Extremal Sasakian Geometry

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Problems:

Given a contact structure or isotopy class of contact structures:

1. Determine the space of compatible Sasakian structures.
2. Determine the (pre)-moduli space of extremal Sasakian structures; those of constant scalar curvature (cscS).

• **Contact Manifold** M (compact). *A* **contact 1-form** η *such that*

$$\eta \wedge (d\eta)^n \neq 0.$$

defines a **contact structure**

$$\eta' \sim \eta \iff \eta' = f\eta$$

for some $f \neq 0$, *take* $f > 0$. *or equivalently a codimension 1 sub-bundle* $\mathcal{D} = \mathrm{Ker}\ \eta$ *of* TM *with a conformal symplectic structure.*

A **contact invariant**: $c_1(\mathcal{D})$

Unique vector field ξ, *called the* **Reeb vector field**, *satisfying*

$$\xi \rfloor \eta = 1, \qquad \xi \rfloor d\eta = 0.$$

The **characteristic foliation** $\mathcal{F}_\xi$ *each leaf of* $\mathcal{F}_\xi$ *passes through any nbd* U *at most* k *times* $\Longleftrightarrow$ **quasi-regular**, $k = 1 \leftrightarrow$ *regular, otherwise* **irregular**

Quasi-regularity is strong, most contact 1-forms are irregular.

Contact bundle $\mathcal{D} \longrightarrow$ *choose* **almost complex structure** J *extend to* Φ *with* $\Phi\xi = 0$
with a compatible metric

$$g = d\eta \circ (\Phi \otimes \mathbb{1}) + \eta \otimes \eta$$

Quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ *called* **contact metric structure**

The pair $(\mathcal{D}, J)$ *is a strictly pseudoconvex almost CR structure.*

Definition: *The structure* $\mathcal{S} = (\xi, \eta, \Phi, g)$ *is* **K-contact** *if* $\pounds_\xi g = 0$ *(or* $\pounds_\xi \Phi = 0$*). It is* **Sasakian** *if in addition* $(\mathcal{D}, J)$ *is integrable.*

Transverse Metric $g_{\mathcal{D}}$ *is Kähler*

Cone (Symplectization)

$C(M) = M \times \mathbb{R}^+$

symplectic form $d(r^2\eta),\ r \in \mathbb{R}^+$.

Cone Metric $g_C = dr^2 + r^2 g$

- g_C *is Kähler* $\iff$ g *is Sasaki* $\iff$ $g_{\mathcal{D}}$ *is Kähler.*

Sasaki-Kähler Sandwich

 $\Leftrightarrow$  $\Leftrightarrow$ 

Symmetries

- **Contactomorphism Group**

$\mathfrak{Con}(M,\mathcal{D}) =$

$\{\phi \in \mathfrak{Diff}(M) \mid \phi_*\mathcal{D} \subset \mathcal{D}\}.$

CR transformation group: $\mathfrak{CR}(\mathcal{D},J)$

$= \{\phi \in \mathfrak{Con}(M,\mathcal{D}) \mid \phi_* J = J\phi_*\}$

Have: $T^k \subset \mathfrak{CR}(\mathcal{D},J) \subset \mathfrak{Con}(M,\mathcal{D})$

T^k *a* **max'l torus** $0 \leq k \leq n+1$.

$\mathcal{J}(\mathcal{D})$ *space of compatible almost CR structures, then a map*

$\mathfrak{Q} : \mathcal{J}(\mathcal{D}) \to \{$*conjugacy classes of maximal tori*$\}$ *in* $\mathfrak{Con}(M,\mathcal{D})$

Bouquets of Sasaki cones

$\mathfrak{t}_k^+(\mathcal{D}, J) = \{\xi \in \mathfrak{t}_k \mid \eta'(\xi) > 0, \}$
s.t. $\mathcal{S} = (\xi, \eta, \Phi, g) \in (\mathcal{D}, J)$ *is Sasakian*

- *finite dim'l moduli of Sasakian structures within CR structure*

$\kappa(\mathcal{D}, J) = \mathfrak{t}_k^+(\mathcal{D}, J)/\mathcal{W}(\mathcal{D}, J)$

A given $\mathcal{D}$ can have many Sasaki cones $\mathfrak{t}_k^+(\mathcal{D}, J_\alpha)$ labelled by complex structures, and $k = k(\alpha)$.

Get **bouquet** $\bigcup_\alpha \mathfrak{t}_{k(\alpha)}^+(\mathcal{D}, J_\alpha)$
union over tori conjugacy classes

- **Extremal Sasakian metrics**

(B-,Galicki,Simanca)

$E(g) = \int_M s_g^2 d\mu_g,$

- Deform contact structure

Vary $\eta \mapsto \eta + td^c\varphi$, φ basic, gives critical point of $E(g) \iff \partial_g^{\#} s_g$ is transversely holomorphic. $s_g =$ scalar curvature.

Special case: constant scalar curvature Sasakian (cscS) . If $c_1(\mathcal{D}) = 0 \Rightarrow$ Sasaki-η-Einstein (SηE)

$\mathrm{Ric}_g = ag + b\eta \otimes \eta,\ a, b$ constants.

Sasaki-Einstein (SE) $b = 0$

Extremal Set $\mathfrak{e}(\mathcal{D}, J)$

$\mathfrak{e}(\mathcal{D}, J) \subset \mathfrak{t}_k^+(\mathcal{D}, J)$ is open in Sasaki cone B-,Galicki,Simanca

If $\mathcal{S} = \mathcal{S}_1 \in \mathfrak{e}(\mathcal{D}, J)$ then entire ray $\mathcal{S}_a = (a^{-1}\xi, a\eta, \Phi, g_a) \in \mathfrak{e}(\mathcal{D}, J)$

When is $\mathfrak{e}(\mathcal{D}, J) = \mathfrak{t}_k^+(\mathcal{D}, J)$?

Many ex's if $\dim \mathfrak{t}_k^+(\mathcal{D}, J) = 1$

- If $\dim \kappa(\mathcal{D}, J) > 1$, sphere, Heisenberg group, $T^2 \times S^3$ have $\mathfrak{e}(\mathcal{D}, J) = \mathfrak{t}_k^+(\mathcal{D}, J) > 1$.

(1) standard CR structure on S^{2n+1}

Toric (dim $\kappa(\mathcal{D}, J) = n+1$.)

$\kappa(\mathcal{D}, J) = \{\mathbf{w} = (w_0, \cdots, w_n) \in \mathbb{R}^{n+1} \mid w_0 \leq w_1 \leq \cdots \leq w_n\}$

All $\mathcal{S}_{\mathbf{w}}$ have extremal representatives, but only Φ-sect. curv. $c > -3$ has (cscS), and only the round sphere ($c = 1$) is SE.

(B,Galicki,Simanca)

(2) The Heisenberg group $\mathfrak{H}^{2n+1}$ with standard CR structure (non-compact), dim $\kappa(\mathcal{D}, J) = n$. (B-)

All $\mathcal{S} \in \kappa(\mathcal{D}, J)$ have extremal representatives, but there is only one with constant scalar curvature, SηE with Φ-holomorphic curvature $= -3$. Here transverse homothety is induced by diffeomorphism.

Probably $\mathfrak{e}(\mathcal{D}, J) = \mathfrak{t}_k^+(\mathcal{D}, J)$ also holds for standard CR structure on the hyperbolic ball $B^n_{\mathbb{C}} \times \mathbb{R}$. Here Φ-sect. curv. $c < -3$ is (cscS).

When: extremal bouquets?

Toric Contact Manifold

$(M^{2n+1}, \mathcal{D})$, effective action of torus T^{n+1} leaving $\mathcal{D}$ invariant.

(1): **Reeb Type** Reeb field ξ lies in $\mathfrak{t}_{n+1}$, Lie algebra of T^{n+1}.

(2): $\xi \not\in \mathfrak{t}_{n+1}$. (less interesting)

Reeb type are Sasakian. B-/Galicki.

Other References: Banyaga/Molino, Lerman, Falcao de Moraes/Tomei.

Complete classification: Lerman.

Toric contact manifolds of Reeb type are classified by certain convex polyhedral cones in $\mathfrak{t}^*_{n+1}$ up to T^{n+1}-equivariant equivalence. (Lerman).

Theorem: Every toric contact structure of Reeb type with $c_1(\mathcal{D}) = 0$ admits a unique Sasaki-Einstein metric (Futaki,Ono,Wang,Cho)

There is a ray of cscS metrics in an open set of extremal rays. How big is $\mathfrak{e}(\mathcal{D}, J)$?

5-manifolds

Barden-Smale classification of **simply connected 5-manifolds** $H_2(M^5, \mathbb{Z})$ torsionfree

$S^5, S^2 \times S^3, X_\infty, k\#(S^2 \times S^3)$, $X_\infty \# k\#(S^2 \times S^3)$.

All admit toric contact structures of Reeb type. (B-/Galicki,Ornea)

All but S^5 admit infinitely many.

All obtained by **Symmetry Reduction** by weighted S^1-action.

weights $\mathbf{p} = (p_1, p_2, p_3, p_4)$.

• S^3-bundles over S^2:
$M^5 = S^2 \times S^3$ or X_∞. Which?
$w_2(M^5) \equiv c_1(\mathcal{D}_{\mathbf{p}})\ mod\ 2$. $\Rightarrow$
$M^5 = S^2 \times S^3 (X_\infty)$ if $c_1(\mathcal{D}_{\mathbf{p}})$ is even (odd).
$c_1(\mathcal{D}_{\mathbf{p}}) = (p_1 + p_2 - p_3 - p_4)\gamma$
Calabi extremal Kähler metrics on Hirzebruch surfaces give a bouquet of extremal Sasakian structures on $S^2 \times S^3$ and X_∞. Moduli space is non-Hausdorff.
The quotient M^5/S^1_ϕ is an orbifold Hirzebruch surface.

•E. Legendre on cscS metrics: M^5 5-manifold with $b_2(M^5) = 1$ with toric contact structure of Reeb type. $\exists$ at least 1 and at most 7 rays of cscS metrics. 2 rays of cscS non-isometric metrics $M^{1,1}_{k,l} \approx S^2 \times S^3$ if $k > 5l$.

Special case of our special case: $Y^{p,q} \approx S^2 \times S^3$. Physicists: Gauntlett,Martelli,Sparks,Waldram. Infinitely many toric contact structures. Each $Y^{p,q}$ admits unique

Sasaki-Einstein metrics.

In our notation $\mathcal{D}_{p-q,p+q,p,p}$ with $\gcd(p,q)=1$ and $1\leq q<p$.

$c_1(\mathcal{D}_{p-q,p+q,p,p})=0$.

Non-equivalence if $p'\neq p$:

Contact homology

(Eliashberg,Givental,Hofer)

(Abreu,Macarini): $Y^{p,1}\not\approx Y^{p',1}$ when $p'\neq p$.

Theorem:(B-)$Y^{p,q}$ and $Y^{p',q'}$ are contact equivalent $\iff p'=p$.

5-manifolds, $\pi_1 \neq \{1\}$

Join Construction: Given quasi-regular Sasakian manifolds
$\pi_i : M_i \rightarrow \mathcal{Z}_i$ for $i = 1, 2$.
Form (k, l)-join (B-,Galicki,Ornea)
$\pi : M_1 \star_{k,l} M_2 \rightarrow \mathcal{Z}_1 \times \mathcal{Z}_2$.
$M_1 \star_{k,l} M_2$ – Sasakian structure.
smooth iff $\gcd(v_1 l, v_2 k) = 1$,
v_i order of orbifold $\mathcal{Z}_i$.
(B-,Tønnesen-Friedman) Construct Sasakian 5-manifolds. Consider $M^3 \star_{k,l} S^3$ where M^3 Sasakian 3-manifold (Belgun)-uniformization.

Hamiltonian circle action

Two Cases:

(1) M^3 circle bundle over Riemann surface Σ_g of genus g.

(2) M^3 homology sphere as link of complete intersection $L(a_0, \cdots, a_n)$.
The $a_i > 1$ pairwise relatively prime.
$M^3 \star_{1,l} S^3$ homology of $S^2 \times S^3$.
$L(a_0, \cdots, a_n) \neq L(2, 3, 5)$ and
$\{1\} \neq \pi_1(M^3 \star_{1,l} S^3)$ perfect, ∞.

Extremal Sasaki metrics

Case (1): Topological rigidity argument of Kreck-Lück $\Rightarrow$ diffeomorphism type:
$M^3 \star_{k,1} S^3 = \Sigma_g \times S^3, \quad \forall k \in \mathbb{Z}^+$.
Extremal Sasaki metrics in general case is in progress.

Case: $g = 1$, that is,
$M^3 \star_{k,1} S^3 = T^2 \times S^3, \ \forall k \in \mathbb{Z}^+$.

Ruled Surfaces $g = 1$

Complex structures (Atiyah, Suwa)

$\mathbb{P}(E) \approx T^2 \times S^2$, rank$(E) = 2$

1. nonsplit case (no extremal Kähler metric)

2. $E = L \oplus 1\!\!1$, degree $L = 0$

3. $E = L \oplus 1\!\!1$, degree $L > 0$

Extremal Kähler metr (Fujiki,Hwang)

Hamiltonian 2-forms: (Apostolov, Calderbank, Gauduchon, Tønnesen-Friedman (ACGT))

2. Degree $L = 0$:

Representation $\rho : \pi_1(T^2) \to S0(3)$

Get S^1-bundle over $T^2 \times_\rho \mathbb{CP}^1$ with CSC Sasaki metrics on $T^2 \times S^3$.

Vary in Sasaki cone, extremal Sasaki metrics exhaust Sasaki cone.

3. Degree $L = 2n > 0$: ACGT method: Kähler metric

$g = \frac{1+r\mathfrak{z}}{r} g_{T^2} + \frac{d\mathfrak{z}^2}{\Theta(\mathfrak{z})} + \Theta(\mathfrak{z})\theta^2$

θ connection 1-form, $d\theta = \omega_{T^2}$,

$0 < r < 1$, $\Theta(\mathfrak{z}) > 0$ in $-1 < \mathfrak{z} < 1, \Theta(\pm 1) = 0, \Theta'(\pm 1) = \mp 2$

Θ 4th order polynomial gives extremal Kähler metric

Deform in Sasaki cone, extremal Sasaki metrics exhaust Sasaki cone. 3rd order polynomial gives cscS metrics. All are quasi-regular.

Summary of Results

(1): $T^2 \times S^3$ admits a countably infinite number of distinct contact structures $\mathcal{D}_k$.

(2): $\mathcal{D}_k$ admits a bouquet of k 2-dimensional Sasaki cones each with a unique ray of constant scalar curvature Sasaki metrics.

(3): Each member of the bouquet in (2) has an extremal Sasaki metric.

(4): There is a Sasaki cone consisting of a single ray that admits no extremal Sasaki metric.

(5): Some results for quotients of the form $(T^2 \times S^3)/\mathbb{Z}_l$

References

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