ON THE TOPOLOGY OF SOME SASAKI-EINSTEIN MANIFOLDS

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Abstract. This is a sequel to our paper [BTF15] in which we concentrate on developing some of the topological properties of Sasaki-Einstein manifolds. In particular, we explicitly compute the cohomology rings for several cases not treated in [BTF15] and give formulae for homotopy equivalence as well as homeomorphism equivalence in one particular 7-dimensional case.

1. Introduction

Recently the authors have been able to obtain many new results on extremal Sasakian geometry [BTF13c, BTF13a, BTF14a, BTF15] by giving a geometric construction that combines the 'join construction' of [BG00, BGO07] with the 'admissible construction of Hamiltonian 2-forms' for extremal Kähler metrics described in [ACG06, ACGTF04, ACGTF08b, ACGTF08a]. The current paper is a result of re-arranging the two previous ArXiv papers [BTF13b, BTF14b]. The basic analysis of both the constant scalar curvature and Sasaki-Einstein cases were combined in [BTF15] which also contains the foundational topological description. The current paper contains further results on the topology of the Sasaki-Einstein manifolds most of which appeared in [BTF13b], but were left out of [BTF15].

The main result concerning Sasaki-Einstein manifolds in [BTF15] is:

Theorem 1.1. *Let $M_{l_1,l_2,\mathbf{w}} = M \star_{l_1,l_2} S^3_{\mathbf{w}}$ be the $S^3_{\mathbf{w}}$-join with a regular Sasaki manifold M which is an S^1-bundle over a compact positive Kähler-Einstein manifold N with a primitive Kähler class $[\omega_N] \in H^2(N,\mathbb{Z})$. Assume that the relatively prime positive integers (l_1, l_2) are*

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the relative Fano indices given explicitly by

$$l_1(\mathbf{w}) = \frac{\mathcal{I}_N}{\gcd(w_1+w_2,\mathcal{I}_N)}, \qquad l_2(\mathbf{w}) = \frac{w_1+w_2}{\gcd(w_1+w_2,\mathcal{I}_N)},$$

where $\mathcal{I}_N$ *denotes the Fano index of* N. *Then for each vector* $\mathbf{w} = (w_1,w_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ *with relatively prime components satisfying* $w_1 > w_2$ *there exists a Reeb vector field* $\xi_{\mathbf{v}}$ *in the 2-dimensional* $\mathbf{w}$*-Sasaki cone on* $M_{l_1,l_2,\mathbf{w}}$ *such that the corresponding Sasakian structure* $\mathcal{S} = (\xi_{\mathbf{v}},\eta_{\mathbf{v}},\Phi,g)$ *is Sasaki-Einstein (SE).*

The procedure involved taking a join of a regular Sasaki-Einstein manifold M with the weighted 3-sphere $S^3_{\mathbf{w}}$, that is, S^3 with its standard contact structure, but with a weighted contact 1-form whose Reeb vector field generates rotations with generally different weights w_1,w_2 for the two complex coordinates z_1,z_2 of $S^3 \subset \mathbb{C}^2$. We call this the $S^3_{\mathbf{w}}$-join. By the $\mathbf{w}$-Sasaki cone we mean the two dimensional subcone of Sasaki cone induced by the Sasaki cone of $S^3_{\mathbf{w}}$. It is denoted by $\mathfrak{t}^+_{\mathbf{w}}$ and can be identified with the open first quadrant in $\mathbb{R}^2$.

Most of the SE structures in Theorem 1.1 are irregular. Such structures have irreducible transverse holonomy [HS12], implying there can be no generalization of the join procedure to the irregular case. We must deform within the Sasaki cone to obtain them. Furthermore, it follows from [RT11, CS12] that constant scalar curvature Sasaki metrics (hence, SE) imply a certain K-semistability.

The SE metrics obtained from Theorem 1.1 were obtained earlier by physicists [GHP03, GMSW04b, GMSW04a, CLPP05, MS05] working on the AdS/CFT correspondence. Their method, particularly that of [GMSW04a], is very closely related to the Hamiltonian 2-form approach of [ACG06] (cf. Section 4.3 of [Spa11]). In fact Thorem 1.1 indicates that the physicist's results fit naturally into our geometric construction. Furthermore, we showed in [BTF15] that our geometric approach leads naturally to an algorithm for computing the cohomology ring of the $2n+3$-manifolds. In the present paper we explicitly compute the cohomology ring of all such examples of SE manifolds in dimension 7 showing that there are a countably infinite number of distinct homotopy types of such manifolds. The case that M is a standard odd dimensional sphere was computed in [BTF15], so here we give the cohomology rings of the joins when M is a circle bundle over one of the remaining del Pezzo surfaces. Explicitly, for $N = \mathbb{CP}^1 \times \mathbb{CP}^1$ we have

Theorem 1.2. *For each relatively prime pair* (w_1,w_2) *of positive integers there exist Sasaki-Einstein metrics on the 7-manifolds* $M^7_{l_1,l_2,\mathbf{w}}$

with cohomology ring

$$\mathbb{Z}[x,y,u,z]/(x^2, l_2(\mathbf{w})xy, w_1w_2l_1(\mathbf{w})^2y^2, z^2, u^2, zu, zx, ux, uy)$$

with $(l_1,l_2)=(2,|\mathbf{w}|)$ *if* $\mathbf{w}$ *is odd, or* $(l_1,l_2)=(1,\frac{|\mathbf{w}|}{2})$ *if* $\mathbf{w}$ *is even, where* x,y *are 2-classes, and* z,u *are 5-classes.*

It is well-known that when N is the blow-up of $\mathbb{CP}^2$ at k generic points, namely $N=\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$ there is a Kähler-Einstein metric precisely for $k=3,\cdots,8$. Then our results give

Theorem 1.3. *For each relatively prime pair* (w_1,w_2) *of positive integers there exist Sasaki-Einstein metrics on the 7-manifolds* $M^7_{k,\mathbf{w}}$ *with cohomology ring*

$$H^q(M^7_{k,\mathbf{w}},\mathbb{Z})\approx\begin{cases}\mathbb{Z} & \text{if } q=0,7;\\ \mathbb{Z}^{k+1} & \text{if } q=2,5;\\ \mathbb{Z}^k_{w_1+w_2}\times\mathbb{Z}_{w_1w_2} & \text{if } q=4;\\ 0 & \text{if otherwise,}\end{cases}$$

with the ring relations determined by $\alpha_i\cup\alpha_j=0, w_1w_2s^2=0$, $(w_1+w_2)\alpha_i\cup s=0$, *and* α_i,s *are the* $k+1$ *two classes with* $i=1,\cdots k$ *where* $k=3,\cdots,8$. *Furthermore, when* $4\leq k\leq 8$ *the local moduli space of Sasaki-Einstein metrics has real dimension* $4(k-4)$.

Of particular interest is the join $M^{2r+3}_{\mathbf{w}}=S^{2r+1}\star_{l_1,l_2}S^3_{\mathbf{w}}$ of the standard odd dimensional sphere with the weighted $S^3_{\mathbf{w}}$ where

$$(l_1,l_2)=\Bigl(\frac{r+1}{\gcd(w_1+w_2,r+1)},\frac{w_1+w_2}{\gcd(w_1+w_2,r+1)}\Bigr). \tag{1}$$

By Theorem 4.5 of [BTF15] its cohomology ring is

$$\mathbb{Z}[x,y]/(w_1w_2l_1(\mathbf{w})^2x^2, x^{r+1}, x^2y, y^2) \tag{2}$$

where x,y are classes of degree 2 and $2r+1$, respectively. Let k be the length of the prime decomposition of w_1w_2. Then for arbitrary r we show that there are 2^{k-1} Sasaki-Einstein manifolds of the form $M^{2r+3}_{\mathbf{w}}$ with cohomology ring given by Equation (2). For the manifolds $M^7_{\mathbf{w}}$ of dimension 7 ($r=2$) much more is known about the topology. These are special cases of what are called generalized Witten spaces in [Esc05]. In particular, the homotopy type was given in [Kru97], while the homeomorphism and diffeomorphism type was given in [Esc05]. For our subclass admitting Sasaki-Einstein metrics we give necessary and sufficient conditions on $\mathbf{w}$ for homotopy equivalence when the order of H^4 is odd in Proposition 3.7 below. Thus, we answer in the affirmative the existence of Einstein metrics on certain generalized Witten manifolds.

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2. The $\mathbf{w}$-Sasaki cone when $c_1(\mathcal{D})=0$

In this section we describe some of the properties of the $\mathbf{w}$-Sasaki cone when $c_1(\mathcal{D})=0$, ending with some examples. Since we require that N be a positive Kähler-Einstein manifold, we have $c_1(N)=\mathcal{I}_N[\omega_N]$ where $\mathcal{I}_N$ is the Fano index. Recall from [BTF15] that the cohomological Einstein condition $c_1(\mathcal{D}_{l_1,l_2,\mathbf{w}})=0$ implies

Lemma 2.1. *Necessary conditions for the Sasaki manifold $M_{l_1,l_2,\mathbf{w}}$ to admit a Sasaki-Einstein metric is that $\mathcal{I}_N>0$, and that*

$$l_2=\frac{|\mathbf{w}|}{\gcd(|\mathbf{w}|,\mathcal{I}_N)},\qquad l_1=\frac{\mathcal{I}_N}{\gcd(|\mathbf{w}|,\mathcal{I}_N)}.$$

The integers l_1,l_2 in Lemma 2.1 were called *relative Fano indices* in [BG00]. For the remainder of the paper we assume that these integers take the values given by Lemma 2.1 unless explicitly stated otherwise. Note that the Fano index $\mathcal{I}_N$ of a Fano manifold of complex dimension r is bounded by $r+1$, thus, l_1 is also bounded by $r+1$. Moreover, $\mathcal{I}_N=r+1$ if and only if the universal cover of the regular Sasaki manifold M is the standard sphere S^{2r+1}.

We shall make use of the following easily verified proposition for low values of $\mathcal{I}_N$.

Proposition 2.2. *Let M be a regular Sasaki-Einstein manifold and consider the join $M\star_{l_1,l_2}S^3_{\mathbf{w}}$. Then*

(1) *If $\mathcal{I}_N=1$ then $(l_1,l_2)=(1,|\mathbf{w}|)$.*

(2) *If $\mathcal{I}_N=2$, then* $(l_1,l_2)=\begin{cases}(2,|\mathbf{w}|) & \text{if } |\mathbf{w}| \text{ is odd;}\\ (1,\frac{|\mathbf{w}|}{2}) & \text{if } |\mathbf{w}| \text{ is even.}\end{cases}$

(3) *If $\mathcal{I}_N=3$, then* $(l_1,l_2)=\begin{cases}(3,|\mathbf{w}|) & \text{if } 3 \text{ does not divide } |\mathbf{w}|;\\ (1,\frac{|\mathbf{w}|}{3}) & \text{if } 3 \text{ divides } |\mathbf{w}|.\end{cases}$

A natural question that arises is whether the $\mathbf{w}$-cone contains a regular Reeb vector field.

Proposition 2.3. *Assume $\mathbf{w}\neq(1,1)$ and let $K=\gcd(\mathcal{I}_N,|\mathbf{w}|)$. Then there are exactly $K-1$ different $\mathbf{w}$-Sasaki cones that have a regular Reeb vector field. These are given by*

$$\mathbf{w}=\Big(\frac{K+n}{\gcd(K+n,K-n)},\frac{K-n}{\gcd(K+n,K-n)}\Big). \tag{3}$$

where $1 \leq n < K$.

Proof. By Proposition 3.4 of [BTF15] a $\mathbf{w}$-Sasaki cone contains a regular Reeb vector field if and only if there is $n \in \mathbb{Z}^+$ such that

$$w_1 - w_2 = n\frac{w_1 + w_2}{\gcd(\mathcal{I}_N, w_1 + w_2)}.$$

Clearly, for a solution we must have $n < \gcd(\mathcal{I}_N, w_1 + w_2)$. Then we have a solution if and only if

$$(K - n)w_1 = (K + n)w_2$$

for all $1 \leq n < K$. Since $w_1 > w_2$ and they are relatively prime we have the unique solution Equation (3) for each integer $1 \leq n < K$. □

We have an immediate corollary to Proposition 2.3:

Corollary 2.4. *If* $\mathcal{I}_N = 1$ *there are no regular Reeb vector fields in any* $\mathbf{w}$*-Sasaki cone with* $\mathbf{w} \neq (1,1)$*.*

Example 2.5. Let us determine the $\mathbf{w}$-joins with regular Reeb vector field for $\mathcal{I}_N = 2, 3$. For example, if $\mathcal{I}_N = 2$ for a solution to Equation (3) we must have $K = 2$ which gives $n = 1$ and $\mathbf{w} = (3,1)$. This has as a consequence Corollary 2.7 below. Similarly if $\mathcal{I}_N = 3$ we must have $K = 3$, which gives two solutions $\mathbf{w} = (2,1)$ and $\mathbf{w} = (5,1)$.

Example 2.6. Let p and q be relatively prime positive integers satisfying $p > 1$ and $1 \leq q < p$. Recall that the contact structures $Y^{p,q}$ on $S^2 \times S^3$ were discovered in [GMSW04b], where it is shown that there is a unique Sasaki-Einstein metric in the Sasaki cone of each such $Y^{p,q}$. These SE metrics are most often irregular. From the viewpoint of the present work, $Y^{p,q}$ is a join $M_{l_1,l_2,\mathbf{w}} = M^3 \star_{l_1,l_2} S^3_{\mathbf{w}}$ where $N = S^2$ with its standard (Fubini-Study) Kähler structure. Hence, $\mathcal{I}_N = 2$. This example has been treated in more detail elsewhere [BP14, Boy11, BTF15] so we shall be very brief here[1]. As in [BTF15] we have using Lemma 2.1

$$\mathbf{w} = \frac{(p+q, p-q)}{\gcd(p+q, p-q)}, \qquad l_1 = \gcd(p+q, p-q), \qquad l_2 = p. \tag{4}$$

It follows from Proposition 2.2 that there are two cases depending on whether $|\mathbf{w}|$ is odd or even. In the former case $p = |\mathbf{w}|$, and in the latter $p = \frac{|\mathbf{w}|}{2}$. From Example 2.5 we have

[1]Unfortunately, the conventions are slightly different. In [BP14, Boy11] the convention $w_1 \leq w_2$ is used; whereas, here and in [BTF15] the opposite convention, $w_1 \geq w_2$, is used.

Corollary 2.7. *For $Y^{p,q}$ the* $\mathbf{w}$*-Sasaki cone has a regular Reeb vector field if and only if* $p=2,q=1$ *or equivalently* $\mathbf{w}=(3,1)$.

We remark that the quotient of $Y^{2,1}$ by the regular Reeb vector field is $\mathbb{CP}^2$ blown-up at a point; whereas, we have arrived at it from the $\mathbf{w}$-Sasaki cone of an S^1 orbibundle over $S^2\times\mathbb{CP}^1[3,1]$.

3. The Topology of the Sasaki-Einstein manifolds

We briefly recall the method used in [BTF15] to prove Theorem 1.1. The idea is that if we know the differentials in the spectral sequence of the fibration

$$M\longrightarrow N\longrightarrow \mathsf{B}S^1, \tag{5}$$

we can use the commutative diagram of fibrations

$$\begin{array}{ccccc} M\times S^3_{\mathbf{w}} & \longrightarrow & M_{l_1,l_2,\mathbf{w}} & \longrightarrow & \mathsf{B}S^1 \\ \Big\downarrow{=} & & \Big\downarrow & & \Big\downarrow{\psi} \\ M\times S^3_{\mathbf{w}} & \longrightarrow & N\times\mathsf{B}\mathbb{CP}^1[\mathbf{w}] & \longrightarrow & \mathsf{B}S^1\times\mathsf{B}S^1 \end{array} \tag{6}$$

to compute the cohomology ring of the join $M_{l_1,l_2,\mathbf{w}}$. Here $\mathsf{B}G$ is the classifying space of a group G or Haefliger's classifying space [Hae84] of an orbifold if G is an orbifold.

3.1. Examples in General Dimension.

In this section we mainly give partial topological results for some examples of general dimension.

3.1.1. *M is a Standard Sphere.* The topology of the join when M is a regular Sasakian sphere S^{2r+1} was worked out in [BTF15] and further studied in [BTF14c]. We shall treat the 7-dimensional case in more detail in Section 3.2.1 below; however, before doing so we give the following result for $M^{2r+3}_{\mathbf{w}}=S^{2r+1}\star_{l_1,l_2}S^3_{\mathbf{w}}$ with $(l_1(\mathbf{w}),l_2(\mathbf{w}))$ satisfying Equations (1).

Lemma 3.1. *If* $H^4(M^{2r+3}_{\mathbf{w}},\mathbb{Z})=H^4(M^{2r+3}_{\mathbf{w}'},\mathbb{Z})$*, then* $w_1'w_2'=w_1w_2$ *and* $l_1(\mathbf{w}')=l_1(\mathbf{w})$.

Proof. The equality of the 4th cohomology groups together with the definition of l_1 imply

$$w_1'w_2'l_1\gcd(|\mathbf{w}|,r+1)^2=w_1w_2\gcd(|\mathbf{w}'|,r+1)^2.$$

Set $g_{\mathbf{w}}=\gcd(|\mathbf{w}|,r+1)$ and $g_{\mathbf{w}'}=\gcd(|\mathbf{w}'|,r+1)$. Assume $g_{\mathbf{w}'}>1$. Since $\gcd(w_1',w_2')=1$, $g_{\mathbf{w}'}$ does not divide $w_1'w_2'$. Thus, $g^2_{\mathbf{w}'}$ divides $g^2_{\mathbf{w}}$. Interchanging the roles of $\mathbf{w}'$ and $\mathbf{w}$ gives $g_{\mathbf{w}'}=g_{\mathbf{w}}$ which implies $l_1(\mathbf{w}')=l_1(\mathbf{w})$, and hence, the lemma in the case that $g_{\mathbf{w}'}>1$. Now assume $g_{\mathbf{w}'} = 1$. Then we have $w_1w_2 = w_1'w_2'g_{\mathbf{w}}^2$ which implies that $g_{\mathbf{w}}$ divides w_1w_2. But then since w_1, w_2 are relatively prime, we must have $g_{\mathbf{w}} = 1$. □

Let us set $W = w_1w_2$, and write the prime decomposition of $W = w_1w_2 = p_1^{a_1}\cdots p_k^{a_k}$ Let P_k be the number of partitions of W into the product w_1w_2 of unordered relatively prime integers, including the pair $(w_1w_2, 1)$. Then a counting argument gives $P_k = 2^{k-1}$. Once counted we then order the pair $w_1 > w_2$ as before. Let $\mathcal{P}_W$ denote the set of $(2r+3)$-manifolds $M_{\mathbf{w}}^{2r+3}$ with isomorphic cohomology rings. Then Lemma 3.1 implies that the cardenality of $\mathcal{P}_W$ is $P_k = 2^{k-1}$. This proves

Proposition 3.2. *Let k denote the length of the prime decomposition of w_1w_2, then there are 2^{k-1} simply connected Sasaki-Einstein manifolds $M_{\mathbf{w}}^{2r+3} = S^{2r+1} \star_{l_1,l_2} S_{\mathbf{w}}^3$ of dimension $2r+3$ with isomorphic cohomology rings such that H^4 has order $w_1w_2l_1(\mathbf{w})^2$.*

3.1.2. *M is a Rational Homology Sphere.* If we replace the standard odd dimensional sphere by a rational homology sphere V^{2r+1} with a regular Sasakian structure the computations in [BTF15] immediately give

Proposition 3.3. *The rational cohomology ring of the $S_{\mathbf{w}}^3$-join $V^{2r+1} \star_{l_1,l_2,\mathbf{w}} S_{\mathbf{w}}^3$ of a rational homology sphere V^{2r+1} is*

$$\mathbb{Q}[x,y]/(x^2,y^2)$$

where x, y are classes of degree 2 and $2r+1$, respectively. Here the l_1, l_2 are any positive integers satisfying $\gcd(l_2, w_1w_2l_1) = 1$.

Examples of rational homology spheres with regular Sasaki-Einstein metrics are given in [BGN02]. They are the Sasakian homogeneous Stiefel manifolds $V_2(\mathbb{R}^{2n+1})$ of 2-frames in $\mathbb{R}^{2n+1}$ and the 3-Sasakian homogeneous 11-manifold[2] $G_2/Sp(1)_+$. Since we want the join to have a Sasaki-Einstein metric somewhere it the Sasaki cone, we require that the pair (l_1, l_2) to be the relative Fano indices of Lemma 2.1.

Example 3.4. The Stiefel manifold $V_2(\mathbb{R}^{2n+1})$ of dimension $4n-1$. It is a circle bundle over the odd complex quadric $Q_{2n-1}(\mathbb{C})$. Its Fano index $\mathcal{I}$ is $2n-1$. So the relative Fano indices are

$$l_1(\mathbf{w}) = \frac{2n-1}{\gcd(|\mathbf{w}|, 2n-1)}, \qquad l_2(\mathbf{w}) = \frac{|\mathbf{w}|}{\gcd(|\mathbf{w}|, 2n-1)}. \tag{7}$$

[2]The reason for the subscript + on $Sp(1)$ is that there are two non-conjugate $Sp(1)$ subgroups in the exceptional Lie group G_2 which we denote by the subscripts $\pm$. The quotient by $Sp(1)_-$ is equivalent to $V_2(\mathbb{R}^7)$.

Moreover, the cohomology of $V_2(\mathbb{R}^{2n+1})$ is

$$H^p(V_2(\mathbb{R}^{2n+1}),\mathbb{Z}) \approx \begin{cases} \mathbb{Z} & \text{if } p=0,4n-1; \\ \mathbb{Z}_2 & \text{if } p=2n; \\ 0 & \text{otherwise.} \end{cases}$$

From the long exact homotopy sequence and the commutative diagram one easily obtains the partial results for the join $M_{l_1,l_2,\mathbf{w}}(V) = V_2(\mathbb{R}^{2n+1})\star_{l_1,l_2} S^3_{\mathbf{w}}$ when $n>2$, namely $M_{l_1,l_2,\mathbf{w}}(V)$ is simply connected, $H^3(M_{l_1,l_2,\mathbf{w}}(V),\mathbb{Z}) = H^5(M_{l_1,l_2,\mathbf{w}}(V),\mathbb{Z}) = 0$, and

(8)
$$\pi_1(M_{l_1,l_2,\mathbf{w}}(V)) = H^2(M_{l_1,l_2,\mathbf{w}}(V),\mathbb{Z}) \approx \mathbb{Z}, \quad H^4(M_{l_1,l_2,\mathbf{w}}(V),\mathbb{Z}) \approx \mathbb{Z}_{w_1w_2l_1^2}.$$

Since the Stiefel manifolds $V_2(\mathbb{R}^{2n+1})$ are S^1-bundles over a complex quadric, they are special cases of the next example.

Example 3.5. The homogeneous 3-Sasakian 11-manifold $G_2/Sp(1)_+$. It is a rational homology sphere with a $\mathbb{Z}_3$ in cohomological degrees 4 and 8. By Proposition 2.3 in [BG00] the Fano index $\mathcal{I}$ associated to $G_2/Sp(1)_+$ is 3. Then by Proposition 2.2 we have Sasaki-Einstein metrics on the simply connected 13-manifolds, $G_2/Sp(1)_+ \star_{3,|\mathbf{w}|} S^3_{\mathbf{w}}$ if 3 does not divide $|\mathbf{w}|$, and $G_2/Sp(1)_+ \star_{1,\frac{|\mathbf{w}|}{3}} S^3_{\mathbf{w}}$ if 3 divides $|\mathbf{w}|$. These 13-manifolds are simply connected with $\pi_2 = \mathbb{Z}$ and torsion in H^4.

3.1.3. *M is the link of a Fermat hypersurface.* The projective Fermat hypersurface $F_{d,n+1}$ of degree d in $\mathbb{CP}^{n+1}$ is described in homogeneous coordinates by the equation

$$z_0^d + z_1^d + \cdots + z_{n+1}^d = 0. \tag{9}$$

It is Fano when $d \leq n+1$ with index $\mathcal{I}_{F_{d,n+1}} = n+2-d$ when $d \leq n+1$. Moreover, they have a Kähler-Einstein metric when $\frac{n+1}{2} \leq d \leq n+1$. So for this range of d the Sasakian circle bundle $\mathcal{S}_{d,n+1}$ over $F_{d,n+1}$ has a Sasaki-Einstein metric [BG00]. Note that $F_{2,2n}$ is the complex quadric $Q_{2n-1} \subset \mathbb{CP}^{2n}$ and $\mathcal{S}_{2,2n} = V_2(\mathbb{R}^{2n+1})$ described in Example 3.4 and they are endowed with KE and SE metrics, respectively although d is outside of the range given above. The integral cohomology ring of $F_{d,n+1}$ is well understood [KW80]. It is torsion free with $H_*(F_{d,n+1},\mathbb{Z}) = H_*(\mathbb{P}^n,\mathbb{Z})$ except in the middle dimension n where the nth cohomology group of $F_{d,n+1}$ is $\mathbb{Z}^{b_n}$ when n is odd, and $\mathbb{Z}^{b_n+1}$ when n is even, where

$$b_n = (-1)^n\Big(1 + \frac{(1-d)^{n+2}-1}{d}\Big).$$

Then if $n > 4$ we see as in Example 3.4 that the join $M_{l_1,l_2,\mathbf{w}} = \mathcal{S}_{d,n+1} \star_{l_1,l_2} S^3_{\mathbf{w}}$ is simply connected satisfying the conditions of Equation (8). In order that the join has a Sasaki-Einstein metric in its $\mathbf{w}$-Sasaki cone, we must choose the relative Fano indices to be

$$l_1(\mathbf{w}) = \frac{n+2-d}{\gcd(|\mathbf{w}|, n+2-d)}, \qquad l_2(\mathbf{w}) = \frac{|\mathbf{w}|}{\gcd(|\mathbf{w}|, n+2-d)} \tag{10}$$

with $\frac{n+1}{2} \leq d \leq n+1$ or $d = 2$. For $2 < d < \frac{n+1}{2}$ it is unknown whether there is such an SE metric.

3.2. **Examples in Dimension 7.** We focus attention to dimension seven in which case N is a del Pezzo Surface, namely $\mathbb{CP}^2, \mathbb{CP}^1 \times \mathbb{CP}^1$, and $\mathbb{CP}^2$ blown-up at k generic points with $1 \leq k \leq 8$. Then the $S^3_{\mathbf{w}}$-join a Sasakian circle bundle over N will be a Sasaki 7-manifold.

3.2.1. $M = S^5, N = \mathbb{CP}^2$. For $\mathbb{CP}^2$ with its standard Fubini-Study Kählerian structure, we have $\mathcal{I}_N = 3$. From Example 2.5 we see that we have a regular Reeb vector field in the $\mathbf{w}$-Sasaki cone in precisely two cases, either $\mathbf{w} = (2,1)$, or $\mathbf{w} = (5,1)$. In the first case the relative Fano indices are $(l_1, l_2) = (1,1)$ while in the second case they are $(l_1, l_2) = (1,2)$. In the former case our 7-manifold $M^7_{(2,1)} = S^5 \star_{1,1} S^3_{(2,1)}$ is an S^3-bundle over $\mathbb{CP}^2$; whereas, in the latter case the 7-manifold $M^7_{(5,1)} = S^5 \star_{1,2} S^3_{(5,1)}$ is an $L(2;5,1)$ bundle over $\mathbb{CP}^2$. Moreover, it follows from standard lens space theory that $L(2;5,1)$ is diffeomorphic to the real projective space $\mathbb{RP}^3$. For general $\mathbf{w}$ we have two cases by Proposition 2.2, $H^4(M^7_{\mathbf{w}}, \mathbb{Z}) = \mathbb{Z}_{w_1 w_2}$ if 3 divides $|\mathbf{w}|$ and $H^4(M^7_{\mathbf{w}}, \mathbb{Z}) = \mathbb{Z}_{9 w_1 w_2}$ if 3 does not divide $|\mathbf{w}|$. In both cases the cohomology ring is given by

$$\mathbb{Z}[x,y]/(w_1 w_2 l_1^2 x^2, x^3, x^2 y, y^2)$$

where x, y are classes of degree 2 and 5, respectively. Notice that since 3 must divide $w_1 + w_2$ in the first case and $w_1 w_2$ are relatively prime, the cohomology rings are never isomorphic for the two different cases.

Remark 3.6. Let us make a brief remark about the homogeneous case $\mathbf{w} = (1,1)$ with symmetry group $SU(3) \times SU(2) \times U(1)$. There is a unique solution with a Sasaki-Einstein metric as shown in [BG00]. However, dropping both the Einstein and Sasakian conditions, Kreck and Stolz [KS88] gave a diffeomorphism and homeomorphism classification. Furthermore, using the results of [WZ90], they show that in certain cases each of the 28 diffeomorphism types admits an Einstein metric. If we drop the Einstein condition and allow contact bundles with non-trivial c_1 we can apply the classification results of [KS88] to the Sasakian case. This will be studied elsewhere.

For dimension 7 we see from Proposition 2.2 that if 3 divides w_1+w_2 then the order $|H^4|$ is W. However, if 3 does not divide w_1+w_2 then the order of $|H^4|$ is $9W$. So by Lemma 3.1 $\mathcal{P}_W$ splits into two cases, $\mathcal{P}^0_W$ if $W+1$ is divisible by 3, and $\mathcal{P}^1_W$ if $W+1$ is not divisible by 3. Of course, in either case the cardenality of $\mathcal{P}_W$ is 2^{k-1} where k is the number of prime powers in the prime decomposition of W.

Proposition 3.7. *Suppose the order of H^4 is odd. The elements $M^7_{\mathbf{w}}$ and $M^7_{\mathbf{w}'}$ in $\mathcal{P}^0_W$ are homotopy inequivalent if and only if either*

$$\big(\frac{w_1'+w_2'}{3}\big)^3\equiv\pm\big(\frac{w_1+w_2}{3}\big)^3 \mod \mathbb{Z}_W.$$

The elements $M^7_{\mathbf{w}}$ and $M^7_{\mathbf{w}'}$ in $\mathcal{P}^1_W$ are homotopy inequivalent if and only if

$$(w_1'+w_2')^3\equiv\pm(w_1+w_2)^3 \mod \mathbb{Z}_{9W}.$$

Proof. For $r=2$ consider the E_6 differential $d_6(\beta)=l_2(\mathbf{w})^3s^3$ in the spectral sequence of Thorem 4.5 of [BTF15]. Since l_2 is relatively prime to $l_1(\mathbf{w})^2w_1w_2$, this takes values in the multiplicative group $\mathbb{Z}^*_{l_1^2W}$ of units in $\mathbb{Z}_{l_1^2W}$. Taking into account the choice of generators, it takes its values in $\mathbb{Z}^*_{l_1^2W}/\{\pm1\}$. According to Theorem 5.1 of [Kru97] $M^7_{\mathbf{w}},M^7_{\mathbf{w}'}\in\mathcal{P}_W$ are homotopy equivalent if and only if $l_2(\mathbf{w}')^3=l_2(\mathbf{w})^3$ in $\mathbb{Z}^*_{l_1^2W}/\{\pm1\}$. Of course, this means that $l_2(\mathbf{w}')^3=\pm l_2(\mathbf{w})^3$ in $\mathbb{Z}^*_{l_1^2W}$. Note that the the other two conditions of Theorem 5.1 of [Kru97] are automatically satisfied in our case. □

Using a Maple program we have checked some examples for homotopy equivalence which appears to be quite sparse. So far we haven't found any examples of a homotopy equivalence. However, we have not done a systematic computer search which we leave for future work.

Example 3.8. Our first example is an infinite sequence of pairs with the same cohomology ring. Set $W=3p$ with p an odd prime not equal to 3, which gives $P_k=2$. Then for each odd prime $p\neq3$ there are two manifolds in $\mathcal{P}^1_W$, namely $M^7_{(3p,1)}$ and $M^7_{(p,3)}$. The order of H^4 is $27p$. We check the conditions of Proposition 3.7. We find

$$(3p+1)^3\equiv 9p+1 \mod 27p, \qquad (p+3)^3\equiv p^3+9p^2+27 \mod 27p.$$

First we look for integer solutions of $p^3+9p^2-9p+26\equiv0 \mod 27p$. By the rational root test the solutions could only be $p=2,13,26$ none of which are solutions. Next we check the second condition of Proposition 3.7, namely, $p^3+9p^2+9p+28\equiv0 \mod 27p$. Again by the rational root test we find the only possibilities are $p=2,7,14,28$, from which s we see that there are no solutions. Thus, we see that $M^7_{(3p,1)}$ and $M^7_{(p,3)}$ are not homotopy equivalent for any odd $p \neq 3$.

By the same arguments one can also show that the infinite sequence of pairs of the form $M^7_{(9p,1)}$ and $M^7_{(p,9)}$, with p an odd prime relatively prime to 3, are never homotopy equivalent.

Remark 3.9. In Example 3.8 we do not need to have p a prime, but we do need it to be relatively prime to 3. In this more general case, there will be more elements in $\mathcal{P}^1_W$. For example, if $p = 55$ we have $P_k = 4$ and the pair $(M^7_{(165,1)}, M^7_{(55,3)}$ has the same cohomology ring as $M^7_{(33,5)}$ and $M^7_{(15,11)}$. However, they are not homotopy equivalent to either member of the pair nor to each other.

Example 3.10. A somewhat more involved example is obtained by setting $W = 5 \cdot 7 \cdot 11 \cdot 17$. Here $P_k = 8$, so this gives eight 7-manifolds in $\mathcal{P}^0_W$, namely,

$$M^7_{(6545,1)}, M^7_{(1309,5)}, M^7_{(935,7)}, M^7_{(595,11)}, M^7_{(385,17)}, M^7_{(187,35)}, M^7_{(119,55)}, , M^7_{(85,77)}.$$

One can check that these do not satisfy the conditions for homotopy equivalence of Proposition 3.7. So they are all homotopy inequivalent.

It is easy to get a necessary condition for homeomorphism.

Proposition 3.11. *Suppose $w_1'w_2' = w_1w_2$ is odd and that $M^7_{\mathbf{w}}$ and $M^7_{\mathbf{w}'}$ are homeomorphic. Then in addition to the conditions of Proposition 3.7, we must have*

$$2(w_1' + w_2')^2 \equiv 2(w_1 + w_2)^2 \mod 3w_1w_2.$$

Proof. This is because the first Pontrjagin class p_1 is actually a homeomorphism invariant[3]. From Kruggel [Kru97] we see that if 3 does not divide $|\mathbf{w}|$

$$p_1(M^7_{\mathbf{w}}) \equiv 3|\mathbf{w}|^2 - 9w_1^2 - 9w_2^2 \equiv -6|\mathbf{w}|^2 \mod 9w_1w_2, \tag{11}$$

which implies the result in this case. If 3 divides $|\mathbf{w}|$ we have

$$p_1(M^7_{\mathbf{w}}) \equiv -6\Big(\frac{|\mathbf{w}|}{3}\Big)^2 \mod w_1w_2 \tag{12}$$

and this implies the same result. □

Note that Equations (11) and (12) both imply the third condition of Theorem 5.1 in [Kru97] holds in our case. To determine a full homeomorphism and diffeomorphism classification requires the Kreck-Stolz

[3]This appears to be a folklore result with no proof anywhere in the literature. It is stated without proof on page 2828 of [Kru97] and on page 31 of [KL05]. We thank Matthias Kreck for providing us with a proof that p_1 is a homeomorphism invariant.

invariants [KS88] $s_1, s_2, s_3 \in \mathbb{Q}/\mathbb{Z}$. These can be determined as functions of $\mathbf{w}$ in our case by using the formulae in [Esc05, Kru05]; however, they are quite complicated and the classification requires computer programing which we leave for future work.

It is interesting to compare the Sasaki-Einstein 7-manifolds described in this section with the 3-Sasakian 7-manifolds studied in [BGM94, BG99] for their cohomology rings have the same form. Seven dimensional manifolds whose cohomology rings are of this type were called 7-manifolds of type r in [Kru97] where r is the order of H^4. First recall that the 3-Sasakian 7-manifolds in [BGM94] are given by a triple of pairwise relatively prime positive integers (p_1, p_2, p_3) and H^4 is isomorphic to $\mathbb{Z}_{\sigma_2(\mathbf{p})}$ where $\sigma_2(\mathbf{p}) = p_1p_2 + p_1p_3 + p_2p_3$ is the second elementary symmetric function of $\mathbf{p} = (p_1, p_2, p_3)$. It follows that σ_2 is odd. The following theorem is implicit in [Kru97], but we give its simple proof here for completeness.

Theorem 3.12. *The 7-manifolds $M^7_{\mathbf{p}}$ and $M^7_{\mathbf{w}}$ are not homotopy equivalent for any admissible* $\mathbf{p}$ *or* $\mathbf{w}$.

Proof. These manifolds are distinguished by π_4. Our manifolds $M^7_{\mathbf{w}}$ are quotients of $S^5 \times S^3$ by a free S^1-action, whereas, the manifolds $M^7_{\mathbf{p}}$ of [BGM94] are free S^1 quotients of $SU(3)$. So from their long exact homotopy sequences we have $\pi_i(M^7_{\mathbf{w}}) \approx \pi_i(S^5 \times S^3)$ and $\pi_i(M^7_{\mathbf{p}}) \approx \pi_i(SU(3))$ for all $i > 2$. But it is known that $\pi_4(SU(3)) \approx 0$ whereas, $\pi_4(S^5 \times S^3) \approx \mathbb{Z}_2$. □

3.2.2. $M = S^2 \times S^3, N = \mathbb{CP}^1 \times \mathbb{CP}^1$. Note that this is Example 3.1.3 with $n = d = 2$. We have $\mathcal{I}_N = 2$, so there are two cases: $|\mathbf{w}|$ is odd impying $l_2 = |\mathbf{w}|$ and $l_1 = 2$; and $|\mathbf{w}|$ is even with $l_2 = \frac{|\mathbf{w}|}{2}$ and $l_1 = 1$. In both cases the smoothness condition $\gcd(l_2, l_1 w_i) = 1$ is satisfied. The E_2 term of the Leray-Serre spectral sequence of the top fibration of diagram (6) is

$$E_2^{p,q} = H^p(\mathsf{B}S^1, H^q(S^2 \times S^3 \times S^3_{\mathbf{w}}, \mathbb{Z})) \approx \mathbb{Z}[s] \otimes \mathbb{Z}[\alpha]/(\alpha^2) \otimes \Lambda[\beta, \gamma],$$

which by the Leray-Serre Theorem converges to $H^{p+q}(M_{l_1,l_2,\mathbf{w}}, \mathbb{Z})$. Here α is a 2-class and β, γ are 3-classes. From the bottom fibration in Diagram (6) we have $d_2(\beta) = \alpha \otimes s_1$ and $d_4(\gamma) = w_1 w_2 s_2^2$. From the commutativity of diagram (6) we have $d_2(\beta) = l_2 s$ and $d_4(\gamma_{\mathbf{w}}) = w_1 w_2 l_1^2 s^2$ which gives $E_4^{4,0} \approx \mathbb{Z}_{w_1 w_2 l_1^2}$, $E_4^{0,3} \approx \mathbb{Z}$, $E_4^{2,2} \approx \mathbb{Z}_{l_2}$, and $E_\infty^{0,3} = 0$. Then using Poincaré duality and universal coefficients we obtain

Proposition 3.13. *In this case* $M^7_{l_1,l_2,\mathbf{w}}$ *with either* $(l_1,l_2)=(2,|\mathbf{w}|)$ *or* $(1,\frac{|\mathbf{w}|}{2})$ *has the cohomology ring given by*

$$H^*(M^7_{l_1,l_2,\mathbf{w}},\mathbb{Z})=\mathbb{Z}[x,y,u,z]/(x^2,l_2xy,w_1w_2l_1^2y^2,z^2,u^2,zu,zx,ux,uy)$$

where x,y *are 2-classes, and* z,u *are 5-classes.*

There is only one case with a regular Reeb vector field, and that is $\mathbf{w}=(3,1)$ in which case the relative Fano indices are $(1,2)$. Then the 7-manifold is $(S^2\times S^3)\star_{1,2}S^3_{(3,1)}$ can be realized as an $L(2;3,1)\approx\mathbb{RP}^3$ lens space bundle over $\mathbb{CP}^1\times\mathbb{CP}^1$. Proposition 3.13 and Theorem 1.1 prove Theorem 1.2.

3.2.3. *$M=k(S^2\times S^3), N=\mathbb{CP}^2$ blown-up at k generic points with $k=1,\ldots,8$.* Equivalently we write $N=N_k=\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$. All the Kähler structures have an extremal representative, but for $k=1,2$ they are not CSC. However, for $k=3,\ldots,8$ they are CSC, and hence, Kähler-Einstein. Notice that when $4\leq k\leq 8$ the complex automorphism group has dimension 0, so the $\mathbf{w}$-Sasaki cone is the entire Sasaki cone. Moreover, if $5\leq k\leq 8$ the local moduli space has positive dimension, and we can choose any of the complex structures. By a theorem of Kobayashi and Ochiai [KO73] we have $\mathcal{I}_{N_k}=1$ for all $k=1,\ldots,8$. So $l_1=1,l_2=|\mathbf{w}|$, and by Corollary 2.4 there are no regular Reeb vector fields in the $\mathbf{w}$-Sasaki cone with $\mathbf{w}\neq(1,1)$. In particular, if $4\leq k\leq 8$, there are no regular Reeb vector fields in the Sasaki cone. Generally, these are $L(|\mathbf{w}|;w_1,w_2)$ lens space bundles over N_k. Of course, the case $\mathbf{w}=(1,1)$ is just an S^1-bundle over $N_k\times\mathbb{CP}^1$ with the product complex structure which is automatically regular. These were studied in [BG00]. Let $\mathcal{S}_k$ denote the total space of the principal S^1-bundle over N_k corresponding to the anticanonical line bundle K^{-1} on N_k. By a well-known result of Smale $\mathcal{S}_k$ is diffeomorphic to the k-fold connected sum $k(S^2\times S^3)$. We consider the join $\mathcal{S}_k\star_{1,|\mathbf{w}|}S^3_{\mathbf{w}}$. The case $\mathbf{w}=(1,1)$ was studied in [BG00] where it is shown to have a Sasaki-Einstein metric when $3\leq k\leq 8$. Moreover, in this case we have determined the integral cohomology ring (see Theorem 5.4 of [BG00]). Here we generalize this result.

Proposition 3.14. *The integral cohomology ring of the 7-manifolds* $M^7_{k,\mathbf{w}}=\mathcal{S}_k\star_{1,|\mathbf{w}|}S^3_{\mathbf{w}}$ *is given by*

$$H^q(M^7_{k,\mathbf{w}},\mathbb{Z})\approx\begin{cases}\mathbb{Z} & \text{if } q=0,7;\\ \mathbb{Z}^{k+1} & \text{if } q=2,5;\\ \mathbb{Z}^k_{|\mathbf{w}|}\times\mathbb{Z}_{w_1w_2} & \text{if } q=4;\\ 0 & \text{if otherwise,}\end{cases}$$

with the ring relations determined by $\alpha_i \cup \alpha_j = 0, w_1w_2s^2 = 0, |\mathbf{w}|\alpha_i \cup s = 0$, *where* α_i, s *are the* $k+1$ *two classes with* $i = 1, \cdots k$.

Proof. As before the E_2 term of the Leray-Serre spectral sequence of the top fibration of diagram (6) is

$$E_2^{p,q} = H^p(\mathsf{B}S^1, H^q(\mathcal{S}_k \times S^3_{\mathbf{w}}, \mathbb{Z})) \approx \mathbb{Z}[s] \otimes \prod_i \Lambda[\alpha_i, \beta_i, \gamma]/\mathfrak{I},$$

where $\alpha_i, \beta_j, \gamma$ have degrees $2, 3, 3$, respectively, and $\mathfrak{I}$ is the ideal generated by the relations $\alpha_i \cup \beta_i = \alpha_j \cup \beta_j, \alpha_i \cup \alpha_j = \beta_i \cup \beta_j = 0$ for all i, j, $\alpha_i \cup \beta_j = 0$ for $i \neq j$ and $\gamma^2 = 0$.

Consider the lower product fibration of diagram (6). As in the previous case the first non-vanishing differential of the second factor is d_4, and as in that case $d_4(\gamma) = w_1w_2s_2^2$. For the first factor we know from Smale's classification of simply connected spin 5-manifolds that $\mathcal{S}_k$ is diffeomorphic to the k-fold connected sum $k(S^2 \times S^3)$. Moreover, since $N = \mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$, the first factor fibration is

$$k(S^2 \times S^3) \longrightarrow \mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2 \longrightarrow \mathsf{B}S^1.$$

Here the first non-vanishing differential is $d_2(\beta_i) = \alpha_i \otimes s$. Again from the commutativity of diagram (6) for the top fibration we have $d_2(\beta_i) = |\mathbf{w}|\alpha_i \otimes s$ at the E_2 level and $d_4(\gamma) = w_1w_2s^2$ at the E_4 level. One easily sees that the $k+1$ 2-classes $\alpha_i \in E_2^{2,0}$ and $s \in E_2^{0,2}$ live to E_∞ and there is no torsion in degree 2. Moreover, there is nothing in degree 1, and the 3-classes $\beta_i \in E_2^{3,0}$ and $\gamma \in E_4^{3,0}$ die, so there is nothing in degree 3. However, there is torsion in degree 4, namely $\mathbb{Z}^k_{|\mathbf{w}|} \times \mathbb{Z}_{w_1w_2}$. The remainder follows from Poincaré duality and dimensional considerations. □

This generalizes Theorem 5.4 of [BG00] where the case $\mathbf{w} = (1,1)$ is treated and together with Theorem 1.1 proves Theorem 1.3.

Remark 3.15. Since $|\mathbf{w}|$ and w_1w_2 are relatively prime, $H^4(M^7_{k,\mathbf{w}}, \mathbb{Z}) \approx \mathbb{Z}^{k-1}_{|\mathbf{w}|} \times \mathbb{Z}_{w_1w_2|\mathbf{w}|}$. We can ask the question: when can $M^7_{k,\mathbf{w}}$ and $M^7_{k',\mathbf{w}'}$ have isomorphic cohomology rings? It is interesting and not difficult to see that there is only one possibility, namely $M^7_{1,(3,2)}$ and $M^7_{1,(5,1)}$ in which case $H^4 \approx \mathbb{Z}_{30}$.

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