, \eta_{\alpha,\varphi}, \Phi_{\alpha,\varphi}, g_{\alpha,\varphi}) = (\xi, \eta, \Phi, g)$. The restriction on $l$ ensures that the curvature tensors of $g_{\alpha,\varphi}$ are all well-defined because, under such a constraint, $L^2_{B,l,T_k}$ is a Banach algebra.

We let $\mathcal{U} \subset \mathfrak{cr}(\mathcal{D}, J) \times L^2_{B,l+4,T_k}(M)$ be the open neighborhood of $(0, 0)$ where the two-parameter family of deformations $g_{\alpha,\varphi}$ of $g$ is well-defined, and consider the scalar curvature map

$$
\begin{array}{rcccl}
\mathfrak{cr}(\mathcal{D}, J) \times L^2_{B,l+4,T_k}(M) \supset & \mathcal{U} & \xrightarrow{S} & L^2_{B,l,T_k}(M) \\
& (\alpha, \varphi) & \mapsto & s_{g_{\alpha,\varphi}}\, ,
\end{array}
\tag{37}
$$

where $s_{g_{\alpha,\varphi}}$ is the scalar curvature of the metric $g_{\alpha,\varphi}$.

**Proposition 7.2.** *For $l > n$, the map (37) is well-defined and $C^1$, with Fréchet derivative at the origin given by*

$$DS_{(0,0)} = \begin{bmatrix} -n\Delta_B\,(\eta(\,\cdot\,)) & -\frac{1}{2}(\Delta_B^2 + 2r^T \cdot \nabla_T\nabla_T) \end{bmatrix}\,, \tag{38}$$

*where the quantities in the right are associated to the transversal metric $g^T$ defined by $g$, and $r^T \cdot \nabla_T\nabla_T$ denotes the full contraction of the Ricci tensor and two covariant derivatives of $g^T$.*

*Proof.* Notice that when deforming $(\xi, \eta, \Phi, g)$ to $(\tilde{\xi}, \tilde{\eta}, \tilde{\Phi}, \tilde{g})$ while preserving the underlying CR structure $(\mathcal{D}, J)$, the transversal Kähler form $d\eta$ changes by the conformal factor $f(\tilde{\xi}) = 1/\eta(\tilde{\xi})$. The first component of the Fréchet derivative above follows via a simple calculation, after observing that the Ricci tensor of the transversal Kähler metric is computed in a holomorphic frame by $-i\partial\overline{\partial}\log\det(g^T_{i\bar{k}})$. The second component of the Fréchet derivative follows by Proposition 3.2. □

For any integer $l$, we let $I_l \subset L^2_{B,l,T_k}$ denote the orthogonal complement of the kernel of the operator $L^B_g = (\bar{\partial}\partial^{\#}_g)^*\bar{\partial}\partial_g$ in (11), and set $\mathcal{V} = \mathcal{U} \cap (\mathfrak{cr}(\mathcal{D}, J) \times I_{k+4})$, where $\mathcal{U}$ is the neighborhood of $(0,0)$ in $\mathfrak{cr}(\mathcal{D}, J) \times L^2_{B,k+4,T_k}$ in (37), shrunk if necessary so that

$$\ker(1 - \pi_g)(1 - \pi_{g_{\alpha,\varphi}}) = \ker(1 - \pi_{g_{\alpha,\varphi}})$$

whenever we have a Sasaki metric $g_{\alpha,\varphi}$ of the type indicated above, parametrized by some $(\alpha, \varphi) \in \mathcal{U}$. Here, $\pi_g$ is the projection onto the finite dimensional space of functions (36) introduced in Definition 7.1. It is clear that the range of this projection changes smoothly with the metric.

Since a Sasaki metric $g$ is canonical if, and only if, its scalar curvature is annihilated by the projection operator $1 - \pi_g$, we introduce the map

$$\begin{array}{rcl} \mathfrak{cr}(\mathcal{D}, J) \times I_{l+4} \supset \mathcal{V} & \stackrel{\mathcal{S}}{\longrightarrow} & \mathfrak{cr}(\mathcal{D}, J) \times I_k \\ \mathcal{S}(\alpha, \varphi) & := & (\alpha, (1 - \pi_g)(1 - \pi_{g_{\alpha,\varphi}})\, S(\alpha, \varphi)\,) \end{array}\,, \tag{39}$$

where $S(\alpha, \varphi)$ is the map in (37).

We have the following.

**Lemma 7.3.** *Suppose that $g$ is a canonical Sasaki metric representing the polarization of $M$ given by $(\xi, \bar{J})$, and that $g_t = g_{t,\alpha,\varphi}$ is a curve of Sasakian metrics of the type above that starts at $g$ when $t = 0$, and is parametrized by $(\alpha, \varphi) \in \mathfrak{cr}(\mathcal{D}, J) \times L^2_{B,l+4,T_k}$. Then*

$$(1 - \pi_g)\,(\frac{d}{dt}\pi_{g_t})\bigg|_{t=0} s_g = (1 - \pi_g)[-2iG_g\overline{\partial}^*\left((\eta(\alpha))\,\overline{\partial}s_g\right) + (\partial s_g \lrcorner \partial^{\#}\varphi)]\,,$$

*where $G_g$ is the Green's operator of $g$.*

*Proof.* The result is clear if the basic scalar curvature function $s_g$ is constant. For the general case, we refer the reader to the original argument in [22] for the Kähler case (see also the significantly improved understanding, and related discussions, given in [35]). Its required extension follows by the observation that the transversal metric of $g$ is deformed by the conformal factor $1/\eta(\xi_{t,\alpha,\varphi})$, as noted before. □

**Proposition 7.4.** *For $l > n$, the map* (39) *is $C^1$ with Fréchet derivative at the origin given by*

(40)

$$D\mathcal{S}_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1-\pi_g \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -n\Delta_B\,(\eta(\,\cdot\,)) + 2iG_g\overline{\partial}^*\big((\eta(\,\cdot\,))\,\overline{\partial}s_g\big) & -2L_g^B \end{pmatrix},$$

*where $L_g^B = (\bar\partial\partial^{\#})^*\bar\partial\partial^{\#}$.*

**Proposition 7.5.** *Let $M$ be a closed manifold, and $(\mathcal{D}, J)$ be a* CR *structure of Sasaki type on it. Then the map $\mathcal{S}$ defined in* (39) *becomes a diffeomorphism when restricted to a sufficiently small neighborhood of the origin.*

*Proof.* We apply the inverse function theorem for Banach spaces. Hence, we just need to prove that $D\mathcal{S}_{(0,0)}$ has trivial kernel and cokernel.

Suppose that $(\alpha, \varphi)$ is in the kernel of $D\mathcal{S}_{(0,0)}$. By (40), we see that $\alpha = 0$, and that

$$(1-\pi_g)L_g^B\varphi = 0\,.$$

It follows that $L_g^B\varphi = (\bar\partial\partial^{\#})^*\bar\partial\partial^{\#}\varphi$ is a holomorphy potential, and consequently, it can be written as

$$(\bar\partial\partial^{\#})^*\bar\partial\partial^{\#}\varphi = \sum_j c_j f_g^j\,,$$

in terms of an orthonormal basis of the space spanned by the functions in (36). If we take the inner product of this expression with $f_g^j$, and dualize the symmetric map $(\bar\partial\partial^{\#})^*\bar\partial\partial^{\#}$, we see that $c_j = 0$. Thus, $(\bar\partial\partial^{\#})^*\bar\partial\partial^{\#}\varphi = 0$. But $\varphi \in I_{l+4}$, space orthogonal to the kernel of $(\bar\partial\partial^{\#})^*\bar\partial\partial^{\#}$. So $\varphi$ must be zero, and the kernel of $D\mathcal{S}_{(0,0)}$ consists of the point $(0, 0)$.

Suppose now that $(\beta, \psi)$ is orthogonal to every element in the image of $D\mathcal{S}_{(0,0)}$. Then, it must be orthogonal to the image of $(0, \varphi)$ for any $\varphi \in I_{l+4}$, and therefore,

$$\langle (1-\pi_g)(\bar\partial\partial^{\#})^*\bar\partial\partial^{\#}\varphi, \psi\rangle = \langle \varphi, (\bar\partial\partial^{\#})^*\bar\partial\partial^{\#}(1-\pi_g)\psi\rangle = 0$$

for all such $\varphi$. It follows that the component of $(\bar\partial\partial^{\#})^*\bar\partial\partial^{\#}(1-\pi_g)\psi$ perpendicular to the kernel of $(\bar\partial\partial^{\#})^*\bar\partial\partial^{\#}$ is zero, and thus, $(\bar\partial\partial^{\#})^*\bar\partial\partial^{\#}(1-$

$\pi_g)\psi = \sum c_j f_g^j$. The same argument used above implies that $c_j = 0$, and so, $(1-\pi_g)\psi$ is the kernel of $(\bar{\partial}\partial^{\#})^*\bar{\partial}\partial^{\#}$. But the image of $1-\pi_g$ is orthogonal to this kernel. Hence, $\psi = 0$. Using this fact, we may now conclude that $\beta$ must be such that $\langle \beta, \alpha\rangle = 0$ for all $\alpha \in \mathfrak{cr}(\mathcal{D}, J)$, and so $\beta = 0$. The cokernel of $D\mathcal{S}_{(0,0)}$ is trivial. □

We now prove the following result, analogous to the openness of the extremal cone in Kähler geometry [22].

**Theorem 7.6.** *Let* $(\mathcal{D}, J)$ *be a* CR *structure of Sasaki type on* $M$. *Then the canonical Sasaki set* $\mathfrak{e}(\mathcal{D}, J)$ *is an open subset of the Sasaki cone* $\kappa(\mathcal{D}, J)$.

*Proof.* Let $\mathcal{V}_0 \subset \mathcal{V}$ be a neighborhood of $(0, 0) \in \mathfrak{cr}(\mathcal{D}, J) \times I_{k+4}$ such that $\mathcal{S}\mid_{\mathcal{V}_0}$ is a diffeomorphism from $\mathcal{V}_0$ onto an open neighborhood of the origin in $\mathfrak{cr}(\mathcal{D}, J) \times I_k$. For any point $\alpha$ in $\mathcal{S}(\mathcal{V}_0) \cap (\mathfrak{cr}(\mathcal{D}, J) \times \{0\})$, we define $\varphi(\alpha)$ to be the projection onto $I_{k+4}$ of $(\mathcal{S}\mid_{\mathcal{V}_0})^{-1}(\alpha)$. Then we have

$$(\alpha, 0) = \mathcal{S}(\alpha, \varphi(\alpha)) = (\alpha, (1-\pi_g)(1-\pi_{g_{\alpha,\varphi(\alpha)}})s_{g_{\alpha,\varphi(\alpha)}})\,,$$

where $g_{\alpha,\varphi(\alpha)}$ is the deformation of the metric $g$ associated to the parameters $(\alpha, \varphi(\alpha))$. Since the kernel of $(1-\pi_g)(1-\pi_{g_{\alpha,\varphi(\alpha)}})$ equals the kernel of $1-\pi_{g_{\alpha,\varphi(\alpha)}}$, it follows that $(1-\pi_{g_{\alpha,\varphi(\alpha)}})s_{g_{\alpha,\varphi(\alpha)}} = 0$. We then have that the scalar curvature of the Sasaki metric $g_{\alpha,\varphi(\alpha)}$ is a holomorphy potential, and so this metric is a canonical representative of $\mathcal{S}(\xi + \alpha, \bar{J})$. This completes the proof. □

**Example 7.7**. Let us take coordinates $z = (z_0, \ldots, z_n)$ in $\mathbb{C}^{n+1}$, and consider the unit sphere

$$\mathbb{S}^{2n+1} = \{z \in \mathbb{C}^{n+1} : \ |z| = 1\}\,.$$

If $z_k = x_k + iy_k$ is the decomposition of $z_k$ into real and imaginary parts, then the vector fields $H_k = (y_k\partial_{x_k} - x_k\partial_{y_k})$, $k = 0, \ldots, n$, form a basis for the Lie algebra $\mathfrak{t}_{n+1} = \mathbb{R}^{n+1}$ of a maximal torus in the automorphism group $\mathbb{U}(n+1)$ of the standard Sasakian structure on $\mathbb{S}^{2n+1}$. The latter is given by the contact form $\eta = \sum_{k=0}^{n}(y_k dx_k - x_k dy_k)$, Reeb vector field $\xi = \sum_{k=0}^{n} H_k$, and $(1, 1)$-tensor $\Phi$ defined by the restriction $J$ of the complex structure on $\mathbb{C}^{n+1}$ to $\mathcal{D} = \ker \eta$, and the fact that $\Phi(\xi) = 0$. The ensuing compatible metric (4) defined by $(\xi, \eta, \Phi)$ is the standard metric $g$ on $\mathbb{S}^{2n+1}$, which is Einstein with $r_g = 2ng$. Thus, the metric $g$ yields a canonical representative of $\mathcal{S}(\xi, J)$. We have that the automorphism group of $(\xi, \eta, \Phi, g)$ is $\mathbb{U}(n+1)$. Its maximal torus $\mathfrak{T}_{n+1}$ has Lie algebra $\mathfrak{t}_{n+1}$ with basis $\{H_0, \ldots, H_n\}$.

We now fix this CR structure $(\mathcal{D}, J)$ on $\mathbb{S}^{2n+1}$, and consider the set $\mathcal{S}(\mathcal{D}, J)$ of all Sasakian structures associated with it. A vector field $X$ is positive if, and only if, $\eta(X) > 0$, and such a vector field is conjugate to a positive vector in the Lie algebra $\mathfrak{t}_{n+1}$. We use the basis $\{H_0, \ldots, H_n\}$ to identify this Lie algebra with $\mathbb{R}^{n+1}$, so the point $w = (w_0, \ldots, w_n)$ in $\mathbb{R}^{n+1}$ yields the vector $\xi_w = \sum w_k H_k$. Then we have that

$$\eta(\xi_w) = \sum_{i=0}^{n} w_i\big(x_i^2 + y_i^2\big) = \sum_{i=0}^{n} w_i |z_i|^2 \,,$$

and so, the set of positive elements of $\mathfrak{t}_{n+1}$ is just $\mathbb{R}^{n+1}_+$. By Theorem 6.5, the Sasaki cone $\kappa(\mathcal{D}, J)$ is equal to $\mathbb{R}^{n+1}_+$. If $w \in \mathbb{R}^{n+1}_+$, this vector gives rise to the Sasakian structure $(\xi_w, \eta/\eta(\xi_w), \Phi_w, g_w)$, where $\Phi_w$ is defined by the conditions $\Phi_w \mid_{\mathcal{D}} = J$ and $\Phi_w(\xi_w) = 0$, respectively, and $g_w$ is determined by the expression (4) in terms of $\xi_w$, $\eta/\eta(\xi_w)$ and $\Phi_w$.

For any $w \in \mathbb{Z}^{n+1}_+$, the Sasakian structure $(\xi_w, \eta/\eta(\xi_w), \Phi_w, g_w)$ is quasi-regular, and its transversal is a manifold with orbifold singularities, the weighted projective space $\mathbb{CP}^n_w$. The space of metrics $\mathfrak{M}(\xi_w, \bar{J})$ associated with the polarized Sasakian manifold $(\mathbb{S}^{2n+1}, \xi_w, \bar{J})$ has a representative $g_w$ whose transverse Kähler metric $g_w^T$ is Bochner flat [8] on $\mathbb{CP}^n_w$, and thus, extremal. Computing in an affine orbifold chart, it can be determined [16] that the scalar curvature of $g_w^T$ is given by

$$s_{g_w^T} = 4(n+1)\frac{\sum_{j=0}^n w_j(2(\sum_{k=0}^n w_k) - (n+2)w_j)|z_j|^2}{\sum_{j=0}^n w_j|z_j|^2} \,,$$

at $z \in \mathbb{S}^{2n+1}$. Since the volume $\mu_{g_w^T}(\mathbb{CP}^n_w) = \pi^n/(n!\prod_{j=0}^n w_j)$ [16], the volume of $\mathbb{S}^{2n+1}$ in the Sasakian metric $g_w$ is just

$$\mu_{g_w}(\mathbb{S}^{2n+1}) = 2\frac{\pi^{n+1}}{n!}\frac{1}{\prod_{j=0}^n w_j} \,.$$

Similarly, since $s_{g_w} = s_{g_w^T} - 2n$, and since the mean transverse scalar curvature of $g_w^T$ is $4n\sum_{j=0}^n w_j$ [16], we have that the projection $s^0_{g_w}$ of $s_{g_w}$ onto the constants is given by

$$s^0_{g_w} = 2n(2\sum_{j=0}^n w_j - 1) \,.$$

Thus,

$$s_{g_w} - s^0_{g_w} = 4(n+2)\frac{\sum_{j=0}^n w_j((\sum_{k=0}^n w_k) - (n+1)w_j)|z_j|^2}{\sum_{j=0}^n w_j|z_j|^2} \,,$$

Notice that if the weight vector $w$ is of the form $w = l(1, \ldots, 1)$, then $s_{g_w^T} = 4(n+1)nl$, and this yields the scalar curvature of the Fubini-Study metric when $l = 1$, as it should.

For the Sasaki-Futaki character of the polarization $(\xi_w, \bar{J})$, it suffices to determine its value on vector fields $X$ that commute with $\xi_w$, and that are of the form $X = \partial^{\#} f$ for $f$ a basic real holomorphy potential. In that case, we have

$$\mathfrak{F}_{(\xi_w,\bar{J})}(X) = -\int_{\mathbb{S}^{2n+1}} f(s_{g_w} - s_{g_w^0}) d\mu_{g_w} .$$

For convenience, let us set $A_j = \sum_{k=0}^{n} w_k - (n+1)w_j$. Working in an affine orbifold chart for $\mathbb{CP}_w$, we then see that if $f = \sum_{i=0}^{n} b_i |z_i|^2$ we have that

$$\begin{aligned}
\mathfrak{F}_{(\xi_w,\bar{J})}(X) &= -8(n+2)\pi^{n+1} \int_{\mathbb{R}^n_+} \frac{(b_0 + \sum_{j=1}^{n} b_j x_j)(w_0 A_0 + \sum_{j=1}^{n} w_j A_j x_j)}{(w_0 + \sum_{j=1}^{n} w_j x_j)^{n+3}} dx_1 \ldots dx_n \\
&= -16 \frac{\pi^{n+1}}{(n+1)!} \left( \sum_{i=0}^{n} \frac{b_i}{w_i} A_i + \frac{1}{2} \sum_{i \neq j} \frac{b_i}{w_i} A_j \right) \frac{1}{\prod_{j=0}^{n} w_j} .
\end{aligned}$$

**Theorem 7.8.** *Let $(\mathcal{D}, J)$ be the standard* CR *structure on the unit sphere $\mathbb{S}^{2n+1}$, and $\kappa(\mathcal{D}, J)$ and $\mathfrak{e}(\mathcal{D}, J)$ be the associated Sasaki cone and canonical Sasaki set, respectively. Then $\mathfrak{e}(\mathcal{D}, J) = \kappa(\mathcal{D}, J)$, and the only canonical points in $\mathfrak{e}(\mathcal{D}, J)$ that yield metrics of constant scalar curvature are those representing transverse homotheties of the standard Riemannian Hopf fibration. Of these, the metric of constant sectional curvature one is the only Sasaki-Einstein metric whose underlying* CR *structure is $(\mathcal{D}, J)$.*

*Proof.* We have identified above $\kappa(\mathcal{D}, J)$ with points $w$ in $\mathbb{R}^{n+1}_+$. If $w \in \mathbb{Z}^{n+1}_+$, the polarized Sasakian manifold $(\mathbb{S}^{2n+1}, \xi_w, \bar{J})$ admits a representative $g_w$ with transverse Bochner flat metric on the transverse space $\mathbb{CP}^n_w$. Thus, these weights $w$ belong to $\mathfrak{e}(\mathcal{D}, J)$. Using the homotheties (35), we may obtain canonical representatives of the polarized Sasakian manifold $(\mathbb{S}^{2n+1}, \xi_w, \bar{J})$ for any weight $w \in \mathbb{Q}^{n+1}_+$. Applying Theorem 7.6, we obtain the same result for arbitrary weights $w$ in $\mathbb{R}^{n+1}_+$. Thus, $\mathfrak{e}(\mathcal{D}, J) = \kappa(\mathcal{D}, J)$.

The expressions computed above for the scalar curvature, volume, and Sasaki-Futaki character for $w = (w_0, \ldots, w_n) \in \mathbb{Z}^{n+1}_+$ are rational functions of the weights $w_j$, so they also define the scalar curvature, volume, and Sasaki-Futaki character when $w$ is an arbitrary vector in $\mathbb{R}^{n+1}_+$, regardless of the fact that there may not be a transversal manifold to speak of in this general situation.

The assertion about the scalar curvatures of the canonical representatives $g_w$ follows by the expression for $\mathfrak{F}_{(\xi_w,\bar{J})}$ given above, and Proposition 5.2. Indeed, the character $\mathfrak{F}_{(\xi_w,\bar{J})}$ is identically zero if, and only if, $A_j = A_k$ for all pairs of indices $j, k$. This only happens if the vector of weights $w$ is of the form $w = l(1, \ldots, 1)$. The only one of these metrics that is Sasaki-Einstein is the standard metric. This follows by a simple analysis of the change of the Ricci curvature under homotheties of the metric. □

In this example, the first Chern class of the sub-bundle $\mathcal{D}$ is trivial. Thus, for any $w$ in the Sasaki cone, the basic first Chern class of the resulting foliated manifold is proportional to the basic class defined by the transversal Kähler form $d\eta_w$. However, only for the weight $w = (1, \ldots, 1)$ there exists a Sasaki-Einstein representative.

In general, given a CR structure $(\mathcal{D}, J)$ of Sasaki type on a closed manifold $M$, we do not expect the equality $\mathfrak{e}(\mathcal{D}, J) = \kappa(\mathcal{D}, J)$ to hold, though this is likely to be so in the toric case.

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