THE SASAKI JOIN, HAMILTONIAN 2-FORMS, AND CONSTANT SCALAR CURVATURE

CHARLES P. BOYER AND CHRISTINA W. TØNNESEN-FRIEDMAN

Abstract. We describe a general procedure for constructing new Sasaki metrics of constant scalar curvature (CSC) from old ones. Explicitly, we begin with a regular Sasaki metric of constant scalar curvature on a $2n+1$-dimensional compact manifold M and construct a sequence of rays of CSC Sasaki metrics on a compact Sasaki manifolds $M_{l_1,l_2,\mathbf{w}}$ of dimension $2n+3$ which depend on four integral parameters l_1, l_2, w_1, w_2. Most of the CSC Sasaki metrics are irregular. We also give examples which show that the CSC rays are often not unique on the underlying fixed strictly pseudoconvex CR manifold.

1. Introduction

The purpose of this paper is to present a general geometric construction that combines the Sasaki join construction of [BG00, BGO07] with the Hamiltonian 2-form formalism of [ACG06, ACGTF04, ACGTF08] to construct many new Sasaki metrics of constant scalar curvature. This method has already been used by the authors in special cases [BTF13a, BTF13c, BTF14a, BTF13b]. The method is the following: consider a regular Sasaki manifold M with its 'Boothby-Wang circle bundle' $S^1 \longrightarrow M \longrightarrow N$ over the Kähler manifold N. For each pair of relatively prime positive integers (l_1, l_2) we form the Sasaki join $M_{l_1,l_2,\mathbf{w}}$ of M with a weighted 3-sphere $S^3_{\mathbf{w}}$ (cf. [BG08], Example 7.1.12), where the components of the weight vector $\mathbf{w} = (w_1, w_2)$ are relatively prime positive integers satisfying $w_1 \geq w_2$. The latter has a 2-dimensional Sasaki cone $\mathfrak{t}^+_{\mathbf{w}}$ we call the $\mathbf{w}$-Sasaki cone. Now we can deform within $\mathfrak{t}^+_{\mathbf{w}}$ to obtain other Sasakian structures. The quasi-regular ones will fiber over a ruled orbifold (S_n, Δ) with the following structure. S_n is a $\mathbb{CP}^1$-bundle over N with an orbifold structure on its fibers giving rise to a branch divisor Δ. The orbifold (S_n, Δ) is a projectivization

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$\mathbb{P}(\mathbb{1} \oplus L_n)$ where L_n is certain line bundle over N, and it admits a Hamiltonian 2-form. The explicit nature of this formalism allows us to obtain extremal (or constant scalar curvature) Kähler orbifold metrics on (S_n, Δ). Then by a well known procedure we obtain extremal (constant scalar curvature) Sasaki metrics on the join $M_{l_1,l_2,\mathbf{w}}$. This approach was initiated in [BTF14a] for the case $l_2 = 1$ (the case of arbitrary l_2 is being worked out in [Cas14]), and continued in [BTF13b] for Sasaki-Einstein metrics.

Our main theorem is:

Theorem 1.1. *Let $M_{l_1,l_2,\mathbf{w}} = M \star_{l_1,l_2} S^3_{\mathbf{w}}$ be the $S^3_{\mathbf{w}}$-join with a regular Sasaki manifold M which is an S^1-bundle over a compact Kähler manifold N with constant scalar curvature. Then for each vector $\mathbf{w} = (w_1, w_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ with relatively prime components satisfying $w_1 > w_2$ there exists a Reeb vector field $\xi_{\mathbf{v}}$ in the 2-dimensional $\mathbf{w}$-Sasaki cone on $M_{l_1,l_2,\mathbf{w}}$ such that the corresponding ray of Sasakian structures $\mathcal{S}_a = (a^{-1}\xi_{\mathbf{v}}, a\eta_{\mathbf{v}}, \Phi, g_a)$ has constant scalar curvature.*

The manifolds $M_{l_1,l_2,\mathbf{w}}$ can also be realized as certain three dimensional lens space bundles over N.

Suppose in addition that the scalar curvature of N satisfies $s_N \geq 0$, then we obtain more information about extremal Sasaki metrics. In fact, we have

Theorem 1.2. *Suppose that in addition to the hypothesis of Theorem 1.1 the scalar curvature of N satisfies $s_N \geq 0$, then the $\mathbf{w}$-Sasaki cone is exhausted by extremal Sasaki metrics. In particular, if the Kähler structure on N admits no Hamiltonian vector fields, then the entire Sasaki cone κ of the join $M_{l_1,l_2,\mathbf{w}} = M \star_{l_1,l_2} S^3_{\mathbf{w}}$ can be represented by extremal Sasaki metrics.*

A particular example of interest when the hypothesis of the last sentence of Theorem 1.2 is satisfied is when N is an algebraic K3 surface. In this case there are many choices of complex structures and many choices of line bundles. But in all cases $M = 21\#(S^2 \times S^3)$. It is interesting to contemplate the possible diffeomorphism types of the 7-manifolds $21\#(S^2 \times S^3) \star_{l_1,l_2} S^3_{\mathbf{w}}$ in this case.

We also give examples where there are more than one CSC ray in the same $\mathbf{w}$-Sasaki cone. Indeed, generally we have

Theorem 1.3. *Suppose that in addition to the hypothesis of Theorem 1.1 the scalar curvature of N satisfies $s_N > 0$. Then for sufficiently large l_2 there are at least three CSC rays in the $\mathbf{w}$-Sasaki cone of the join $M_{l_1,l_2,\mathbf{w}}$.*

In particular, Theorem 5.12 below gives a countable infinity of inequivalent contact structures on the two S^5-bundles over S^2 such that there are at least three CSC rays of Sasaki metrics. However, the bouquet phenomenon which is related to distinct underlying CR structures and appears for S^3-bundles over Riemann surfaces [Boy11, Boy13, BTF13a, BTF14a] seems not to occur in these more general cases. This is related to the topological rigidity of the Boothby-Wang base space as discussed briefly in Section 2.2 below. The non-uniqueness described in Theorem 1.3 occurs on a fixed strictly pseudoconvex CR manifold and a fixed contact manifold. The former also illustrates non-uniqueness on the sub-Riemannian level.

It should be mentioned that generally the CSC rays are most often irregular, that is the closure of a generic Reeb orbit is a torus of dimension greater than one. In this regard in Section 5.1 we fill in a gap that occured in the first version of [BTF13b] concerning the application of the admissibility conditions to irregular Sasakian structures. This was kindly pointed to us by an anonymous referee. It has been shown recently that irregular Sasakian structures have irreducible transverse holonomy [HS12], and that the corresponding Kähler cone is K-semistable [CS12] for CSC Sasaki metrics. Also the non-uniqueness phenomenon of CSC Sasakian structures was first shown to occur for the case of S^3-bundles over S^2 by a different method in [Leg11]. Theorem 1.3 shows that this is fairly common.

2. Ruled Manifolds

In this section we consider ruled manifolds of the following form. Let (N,ω_N) be a compact Kähler manifold with primitive integer Kähler class $[\omega_N]$, that is, a Hodge manifold. Consider a rank two complex vector bundle of the form $E=\mathbb{1}\oplus L$ where L is a complex line bundle on N and $\mathbb{1}$ denotes the trivial bundle. By a ruled manifold we shall mean the projectivization $S=\mathbb{P}(\mathbb{1}\oplus L)$. We can view S as a compactification of the complex line bundle L on N by adding the 'section at infinity'. For $x\in N$ we let (c,z) denote a point of the fiber $E_x=\mathbb{1}\oplus L_x$. There is a natural action of $\mathbb{C}^*$ (hence, S^1) on E given by $(c,z)\mapsto (c,\lambda z)$ with $\lambda\in\mathbb{C}^*$. The action $z\mapsto\lambda z$ is a complex irreducible representation of $\mathbb{C}^*$ determined by the line bundle L. Such representations (characters) are labeled by the integers $\mathbb{Z}$. Thus, we write $L=L_n$ for $n\in\mathbb{Z}$ and refer to n as the 'degree' of L.

2.1. **A Construction of Ruled Manifolds.**

We now give a construction of such manifolds. Let $S^1\longrightarrow M\longrightarrow N$ be the circle bundle over N determined by the class $[\omega_N]\in H^2(N,\mathbb{Z})$. We denote the

S^1-action by $(x,u)\mapsto(x,e^{i\theta}u)$. Now represent $S^3\subset\mathbb{C}^2$ as $|z_1|^2+|z_2|^2=1$ and consider an S^1-action on $M\times S^3$ given by $(x,u;z_1,z_2)\mapsto(x,e^{i\theta}u;z_1,e^{in\theta}z_2)$. There is also the standard S^1-action on S^3 given by $(z_1,z_2)\mapsto(e^{i\chi}z_1,e^{i\chi}z_2)$ giving a T^2-action on $M\times S^3$ defined by

$$(x,u;z_1,z_2)\mapsto(x,e^{i\theta}u;e^{i\chi}z_1,e^{i(\chi+n\theta)}z_2). \tag{1}$$

Lemma 2.1. *The quotient by the T^2-action of Equation (1) is the projectivization $S_n=\mathbb{P}(\mathbb{1}\oplus L_n)$.*

Proof. First we see from (1) that the action is free, so there is a natural bundle projection $(M\times S^3)/T^2\longrightarrow N$ defined by $\pi(x,[u;z_1,z_2])=x$ where the bracket denotes the T^2 equivalence class. The fiber is $\pi^{-1}(x)=[u;z_1,z_2]$ which since u parameterizes a circle is identified with $S^3/S^1=\mathbb{CP}^1$. This bundle is trivial if and only if $n=0$ and n labels the irreducible representation of S^1 on the line bundle L_n. □

We can take the line bundle L_1 to be any primitive line bundle in $\mathrm{Pic}(N)$. In particular, we are interested in the taking L_1 to be the line bundle associated to the primitive cohomology class $[\omega_N]\in H^2(N,\mathbb{Z})$. Then we have

Lemma 2.2. *The following relation holds: $c_1(L_n)=n[\omega_N]$.*

Proof. Equation (1) implies that the S^1-action on the line bundle L_n is given by $z\mapsto e^{in\theta}z$. But we know that the definition of M that it is the unit sphere in the line bundle over N corresponding to $n=1$, and this corresponds to the class $[\omega_N]$, that is $c_1(L_1)=[\omega_N]$. Thus, $c_1(L_n)=n[\omega_N]$. □

2.2. **Ruled Manifolds with known Diffeomorphism Type.** There are several cases when the diffeomorphism type of the ruled manifold can be ascertained. First we have the case when $N=\Sigma_g$ a Riemann surface of genus g. It is well known [MS98] that in this case there are precisely two diffeomorphism types. They are distinguished by their second Stiefel-Whitney class. This gives rise to inequivalent Kähler structures belonging to the same underlying symplectic structure (up to symplectomorphism). It also gives rise to non-conjugate maximal tori in the symplectomorphism group, a fact that was exploited in [Boy11, Boy13, BP14, BTF13a, BTF14a].

On the other hand it appears that this phenomenon changes in higher dimension. It is still known to occur as witnessed by the polygon spaces of [HT03] and described in Example 8.5 of [Boy13]. However, it has been shown recently [CMS10, CPS12] that for $N=\mathbb{CP}^p$ with $p>1$ the two ruled manifolds S_n and $S_{n'}$ are diffeomorphic if and only if

$|n'| = |n|$. Indeed,the diffeomorphism type is determined completely by its cohomology ring which takes the form

$$H^*(S_n, \mathbb{Z}) = \mathbb{Z}[x_1, x_2]/\big(x_1^{p+1}, (x_2(nx_1 + x_2)\big) \tag{2}$$

where x_1, x_2 have degree 2. So the Hirzebruch-type phenomenon involving inequivalent complex structures on the same manifold does not generalize when $p > 1$.

2.3. **The Admissible Construction.** We will now assume that n from Section 2.1 is non-zero and (N, ω_N) defines a Kähler structure with CSC Kähler metric g_N. Then $(\omega_{N_n}, g_{N_n}) := (2n\pi\omega_N, 2n\pi g_N)$ satisfies that (g_{N_n}, ω_{N_n}) or $(-g_{N_n}, -\omega_{N_n})$ is a Kähler structure (depending on the sign of n). In either case, we let $(\pm g_{N_n}, \pm\omega_{N_n})$ refer to the Kähler structure. We denote the real dimension of N by $2d_N$ and write the scalar curvature of $\pm g_{N_n}$ as $\pm 2d_{N_n}s_{N_n}$. [So, if e.g. $-g_{N_n}$ is a Kähler structure with positive scalar curvature, s_{N_n} would be negative.]

Now Lemma 2.2 implies that $c_1(L_n) = [\omega_{N_n}/2\pi]$. Then, following [ACGTF08], the total space of the projectivization $S_n = \mathbb{P}(\mathbb{1} \oplus L_n)$ is called *admissible*.

On these manifolds, a particular type of Kähler metric on S_n, also called *admissible*, can now be constructed [ACGTF08]. We shall describe this construction in Section 4 where we will use it to prove Theorems 1.1 and 1.2 of the Introduction. An admissible Kähler manifold is a special case of a Kähler manifold admitting a so-called Hamiltonian 2-form [ACG06]. More specifically, the admissible metrics as described in section 4 admit a Hamiltonian 2-form of order one.

Remark 2.3. In the special case where (N, ω_N) is Kähler-Einstein with Kähler metric g_N and Ricci form $\rho_N = 2\pi\mathcal{I}_N\omega_N$, where $\mathcal{I}_N$ denotes the Fano index, there is a simple relationship between the value of s_{N_n} and the value of n. Since the (scale invariant) Ricci form is given by $\rho_N = s_{N_n}\omega_{N_n}$, it is easy to see that $s_{N_n} = \mathcal{I}_N/n$. For the general CSC case this will be more complicated and will need to be handled case by case. We do know that if we write $s_{N_n} = p_n/n$, then $p_n \leq d_N + 1$ (see Remark 1 in [ACGTF08]).

3. The $S^3_{\mathbf{w}}$-Join Construction

The join construction was first introduced in [BG00] for Sasaki-Einstein manifolds, and later generalized to any quasi-regular Sasakian manifolds in [BGO07] (see also Section 7.6.2 of [BG08]). However, as pointed out in [BTF13a] it is actually a construction involving the orbifold Boothby-Wang construction [BW58, BG00], and so applies to quasi-regular strict contact structures. Although it is quite natural to do so, we do not need to fix the transverse (almost) complex structure. Moreover, in [BTF14a] it was shown that in the special case of S^3-bundles over Riemann surfaces a twisted transverse complex structure on a regular Sasakian manifold can be realized by a product transverse complex structure on a certain quasi-regular Sasakian structure in the same Sasaki cone.

We consider a generalization of the join construction used in previous work [BTF13a, BTF14a, BTF13b]. We refer to [BGO07, BG08] for a thorough discussion of the join construction. Here we let M be a regular Sasakian manifold with constant scalar curvature, and consider the join $M_{l_1,l_2,\mathbf{w}} = M \star_{l_1,l_2} S^3_{\mathbf{w}}$ with the weighted 3-sphere $S^3_{\mathbf{w}}$ (a sphere with a weighted circle action) where both $\mathbf{w} = (w_1, w_2)$ and $\mathbf{l} = (l_1, l_2)$ are pairs of relatively prime positive integers. We can assume that the weights (w_1, w_2) are ordered, namely they satisfy $w_1 \geq w_2$. Furthermore, $M_{l_1,l_2,\mathbf{w}}$ is a smooth manifold if and only if $\gcd(l_2, l_1w_1w_2) = 1$ which is equivalent to $\gcd(l_2, w_i) = 1$ for $i = 1, 2$. Henceforth, we shall assume these conditions.

The join is constructed from the following commutative diagram

$$
\begin{array}{ccc}
M \times S^3_{\mathbf{w}} & & \\
 & \searrow \pi_L & \\
\Big\downarrow \pi_2 & & M_{l_1,l_2,\mathbf{w}} \\
 & \swarrow \pi_1 & \\
N \times \mathbb{CP}^1[\mathbf{w}] & &
\end{array}
\tag{3}
$$

where the πs are the obvious projections. Here M has a regular contact form η_1 with Reeb vector field ξ_1, and $S^3_{\mathbf{w}}$ has the weighted contact form η_2 with Reeb vector field $\xi_2 = w_1H_1 + w_2H_2$ where H_i is the infinitesimal generators of the S^1 action on

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$$

given by sending z_i to $e^{i\theta}z_i$. The circle projection π_L is generated by the vector field

$$L_{l_1,l_2,\mathbf{w}} = \frac{1}{2l_1}\xi_1 - \frac{1}{2l_2}\xi_2. \tag{4}$$

Moreover, the 1-form $\eta_{l_1,l_2,\mathbf{w}} = l_1\eta_1 + l_2\eta_2$ on $M \times S^3$ passes to the quotient $M_{l_1,l_2,\mathbf{w}}$ and gives it a contact structure. The Reeb vector field of $\eta_{l_1,l_2,\mathbf{w}}$ is the vector field

$$\xi_{l_1,l_2,\mathbf{w}} = \frac{1}{2l_1}\xi_1 + \frac{1}{2l_2}\xi_2. \tag{5}$$

The base orbifold $N \times \mathbb{CP}^1[\mathbf{w}]$ has a natural Kähler structure, namely the product structure, and this induces a Sasakian structure $\mathcal{S}_{l_1,l_2,\mathbf{w}} = (\xi_{l_1,l_2,\mathbf{w}}, \eta_{l_1,l_2,\mathbf{w}}, \Phi, g)$ on $M_{l_1,l_2,\mathbf{w}}$. The transverse complex structure $J = \Phi|_{\mathcal{D}_{l_1,l_2,\mathbf{w}}}$ is the lift of the product complex structure on $N \times \mathbb{CP}^1[\mathbf{w}]$.

It follows from Proposition 7.6.7 of [BG08] that the join $M_{l_1,l_2,\mathbf{w}}$ can also be realized as a fiber bundle over N associated to the principal S^1-bundle $M \longrightarrow N$ with fiber the lens space $L(l_2; l_1w_1, l_1w_2)$. It is easy to see that the join of extremal (CSC) Sasaki metrics gives an extremal (CSC) Sasaki metric induced by the product extremal (CSC) Kähler metrics. Thus, since weighted projective spaces have extremal orbifold metrics, we can take the Sasakian structure $\mathcal{S}_{l_1,l_2,\mathbf{w}}$ to be extremal. However, most of the CSC Sasaki metrics of interest in this work are not induced by the product of CSC Kähler metrics.

3.1. **The First Chern Class.** Let us compute the first Chern class of our induced contact structure $\mathcal{D}_{l_1,l_2,\mathbf{w}}$ on $M \star_{l_1,l_2} S^3_{\mathbf{w}}$. The orbifold first Chern class of the base is

$$c_1^{orb}(N \times \mathbb{CP}^1[\mathbf{w}]) = c_1(N) + \frac{|\mathbf{w}|}{w_1w_2} PD(D) \tag{6}$$

as an element of $H^2(N \times \mathbb{CP}^1[\mathbf{w}], \mathbb{Q}) \approx H^2(N, \mathbb{Q}) \oplus H^2(\mathbb{CP}^1[\mathbf{w}]), \mathbb{Q})$ where D a divisor given by $z_1 = 0$ or $z_2 = 0$ and PD denotes Poincaré dual. The Kähler form on $N \times \mathbb{CP}^1[\mathbf{w}]$ is $\omega_{l_1,l_2} = l_1\omega_N + l_2\omega_{\mathbf{w}}$ where $\omega_{\mathbf{w}}$ is the standard Kähler form on $\mathbb{CP}^1[\mathbf{w}]$ which satisfies $[\omega_{\mathbf{w}}] = \frac{[\omega_0]}{w_1w_2}$ where ω_0 is the standard volume form on $\mathbb{CP}^1$. Note that $PD(D) = [\omega_0]$. Pulling ω_{l_1,l_2} back to the join $M_{l_1,l_2,\mathbf{w}} = M \star_{l_1,l_2} S^3_{\mathbf{w}}$ we have $\pi^*\omega_{l_1,l_2} = d\eta_{l_1,l_2,\mathbf{w}}$ implying that $l_1\pi^*[\omega_N] + l_2\pi^*[\omega_{\mathbf{w}}] = 0$ in $H^2(M_{l_1,l_2,\mathbf{w}}, \mathbb{Z})$. So taking $\pi^*[\omega_N] = l_2\gamma$ and $\pi^*[\omega_{\mathbf{w}}] = -l_1\gamma$ for some generator $\gamma \in H^2(M_{l_1,l_2,\mathbf{w}}, \mathbb{Z})$ we have

$$c_1(\mathcal{D}_{l_1,l_2,\mathbf{w}}) = \pi^*c_1(N) - l_1|\mathbf{w}|\gamma. \tag{7}$$

Taking the mod 2 reduction gives the second Stiefel-Whitney class of $M_{l_1,l_2,\mathbf{w}}$, viz.

$$w_2(M_{l_1,l_2,\mathbf{w}}) = \pi^*w_2(N) - \rho(l_1|\mathbf{w}|\gamma) \tag{8}$$

where ρ is the reduction mod 2 map. This implies

Corollary 3.1. *If l_1 is even or if w_i are both odd for $i = 1, 2$, then $M_{l_1,l_2,\mathbf{w}}$ is a spin manifold if and only if N is a spin manifold. On the other hand if both l_1 and $|\mathbf{w}|$ are odd, then $M_{l_1,l_2,\mathbf{w}}$ is a spin manifold if and only if N is not a spin manifold.*

Equation (7) reduces further in the special case that $[\omega_N]$ is monotone. Actually we are interested in a generalization. We say that $[\omega_N]$

is *quasi-monotone* if $c_1(N) = \mathcal{I}_N[\omega_N]$ for some integer $\mathcal{I}_N$. Here $\mathcal{I}_N$ is the *Fano index* when $\mathcal{I}_N$ is positive (the monotone case) and the *canonical index* when it is negative. We also allow the case $\mathcal{I}_N = 0$. So when $[\omega_N]$ is quasi-monotone we have

$$c_1(\mathcal{D}_{l_1,l_2,\mathbf{w}}) = (l_2\mathcal{I}_N - l_1|\mathbf{w}|)\gamma. \tag{9}$$

From the homotopy exact sequence of the fibration $S^1 \longrightarrow M \longrightarrow N$ we see that if M is 2-connected then N is quasi-monotone.

Example 3.2. Let $M = S^{2p+1}$ so we have the Hopf fibration with $N = \mathbb{CP}^p$ which is monotone with $\mathcal{I}_N = p + 1$. In this case Equation (9) becomes

$$c_1(\mathcal{D}_{l_1,l_2,\mathbf{w}}) = (l_2(p+1) - l_1|\mathbf{w}|)\gamma. \tag{10}$$

The cohomology ring for this case was computed in [BTF13b], viz.

$$H^*(M_{l_1,l_2,\mathbf{w}}, \mathbb{Z}) \approx \mathbb{Z}[x,y]/(w_1w_2l_1^2x^2, x^{p+1}, x^2y, y^2)$$

where x, y are classes of degree 2 and $2p+1$, respectively. The topology of this case is studied further in [BTF14b], and more specific information about the Sasakian geometry is treated in Section 5.3 below.

3.2. **The Sasaki Cone.** Since for any Sasakian structure $\mathcal{S}$ the Reeb vector field lies in the center of the Lie algebra $\mathfrak{aut}(\mathcal{S})$ of the Sasaki automorphism group $\mathfrak{Aut}(\mathcal{S})$, it follows from the join construction that the Lie algebra $\mathfrak{aut}(\mathcal{S}_{l_1,l_2,\mathbf{w}})$ of the automorphism group of the join satisfies $\mathfrak{aut}(\mathcal{S}_{l_1,l_2,\mathbf{w}}) = \mathfrak{aut}(\mathcal{S}_1) \oplus \mathfrak{aut}(\mathcal{S}_\mathbf{w})$ where $\mathcal{S}_1$ is the Sasakian structure on M, and $\mathcal{S}_\mathbf{w}$ is the Sasakian structure on $S^3_\mathbf{w}$. Now the unreduced Sasaki cone [BGS08] $\mathfrak{t}^+$ of $\mathcal{S} = (\xi, \eta, \Phi, g)$ is by definition the positive cone in the Lie algebra $\mathfrak{t}$ of a maximal torus in $\mathfrak{Aut}(\mathcal{S})$, i.e.

$$\mathfrak{t}^+ = \{X \in \mathfrak{t} \mid \eta(X) > 0\}. \tag{11}$$

Thus, the Sasaki cone $\mathfrak{t}^+_{l_1,l_2,\mathbf{w}}$ of the join $M_{l_1,l_2,\mathbf{w}}$ satisfies

$$\mathfrak{t}^+_{l_1,l_2,\mathbf{w}} = \{X \in \mathfrak{t}_{l_1,l_2,\mathbf{w}} \mid \eta_{l_1,l_2\mathbf{w}}(X) > 0\} = \mathfrak{t}^+_1 + \mathfrak{t}^+_\mathbf{w}. \tag{12}$$

If the Lie algebra of a maximal torus of the automorphism group of $\mathcal{S}_1$ has dimension k, then $\dim \mathfrak{t}^+_{l_1,l_2,\mathbf{w}} = k+2$, since the $\mathfrak{t}_\mathbf{w}$ has dimension 2. However, in this paper we are mainly concerned with the 2-dimensional subcone $\mathfrak{t}^+_\mathbf{w}$, which we call the $\mathbf{w}$-Sasaki cone, of the full Sasaki cone $\mathfrak{t}^+_{l_1,l_2,\mathbf{w}}$. The $\mathbf{w}$-Sasaki cone $\mathfrak{t}^+_\mathbf{w}$ can be identified with the first quadrant in $\mathbb{R}^2$ with coordinates v_1, v_2 for all $\mathbf{w}$, viz.

$$\mathfrak{t}^+_\mathbf{w} = \{(v_1, v_2) \in \mathbb{R}^2 \mid v_1, v_2 > 0\}. \tag{13}$$

We are also interested in the full reduced Sasaki cone κ which is $\mathfrak{t}^+/\mathcal{W}$ where $\mathcal{W}$ is the Weyl group of the Sasaki automorphism group

$\mathfrak{Aut}(\mathcal{S})$. One can think of κ as the moduli space of Sasakian structures with a fixed underlying CR structure $(\mathcal{D}, J)$.

3.3. **The Tori Actions.** Consider the action of the 3-dimensional torus T^3 on the product $M \times S^3_{\mathbf{w}}$ defined by

$$(x, u; z_1, z_2) \mapsto (x, e^{il_2\theta}u; e^{i(\phi_1 - l_1 w_1 \theta)} z_1, e^{i(\phi_2 - l_1 w_2 \theta)} z_2). \tag{14}$$

The Lie algebra $\mathfrak{t}_3$ of T^3 is generated by the vector fields $L_{l_1,l_2,\mathbf{w}}, H_1, H_2$.

Following the analysis of Section 3.3 of [BTF13b] we see that our join $M_{l_1,l_2,\mathbf{w}}$ is the fiber bundle

$$M_{l_1,l_2,\mathbf{w}} = M \times_{S^1} L(l_2; l_1 w_1, l_1 w_2)$$

over the Kähler manifold N associated to the principal S^1-bundle $M \longrightarrow N$ with fiber the lens space $L(l_2; l_1 w_1, l_1 w_2)$ over the Kähler manifold N. The S^1 action on the lens space is accomplished in two stages. First, represent $L(l_2; l_1 w_1, l_1 w_2)$ as a $\mathbb{Z}_{l_2}$ quotient of $S^3_{\mathbf{w}}$, then the residual $S^1_\theta / \mathbb{Z}_{l_2} \approx S^1$ action is

$$(x, u; z_1, z_2) \mapsto (x, e^{i\theta}u; [e^{-i\frac{l_1 w_1}{l_2}\theta)} z_1, e^{-i\frac{l_1 w_2}{l_2}\theta)} z_2]). \tag{15}$$

The brackets in Equation (15) denote the equivalence class defined by $(z_1', z_2') \sim (z_1, z_2)$ if $(z_1', z_2') = (\lambda^{l_1 w_1} z_1, \lambda^{l_1 w_2} z_2)$ for $\lambda^{l_2} = 1$.

Next consider the T^2 action of $S^1_\phi \times (S^1_\theta / \mathbb{Z}_{l_2})$ on $M \times L(l_2; l_1 w_1, l_1 w_2)$ given by

$$(x, u; z_1, z_2) \mapsto (x, e^{i\theta}u; [e^{i(v_1\phi - \frac{l_1 w_1}{l_2}\theta)} z_1, e^{i(v_2\phi - \frac{l_1 w_2}{l_2}\theta)} z_2]), \tag{16}$$

This gives rise to the commutative diagram

$$\begin{array}{ccc} M \times L(l_2; l_1 w_1, l_1 w_2) & & \\ & \searrow \pi_L & \\ \downarrow \pi_B & & M_{l_1,l_2,\mathbf{w}} \\ & \swarrow \pi_{\mathbf{v}} & \\ B_{l_1,l_2,\mathbf{v},\mathbf{w}} & & \end{array} \tag{17}$$

where $B_{l_1,l_2,\mathbf{v},\mathbf{w}}$ is a bundle over N with fiber a weighted projective space, and π_B denotes the quotient projection by T^2. The Lie algebra of this T^2 is generated by

$$L_{\mathbf{w}} = \frac{1}{2l_1}\xi_M - \sum_{j=0}^{q} \frac{1}{2l_2} w_j H_j, \qquad \xi_{\mathbf{v}} = \sum_j v_j H_j, \tag{18}$$

where ξ_M denotes the Reeb vector field of the regular Sasakian structure on M. Note that $\xi_{\mathbf{v}}$ is a Reeb vector field in the $\mathbf{w}$-Sasaki cone of $M_{l_1,l_2,\mathbf{w}}$.

Let us analyze the behavior of the T^2 action given by Equation (16). We shall see that it it not generally effective. First we notice that the S^1_θ action is free since it is free on the first factor. Next we look for fixed points under a subgroup of the circle S^1_ϕ. Thus, we impose

$$(e^{iv_1\phi}z_1, e^{iv_2\phi}z_2) = (e^{-2\pi\frac{l_1w_1}{l_2}ri}z_1, e^{-2\pi\frac{l_1w_2}{l_2}ri}z_2)$$

for some $r = 0, \ldots, l_2 - 1$. If $z_1z_2 \neq 0$ we must have

$$v_1\phi = 2\pi(-\frac{l_1w_1r}{l_2} + k_1), \qquad v_2\phi = 2\pi(-\frac{l_1w_2r}{l_2} + k_2) \tag{19}$$

for some integers k_1, k_2 which in turn implies

$$l_1r(w_2v_1 - w_1v_2) = l_2(k_2v_1 - k_1v_2).$$

This gives

$$r = \frac{l_2}{l_1}\frac{k_2v_1 - k_1v_2}{w_2v_1 - w_1v_2} \tag{20}$$

which must be a nonnegative integer less than l_2. We can also solve Equations (19) for ϕ by eliminating $\frac{l_1r}{l_2}$ giving

$$\phi = 2\pi\frac{k_1w_2 - k_2w_1}{w_2v_1 - w_1v_2}. \tag{21}$$

Next we write (20) as

$$r = \Big(\frac{l_2}{\gcd(|w_2v_1 - w_1v_2|, l_2)}\Big)\Big(\frac{k_2v_1 - k_1v_2}{l_1\frac{w_2v_1 - w_1v_2}{\gcd(|w_2v_1 - w_1v_2|, l_2)}}\Big) \tag{22}$$

Since v_1 and v_2 are relatively prime, we can choose k_1 and k_2 so that the term in the last parentheses is 1. This determines r as

$$r = \frac{l_2}{\gcd(|w_2v_1 - w_1v_2|, l_2)} \tag{23}$$

Now suppose that $z_2 = 0$. Then generally we have $e^{iv_1\phi} = e^{-2\pi\frac{l_1w_1}{l_2}ri}$ for some $r = 0, \ldots, l_2 - 1$ or equivalently $r = 1, \ldots, l_2$. This gives

$$\phi = 2\pi(-\frac{l_1w_1r}{v_1l_2} + \frac{k}{v_1}). \tag{24}$$

A similar computation at $z_1 = 0$ gives

$$\phi = 2\pi(-\frac{l_1w_2r'}{v_2l_2} + \frac{k'}{v_2}). \tag{25}$$

We are interested in when regularity can occur. For this we need the minimal angle at the two endpoints to be equal. This gives

$$-\frac{l_1w_2r'}{v_2l_2} + \frac{k'}{v_2} = -\frac{l_1w_1r}{v_1l_2} + \frac{k}{v_1}$$

for some choice of integers k, k' and nonnegative integers $r, r' < l_2$. This gives

$$\frac{-l_1w_2r' + k'l_2}{v_2} = \frac{-l_1w_1r + kl_2}{v_1}. \tag{26}$$

3.4. **Periods of Reeb Orbits.** We assume that $\mathbf{w} \neq (1,1)$. We want to determine the periods of the orbits of the flow of the Reeb vector field defined by the weight vector $\mathbf{v} = (v_1, v_2)$. In particular, we want to know when there is a regular Reeb vector field in the $\mathbf{w}$-Sasaki cone.

Let us now generally determine the minimal angle, hence the generic period of the Reeb orbits, on the dense open subset Z defined by $z_1z_2 \neq 0$. For convenience we set $s = \gcd(|w_2v_1 - w_1v_2|, l_2)$ in which case (23) becomes $r = l_2/s$.

Lemma 3.3. *The minimal angle on Z is $\frac{2\pi}{s}$. Thus, $S^1_\phi/\mathbb{Z}_s$ acts freely on the dense open subset Z.*

Proof. We choose k_1, k_2 in Equation (22) so that the last parentheses equals 1. This gives

$$l_1\frac{w_2v_1 - w_1v_2}{s} = k_2v_1 - k_1v_2.$$

Rearranging this becomes

$$(sk_2 - l_1w_2)v_1 = (sk_1 - l_1w_1)v_2.$$

Since v_1 and v_2 are relatively prime this equation implies $sk_i = l_1w_i + mv_i$ for $i = 1, 2$ and some integer m. Putting this into Equation (21) gives $\phi = \frac{2\pi m}{s}$, so the minimal angle is $\frac{2\pi}{s}$. □

We now investigate the endpoints defined by $z_2 = 0$ and $z_1 = 0$.

Proposition 3.4. *The following hold:*

(1) *The period on Z, namely $\frac{2\pi}{s}$, is an integral multiple of the periods at the endpoints. Hence, $S^1_\phi/\mathbb{Z}_s$ acts effectively on $M_{l_1,l_2,\mathbf{w}}$.*
(2) *The period at the endpoint $z_j = 0$ is $\frac{2\pi}{v_il_2}$ where $i \equiv j+1 \mod 2$. So the end points have equal periods if and only if $\mathbf{v} = (1,1)$.*
(3) *The $\mathbf{w}$-Sasaki cone contains a regular Reeb vector field if and only if l_2 divides $w_1 - w_2$, and in this case it is given by $\mathbf{v} = (1,1)$.*

Proof. A Reeb vector field will be regular if and only if the period of its orbit is the same at all points. We know that it is $\frac{2\pi}{s}$ on Z. We need to determine the minimal angle at the endpoints. From Equation (24) the angle at $z_2=0$ is

$$\phi = 2\pi(\frac{-l_1w_1r+kl_2}{v_1l_2}).$$

Now $\gcd(l_2,l_1w_1)=1$, so we can choose k and r such that numerator of the term in the large parentheses is 1. This gives period $\frac{2\pi}{v_1l_2}$. Similarly, at $z_1=0$ we have the period $\frac{2\pi}{v_2l_2}$. So the period is the same at the endpoints if and only if $v_1=v_2$ which is equivalent to $\mathbf{v}=(1,1)$ since v_1 and v_2 are relatively prime which proves (2).

Moreover, the period is the same at all points if and only if

$$\mathbf{v}=(1,1), \qquad l_2=s=\gcd(|w_2v_1-w_1v_2|,l_2). \tag{27}$$

But the last equation holds if and only if l_2 divides w_1-w_2 proving (3).

(1) follows from the fact that for each $i=1,2$, v_il_2 is an integral multiple of $\gcd(|w_2v_1-w_1v_2|,l_2)=s$. □

In contrast to the 2-dimensional Sasaki cones in [BTF14a], not every $\mathbf{w}$-Sasaki cone has a regular Reeb vector field. Nevertheless, it does have a special Reeb vector field, namely that given by $\mathbf{v}=(1,1)$. For this there can be, as usual, two branch divisors, but they have the same ramification index, namely $m=l_2/s$. We refer to this Reeb field as *almost regular*. Clearly, there is precisely one almost regular Reeb vector field in each $\mathbf{w}$-Sasaki cone of $M_{l_1,l_2,\mathbf{w}}$.

Example 3.5. Regular Reeb vector fields. As stated in (c) of Proposition 3.4 when l_2 divides w_1-w_2 we always have a regular Reeb vector field in the $\mathbf{w}$-Sasaki cone by taking $\mathbf{v}=(1,1)$. (This was the case in [BTF14a] where $l_2=1$.) We obtain $M_{l_1,l_2,\mathbf{w}}$ as a principle S^1 bundle over the smooth quotient $B_{l_1,l_2,1,\mathbf{w}}=S_n=\mathbb{P}(\mathbb{1}\oplus L_n)$ with $n=l_1\frac{w_1-w_2}{l_2}$.

3.5. $B_{l_1,l_2,\mathbf{v},\mathbf{w}}$ **as a Log Pair.** We follow the analysis in Section 3 of [BTF14a]. We have the action of the 2-torus $S^1_\phi/\mathbb{Z}_s\times(S^1_\theta/\mathbb{Z}_{l_2})$ on $M\times L(l_2;l_1w_1,l_1w_2)$ given by Equation (16), and denoted by $\mathcal{A}_{\mathbf{v},l,\mathbf{w}}$, whose quotient space is $B_{l_1,l_2,\mathbf{v},\mathbf{w}}$. It follows from Equation (16) that $B_{l_1,l_2,\mathbf{v},\mathbf{w}}$ is a bundle over N with fiber a weighted projective space of complex dimension one. By (1) of Proposition 3.4 the generic period is an integral multiple, say m_i, of the period at the divisor D_i. Thus, for $i=1,2$ we have

$$m_i=v_i\frac{l_2}{s}=v_im. \tag{28}$$

Note that from its definition $m = \frac{l_2}{s}$, so m_i is indeed a positive integer. It is the ramification index of the branch divisor D_i. We think of D_1 as the zero section and D_2 as the infinity section of the bundle $B_{l_1,l_2,\mathbf{v},\mathbf{w}}$. Thus, $B_{l_1,l_2,\mathbf{v},\mathbf{w}}$ is a fiber bundle over N with fiber $\mathbb{CP}^1[v_1,v_2]/\mathbb{Z}_m \approx \mathbb{CP}^1$. The isomorphism is simply $[z_1,z_2] \mapsto [z_1^{m_2}, z_2^{m_1}]$ where the brackets denote the obvious equivalence classes on $\mathbb{CP}^1[v_1,v_2]/\mathbb{Z}_m$. The complex structure of $B_{l_1,l_2,\mathbf{v},\mathbf{w}}$ is the projection of the transverse complex structure on $M_{l_1,l_2,\mathbf{w}}$ which in turn is the lift of the product complex structure on $N \times \mathbb{CP}^1[\mathbf{w}]$. However, $B_{l_1,l_2,\mathbf{v},\mathbf{w}}$ is not generally a product as a complex orbifold, nor even topologically.

Now we can follow the analysis leading to Lemma 3.14 of [BTF14a]. So we define the map

$$\tilde{h}_{\mathbf{v}} : M \times L(l_2; l_1w_1, l_1w_2) \longrightarrow M \times L(l_2; l_1w_1v_2, l_1w_2v_1)$$

by

$$\tilde{h}_{\mathbf{v}}(x,u;[z_1,z_2]) = (x,u;[z_1^{m_2}, z_2^{m_1}]). \tag{29}$$

It is a mv_1v_2-fold covering map. Similar to [BTF14a] we get a commutative diagram:

$$\begin{array}{ccc} M \times L(l_2; l_1w_1, l_1w_2) & \xrightarrow{\mathcal{A}_{\mathbf{v},l,\mathbf{w}}(\lambda,\tau)} & M \times L(l_2; l_1w_1, l_1w_2) \\ \Big\downarrow \tilde{h}_{\mathbf{v}} & & \Big\downarrow \tilde{h}_{\mathbf{v}} \\ M \times L(l_2; l_1w_1', l_1w_2') & \xrightarrow{\mathcal{A}_{(1,1),l,\mathbf{w}'}(\lambda,\tau^{mv_1v_2})} & M \times L(l_2; l_1w_1', l_1w_2'), \end{array} \tag{30}$$

where $\mathbf{w}' = (v_2w_1, v_1w_2)$ and $\tau = e^{i\phi}, \lambda = e^{i\theta}$. So $B_{l_1,l_2,\mathbf{v},\mathbf{w}}$ is the log pair $(B_{l_1,l_2,1,\mathbf{w}'}, \Delta)$ with branch divisor

$$\Delta = (1 - \frac{1}{m_1})D_1 + (1 - \frac{1}{m_2})D_2, \tag{31}$$

where $B_{l_1,l_2,1,\mathbf{w}'}$ is a $\mathbb{CP}^1$-bundle over N. Now $B_{l_1,l_2,\mathbf{v},\mathbf{w}}$ is the quotient $\big(M \times L(l_2; l_1w_1, l_1w_2)\big)/\mathcal{A}_{\mathbf{v},l,\mathbf{w}}(\lambda,\tau)$, and $B_{l_1,l_2,1,\mathbf{w}'}$ is the quotient $\big(M \times L(l_2; l_1w_1', l_1w_2')\big)/\mathcal{A}_{(1,1),l,\mathbf{w}'}(\lambda,\tau^{mv_1v_2})$. So $\tilde{h}_{\mathbf{v}}$ induces a map $h_{\mathbf{v}} : B_{l_1,l_2,\mathbf{v},\mathbf{w}} \longrightarrow B_{l_1,l_2,1,\mathbf{w}'}$ defined by

$$h_{\mathbf{v}}([x,u;[z_1,z_2]]) = [x,u;[z_1^{m_2}, z_2^{m_1}]], \tag{32}$$

where the outer brackets denote the equivalence class with respect to the corresponding T^2 action. We have

Lemma 3.6. *The map $h_{\mathbf{v}} : B_{l_1,l_2,\mathbf{v},\mathbf{w}} \longrightarrow B_{l_1,l_2,1,\mathbf{w}'}$ defined by Equation (32) is a biholomorphism.*

Proof. The map is ostensibly holomorphic. Now $\tilde{h}_{\mathbf{v}}$ is the identity map on M and a mv_1v_2-fold covering map on the corresponding lens spaces. From the commutative diagram (30) the induced map $h_{\mathbf{v}}$ is fiber preserving and is a bijection on the fibers with holomorphic inverse. □

Remark 3.7. It is well known that a weighted projective line $\mathbb{CP}^1[w_1,w_2]$ is biholomorphic to the projective line itself $\mathbb{CP}^1$. Similarly, developable orbifolds of the form $\mathbb{CP}^1/G$ are biholomorphic to $\mathbb{CP}^1$ for any finite reflection group $G\subset \mathfrak{Aut}(\mathbb{CP}^1)$. In the case of our ruled manifolds this gives rise to Galois covers of log pairs

$$(S_n,(1-\frac{1}{m_1})D_1+(1-\frac{1}{m_2}D_2))\longrightarrow(S_n,(1-\frac{1}{m})(D_1+D_2))\longrightarrow(S_n,\emptyset).$$

Set theoretically the maps are the identity maps with the identity Galois group. However, they are inequivalent as orbifolds. For further discussion of this approach see [GK07]. Note also that generally the trivial orbifold $(S_n,\emptyset)$ does not occur as one of our quotients.

Lemma 3.6 allows us to consider the orbifold $B_{l_1,l_2,\mathbf{v},\mathbf{w}}$ as the log pair $(B_{l_1,l_2,1,\mathbf{w}'},\Delta)$ where Δ is given by Equation (31). Notice, as mentioned above, when $\mathbf{v}=(1,1)$ we have an almost regular Reeb vector field. Here the orbifold structure can be non-trivial, namely, $B_{l_1,l_2,(1,1),\mathbf{w}}=(B_{l_1,l_2,1,\mathbf{w}'},\Delta)$ where $m_1=m_2=m=\frac{l_2}{s}$ and the branch divisor is given by

$$\Delta=(1-\frac{1}{m})(D_1+D_2).$$

The T^2 action $\mathcal{A}_{(1,1),l,\mathbf{w}'}:M\times L(l_2;l_1w_1',l_1w_2')\longrightarrow M\times L(l_2;l_1w_1',l_1w_2')$ is given by

$$(x,u;z_1,z_2)\mapsto(x,e^{i\theta}u;[e^{i(\phi-\frac{l_1w_1'}{l_2}\theta)}z_1,e^{i(\phi-\frac{l_1w_2'}{l_2}\theta)}z_2]), \tag{33}$$

Defining $\chi=\phi-\frac{l_1w_1'}{l_2}\theta$ gives

$$(x,u;z_1,z_2)\mapsto(x,e^{i\theta}u;[e^{i\chi}z_1,e^{i(\chi+\frac{l_1}{l_2}(w_1'-w_2')\theta)}z_2]). \tag{34}$$

The analysis above shows that this action is generally not free, but has branch divisors at the zero ($z_2=0$) and infinity ($z_1=0$) sections with ramification indices both equal to m.

Equation (34) tells us that the T^2-quotient space $B_{l_1,l_2,1,\mathbf{w}'}$ is the projectivization of the holomorphic rank two vector bundle $E=\mathbb{1}\oplus L_n$ over N where $\mathbb{1}$ denotes the trivial line bundle and L_n is a line bundle of 'degree' $n=\frac{l_1}{s}(w_1v_2-w_2v_1)$ with $s=\gcd(|w_1v_2-w_2v_1|,l_2)$. So $S_n=\mathbb{P}(\mathbb{1}\oplus L_n)$ is a smooth projective algebraic variety. Next we identify N with the zero section D_1 of L_n, and note that $c_1(L_n)$ is just the restriction of the Poincaré dual of D_1 to D_1, i.e. $PD(D_1)|_{D_1}=c_1(L_n)$.

Summarizing we have

Theorem 3.8. *Let $M_{l_1,l_2,\mathbf{w}} = M \star_{l_1,l_2} S^3_{\mathbf{w}}$ be the join as described in the beginning of the section with the induced contact structure $\mathcal{D}_{l_1,l_2,\mathbf{w}}$. Let $\mathbf{v} = (v_1, v_2)$ be a weight vector with relatively prime integer components and let $\xi_{\mathbf{v}}$ be the corresponding Reeb vector field in the Sasaki cone $\mathfrak{t}^+_{\mathbf{w}}$. Then the quotient of $M_{l_1,l_2,\mathbf{w}}$ by the flow of the Reeb vector field $\xi_{\mathbf{v}}$ is a projective algebraic orbifold written as a the log pair (S_n, Δ) where S_n is the total space of the projective bundle $\mathbb{P}(\mathbb{1} \oplus L_n)$ over the Kähler manifold N with $n = l_1\big(\frac{w_1v_2 - w_2v_1}{s}\big)$, Δ the branch divisor*

$$\Delta = (1 - \frac{1}{m_1})D_1 + (1 - \frac{1}{m_2})D_2, \tag{35}$$

with ramification indices $m_i = v_i\frac{l_2}{s} = v_i m$ and divisors D_1 and D_2 given by the zero section $\mathbb{1} \oplus 0$ and infinity section $0 \oplus L_n$, respectively. The fiber of the orbifold (S_n, Δ) is the orbifold $\mathbb{CP}[v_1, v_2]/\mathbb{Z}_m$.

Next we focus on the projective bundle $S_n = \mathbb{P}(\mathbb{1} \oplus L_n)$. From Equation (34) we see that S_n is a fiber bundle over N with fiber $\mathbb{CP}^1$ associated to the principle S^1-bundle $M \longrightarrow N$. We want to determine the Kähler class $[\omega_B]$ of the orbifold $B_{l_1,l_2,\mathbf{v},\mathbf{w}} = (S_n, \Delta)$ induced by the projection $M_{l_1,l_2,\mathbf{w}} \longrightarrow B_{l_1,l_2,\mathbf{v},\mathbf{w}}$. First consider the following commutative diagram:

$$
\begin{array}{ccccc}
 & & M \times L(l_2; l_1w_1, l_1w_2) & & \\
 & & \downarrow \pi_L & & \\
 & & M_{l_1,l_2,\mathbf{w}} & & \\
 & \swarrow \pi_{\mathbf{w}} & & \searrow \pi_{\mathbf{v}} & \\
N \times \mathbb{CP}^1[\mathbf{w}] & & & & (S_n, \Delta) \\
 & p_{\mathbf{w}} \searrow & & \swarrow p_{\mathbf{v}} & \\
 & & N & &
\end{array}
\tag{36}
$$

where $p_{\mathbf{w}}, p_{\mathbf{v}}$ are the obvious projections. Second, note that we have the following lemma

Lemma 3.9. [BTF13b] *For the log pair (S_n, Δ) with*

$$\Delta = (1 - 1/m_1)D_1 + (1 - 1/m_2)D_2$$

the orbifold Chern class equals

$$c_1^{orb}(S_n, \Delta) = p_{\mathbf{v}}^* c_1(N) + \frac{1}{m_1} PD(D_1) + \frac{1}{m_2} PD(D_2).$$

By (7)

$$\pi_{\mathbf{v}}^*c_1^{orb}(S_n,\Delta)=c_1(\mathcal{D}_{l_1,l_2,\mathbf{w}})=(p_{\mathbf{w}}\circ\pi_{\mathbf{w}})^*c_1(N)-l_1|\mathbf{w}|\gamma.$$

So from Lemma 3.9 we have

$$(p_{\mathbf{v}}\circ\pi_{\mathbf{v}})^*c_1(N)+\frac{1}{m_1}\pi_{\mathbf{v}}^*PD(D_1)+\frac{1}{m_2}\pi_{\mathbf{v}}^*PD(D_2)=(p_{\mathbf{w}}\circ\pi_{\mathbf{w}})^*c_1(N)-l_1|\mathbf{w}|\gamma.$$

We also know that (see e.g. Section 1.3 in [ACGTF08])

$$PD(D_1)-PD(D_2)=np_{\mathbf{v}}^*[\omega_N]$$

and so

$$\pi_{\mathbf{v}}^*PD(D_1)-\pi_{\mathbf{v}}^*PD(D_2)=n(p_{\mathbf{v}}\circ\pi_{\mathbf{v}})^*[\omega_N].$$

From the commutative diagram (36) we see that

$$(p_{\mathbf{v}}\circ\pi_{\mathbf{v}})^*[\omega_N]=(p_{\mathbf{w}}\circ\pi_{\mathbf{w}})^*[\omega_N]=l_2\gamma.$$

and

$$(p_{\mathbf{v}}\circ\pi_{\mathbf{v}})^*c_1(N)=(p_{\mathbf{w}}\circ\pi_{\mathbf{w}})^*c_1(N),$$

so we get the system

$$\begin{aligned}\tfrac{1}{m_1}\pi_{\mathbf{v}}^*PD(D_1)+\tfrac{1}{m_2}\pi_{\mathbf{v}}^*PD(D_2) &= -l_1|\mathbf{w}|\gamma.\\ \pi_{\mathbf{v}}^*PD(D_1)-\pi_{\mathbf{v}}^*PD(D_2) &= l_2n\gamma\end{aligned}$$

which implies that $\pi_{\mathbf{v}}^*PD(D_1)=\frac{\frac{nl_2}{m_2}-l_1|\mathbf{w}|}{\frac{1}{m_1}+\frac{1}{m_2}}=-m_1l_1w_2\gamma$ and $\pi_{\mathbf{v}}^*PD(D_2)=-m_2l_1w_1\gamma$.

We are now ready to prove the following lemma

Lemma 3.10. *The induced Kähler class on $B_{l_1,l_2,\mathbf{v},\mathbf{w}}=(S_n,\Delta)$ takes the form*

$$k_1p_{\mathbf{v}}^*[\omega_N]+k_2PD(D_1)$$

for some positive integers k_1,k_2.

Proof. From the commutative diagram (36) we see that on degree 2 homology $\ker\pi_B^*=(\pi_{\mathbf{v}}\circ\pi_L)^*$ has dimension 2. We claim that $p_{\mathbf{v}}^*[\omega_N]$ and $PD(D_1)$ span $\ker\pi_B^*$. To see this we note that from the definition of the join, that $p_{\mathbf{v}}^*[\omega_N]$ is in $\ker\pi_B^*$. Moreover,

$$(p_{\mathbf{v}}\circ\pi_{\mathbf{v}}\circ\pi_L)^*:H^2(N,\mathbb{R})\longrightarrow H^2(M\times L(l_2;l_1w_1,l_1w_2),\mathbb{R})$$

has a one dimensional kernel. So it must be spanned by $[\omega_N]$. Since $p_{\mathbf{v}}^*[\omega_N]$ is in $\ker\pi_B^*$ and $(p_{\mathbf{v}}\circ\pi_{\mathbf{v}})^*[\omega_N]=l_2\gamma$, we must have that $\pi_L^*\gamma=0$. It follows that $PD(D_1)$ is also in the kernel of π_B^* and since it is clearly independent of $p_{\mathbf{v}}^*[\omega_N]$ we conclude that $p_{\mathbf{v}}^*[\omega_N]$ and $PD(D_1)$ span $\ker\pi_B^*$.

The induced Kähler class on $B_{l_1,l_2,\mathbf{v},\mathbf{w}} = (S_n, \Delta)$ is clearly in $\ker \pi_B^*$ and so the lemma follows. □

In view of Lemma 3.10 we write the induced Kähler class $[\omega_B]$ on (S_n, Δ) as

$$[\omega_B] = k_1 p_{\mathbf{v}}^*[\omega_N] + k_2 PD(D_1) \tag{37}$$

Lemma 3.11. *The following hold:*

(1) $k_2 = l_2$,
(2) $k_1 = m_1 l_1 w_2$

Proof. Since we know that $\pi_{\mathbf{v}}^*[\omega_B]$ is a trivial class in $M_{l_1,l_2,\mathbf{w}}$ and $(p_{\mathbf{v}} \circ \pi_{\mathbf{v}})^*[\omega_N] = l_2\gamma$ while $\pi_{\mathbf{v}}^* PD(D_1) = -m_1 l_1 w_2 \gamma$, we see immediately that $k_1 l_2 - k_2 m_1 l_1 w_2 = 0$ and since $\gcd(k_1, k_2) = m = l_2/s$, we conclude that $k_2 = l_2$ while $k_1 = m_1 l_1 w_2$. □

In the almost regular case this process can be inverted. Given positive integers n, m, k_1, k_2 with $m = \gcd(k_1, k_2)$ we can determine the relatively prime positive integers w_1, w_2 from the equation

$$\frac{w_2}{w_1} = \frac{k_1}{nk_2 + k_1}$$

and the relatively prime positive integers l_1, l_2 from

$$\frac{l_1}{l_2} = \frac{n}{m(w_1 - w_2)}.$$

This gives an analog of diagram (32) of [BTF14a] together with its interpretation depicted in the diagram

$$\begin{array}{ccc} & M_{l_1,l_2,\mathbf{w}} & \\ \pi_{\mathbf{w}} \swarrow & & \searrow \pi_{\mathbf{v}} \\ N \times \mathbb{CP}(\mathbf{w}) & & (S_n, \Delta). \end{array} \tag{38}$$

Thus, we can view $M_{l_1,l_2,\mathbf{w}}$ in two ways. First, the southwest arrow describes an S^1 orbibundle over the Kähler orbifold $N \times \mathbb{CP}^1[\mathbf{w}]$ with its product structure. Second the southeast arrow describes the Kähler structure of a $\mathbb{CP}^1$-bundle over N with twisted complex structure and a mild orbifold structure on the fibers given as a quotient by an almost regular Reeb vector field. Note that in (32) of [BTF14a] the southeast arrow is the quotient by a regular Reeb vector field.

4. Admissible CSC constructions

We now pick up the thread from Section 2.3 and describe the construction (see also [ACGTF08]) of admissible Kähler metrics on S_n (in fact, more generally on log pairs (S_n, Δ)). Consider the circle action on S_n induced by the natural circle action on L_n. It extends to a holomorphic $\mathbb{C}^*$ action. The open and dense set $S_{n0} \subset S_n$ of stable points with respect to the latter action has the structure of a principal $\mathbb{C}^*$ bundle over the stable quotient. The hermitian norm on the fibers induces via a Legendre transform a function $\mathfrak{z} : S_{n0} \to (-1,1)$ whose extension to S_n consists of the critical manifolds $\mathfrak{z}^{-1}(1) = P(\mathbb{1} \oplus 0)$ and $\mathfrak{z}^{-1}(-1) = P(0 \oplus L_n)$. Letting θ be a connection one form for the Hermitian metric on S_{n0}, with curvature $d\theta = \omega_{N_n}$, an admissible Kähler metric and form on the base S_n are given up to scale by the respective formulas

$$g = \frac{1+r\mathfrak{z}}{r} g_{N_n} + \frac{d\mathfrak{z}^2}{\Theta(\mathfrak{z})} + \Theta(\mathfrak{z})\theta^2, \quad \omega = \frac{1+r\mathfrak{z}}{r}\omega_{N_n} + d\mathfrak{z} \wedge \theta, \tag{39}$$

valid on S_{n0}. Here Θ is a smooth function with domain containing $(-1,1)$ and r, is a real number of the same sign as g_{N_n} and satisfying $0 < |r| < 1$. The complex structure yielding this Kähler structure is given by the pullback of the base complex structure along with the requirement $Jd\mathfrak{z} = \Theta\theta$. The function $\mathfrak{z}$ is hamiltonian with $K = J\, grad\, \mathfrak{z}$ a Killing vector field. In fact, $\mathfrak{z}$ is the moment map on S_n for the circle action, decomposing S_n into the free orbits $S_{n0} = \mathfrak{z}^{-1}((-1,1))$ and the special orbits $D_1 = \mathfrak{z}^{-1}(1)$ and $D_2 = \mathfrak{z}^{-1}(-1)$. Finally, θ satisfies $\theta(K) = 1$.

Remark 4.1. Note that on S_n

$$\phi := \frac{-(1+r\mathfrak{z})}{r^2}\omega_{N_n} + \mathfrak{z}d\mathfrak{z} \wedge \theta$$

is a Hamiltonian 2-form of order one.

We can now interpret g as a metric on the log pair (S_n, Δ) with

$$\Delta = (1 - 1/m_1)D_1 + (1 - 1/m_2)D_2$$

if Θ satisfies the positivity and boundary conditions

$$\begin{aligned} &\Theta(\mathfrak{z}) > 0, \quad -1 < \mathfrak{z} < 1, \\ &\Theta(\pm 1) = 0, \\ &\Theta'(-1) = 2/m_2 \qquad \Theta'(1) = -2/m_1. \end{aligned} \tag{40}$$

Remark 4.2. This construction is based on the symplectic viewpoint where different choices of Θ yields different complex structures all compatible with the same fixed symplectic form ω. However, for each Θ there is an S^1-equivariant diffeomorphism pulling back J to the original fixed complex structure on S_n in such a way that the Kähler form of the new Kähler metric is in the same cohomology class as ω [ACGTF08]. Therefore, with all else fixed, we may view the set of the functions Θ satisfying (40) as parametrizing a family of Kähler metrics within the same Kähler class of (S_n, Δ).

The Kähler class $\Omega_{\mathbf{r}} = [\omega]$ of an admissible metric is also called *admissible* and is uniquely determined by the parameter r, once the data associated with S_n (i.e. d_N, s_{N_n}, g_{N_n} etc.) is fixed. In fact, up to scale

$$\Omega_{\mathbf{r}} = [\omega_{N_n}]/r + 2\pi PD[D_1 + D_2], \tag{41}$$

where PD denotes the Poincaré dual. The number r, together with the data associated with S_n will be called *admissible data*.

Define a function $F(\mathfrak{z})$ by the formula $\Theta(\mathfrak{z}) = F(\mathfrak{z})/\mathfrak{p}(\mathfrak{z})$, where $\mathfrak{p}(\mathfrak{z}) = (1+r\mathfrak{z})^{d_N}$. Since $\mathfrak{p}(\mathfrak{z})$ is positive for $-1 \leq \mathfrak{z} \leq 1$, conditions (40) are equivalent to the following conditions on $F(\mathfrak{z})$:

$$\begin{aligned} &F(\mathfrak{z}) > 0, \quad -1 < \mathfrak{z} < 1,\\ &F(\pm 1) = 0, \\ &F'(-1) = 2\,\mathfrak{p}(-1)/m_2 \qquad F'(1) = -2\,\mathfrak{p}(1)/m_1. \end{aligned} \tag{42}$$

4.1. **The CSC condition.** From [ACG06] we have that the scalar curvature of an admissible metric given by (39) equals

$$Scal = \frac{2d_N s_{N_n} r}{1 + r\mathfrak{z}} - \frac{F''(\mathfrak{z})}{\mathfrak{p}(\mathfrak{z})}. \tag{43}$$

Thus the CSC condition is equivalent the ODE

$$F''(\mathfrak{z}) = (2d_N s_{N_n} r - k(1 + r\mathfrak{z}))\,(1 + r\mathfrak{z})^{d_N - 1}, \tag{44}$$

where k is a constant (equal to $Scal$ when (44) is solved). Notice that if (44) has a solution such that the boundary conditions from (42) holds, then it will also follow that $F(\mathfrak{z}) > 0$ for $-1 < \mathfrak{z} < 1$ and thus all of (42) is satisfied. To see this, merely observe that since $(1+r\mathfrak{z})^{d_N-1} > 0$ for $0 < |r| < 1$ and $-1 < \mathfrak{z} < 1$, then $F''(\mathfrak{z})$ can change sign at most once over the interval $-1 < \mathfrak{z} < 1$. Together with this fact, the endpoint conditions rule out any possibility of $F(\mathfrak{z})$ being zero for any $-1 < \mathfrak{z} < 1$.

Integrating and using the conditions of $F'(\pm 1)$ in (42), we immediately get that

$$F'(\mathfrak{z}) = \left(2s_{N_n} - k\frac{1}{r(d_N+1)}(1+r\mathfrak{z})\right)(1+r\mathfrak{z})^{d_N} + c,$$

where

$$c = \frac{2\left(1-r^2\right)^{d_N}\left(m_2(1-r)+m_1(1+r)-2m_1m_2s_{N_n}\right)}{m_1m_2\left((1+r)^{d_N+1}-(1-r)^{d_N+1}\right)} \tag{45}$$

and

$$k = \frac{2(d_N+1)r\left(m_2(1+r)^{d_N}(1+m_1s_{N_n})-m_1(1-r)^{d_N}(-1+m_2s_{N_n})\right)}{m_1m_2\left((1+r)^{d_N+1}-(1-r)^{d_N+1}\right)}. \tag{46}$$

Now we have a solution to (44), namely

$$F(\mathfrak{z}) = \int_{-1}^{\mathfrak{z}}\left(\left(2s_{N_n} - k\frac{1}{r(d_N+1)}(1+rt)\right)(1+rt)^{d_N} + c\right)d\mathfrak{z},$$

satisfying (42) iff

$$\int_{-1}^{1}\left(\left(2s_{N_n} - k\frac{1}{r(d_N+1)}(1+r\mathfrak{z})\right)(1+r\mathfrak{z})^{d_N} + c\right)d\mathfrak{z} = 0. \tag{47}$$

Now we integrate (47) to arrive at the equation

$$\frac{2s_{N_n}\left((1+r)^{d_N+1}-(1-r)^{d_N+1}\right)}{r(d_N+1)} - \frac{k\left((1+r)^{d_N+2}-(1-r)^{d_N+2}\right)}{r^2(d_N+1)(d_N+2)} + 2c = 0. \tag{48}$$

Thus the existence of an admissible CSC Kähler metric on the log pair (S_n, Δ) correspond to solving all three equations (45), (46), and (48).

4.2. **Extremal Kähler metrics.** If we generalize equation (44) to the condition that *Scal* from (43) is a affine function of $\mathfrak{z}$, then we obtain the equation

$$F''(\mathfrak{z}) = (1+r\mathfrak{z})^{d_N-1}(2d_Ns_{N_n}r + (\alpha\mathfrak{z}+\beta)(1+r\mathfrak{z})), \tag{49}$$

where α and β are constants. It is well known that this corresponds to extremal Kähler metrics (see e.g. [ACGTF08]). Moreover, similarly to the smooth case, one easily sees (by integrating and solving for A and B) that (49) has a unique solution $F(\mathfrak{z})$ satisfying the endpoint conditions of (42). Finally, if the Kähler form ω_N on N is assumed to have positive scalar curvature, this polynomial $F(\mathfrak{z})$ also satisfies the positivity condition of (42) by the standard root-counting argument introduced by Hwang [Hwa94] and Guan [Gua95]. For completeness we give the root-counting argument for this special case: Assume for contradiction that the positivity condition of (42) fails. Then, due to the endpoint conditions on $F(\mathfrak{z})$, the function $F(\mathfrak{z})$ has at least two relative maxima and at least one relative minimum inside the interval $(-1,1)$. Thus, in the interval $(-1,1)$, the concavity of $F(\mathfrak{z})$ changes at least twice, i.e. $F''(\mathfrak{z})$ is zero at least twice. Since $(1+r\mathfrak{z})^{d_N-1}>0$ for $-1<\mathfrak{z}<1$, we see that this implies that the second order polynomial $P(\mathfrak{z})=(2d_Ns_{N_n}r+(\alpha\mathfrak{z}+\beta)(1+r\mathfrak{z}))$ has two roots inside $(-1,1)$ and further the concavity changes exactly twice. Thus $F(\mathfrak{z})$ has two relative maxima at $\mathfrak{z}=a_1$ and $\mathfrak{z}=a_3$ and one relative minimum at $\mathfrak{z}=a_2$ such that $-1<a_1<a_2<a_3<1$ and the roots r_1,r_2 of $P(\mathfrak{z})$ are such that $a_1<r_1<a_2<r_2<a_3$. Moreover, $P(a_1)<0$ and $P(a_3)<0$. Now we observe that $P(-1/r)=2d_Ns_{N_n}r$ and thus if $s_{N_n}r\geq 0$, we see that $P(\mathfrak{z})$ must have one more root in either $[-1/r,a_1)$ (if $r>0$) or $(a_3,-1/r]$ (if $r<0$). Obviously $P(\mathfrak{z})$ cannot have three roots and so we have a contradiction. We conclude that the positivity condition of(42) must be satisfied.

This yields the following proposition which also proves Theorem 1.2 as we shall see below.

Proposition 4.3. *Assume that the scalar curvature s_N of (N,ω_N) is non-negative. For any log pair (S_n,Δ), any admissible Kähler class on S_n contains an admissible extremal metric which is smooth in the orbifold sense on (S_n,Δ).*

5. CSC and Extremal Rays

In order to finish the proofs of Theorem 1.1 and Theorem 1.2, we now connect the Kähler geometry of Section 4 with the Sasaki geometry of Section 3. Assume $M_{l_1,l_2,\mathbf{w}}=M\star_{l_1,l_2}S^3_{\mathbf{w}}$ is the join as described in the beginning of the Section 3 with the induced contact structure $\mathcal{D}_{l_1,l_2,\mathbf{w}}$, and now we assume that $w_1>w_2$. Let $\mathbf{v}=(v_1,v_2)$ be a weight vector with relatively prime integer components and let $\xi_{\mathbf{v}}$ be the corresponding Reeb vector field in the $\mathbf{w}$-Sasaki cone $\mathfrak{t}^+_{\mathbf{w}}$. Let the log pair (S_n,Δ) be the quotient of $M_{l_1,l_2,\mathbf{w}}$ by the flow of the Reeb vector field $\xi_{\mathbf{v}}$. Using Theorem 3.8 we have $m_i=v_i\frac{l_2}{s}$ and $n=l_1\big(\frac{w_1v_2-w_2v_1}{s}\big)$, where $s=\gcd(|w_2v_1-w_1v_2|,l_2)$. Writing $[\omega_{N_n}]=2\pi np^*_{\mathbf{v}}[\omega_N]$ and using that

$$PD[D_1+D_2]=2PD(D_1)-PD(D_1-D_2)=2PD(D_1)-n[\omega_N],$$

we see that (41) can be re-written to

$$\Omega_{\mathbf{r}}=4\pi\left(\frac{n(1-r)}{2r}[\omega_N]+PD(D_1)\right)$$

and so $[\omega_B]$ given by Lemma 3.11 is indeed admissible, where r is such that

$$\frac{n(1-r)}{2r} = k_1/k_2 = m_1 l_1 w_2/l_2$$

which gives

$$r = \frac{w_1 v_2 - w_2 v_1}{w_1 v_2 + w_2 v_1}, \tag{50}$$

and

$$\Omega_{\mathbf{r}} = 4\pi\left(\frac{k_1}{l_2}[\omega_N] + PD(D_1)\right) = \frac{4\pi}{l_2}\left(k_1[\omega_N] + k_2 PD(D_1)\right) = \frac{4\pi}{l_2}[\omega_B]. \tag{51}$$

For a description of extremal Sasaki metrics we refer the reader to [BGS08] and Section 4.4 of [BTF14a].

5.1. **Lifting the Admissible Data.** We now want to lift the admissibility conditions on (S_n,Δ) to $M_{\mathbf{l},\mathbf{w}}$ using Theorem 3.8, and we need to determine the scale factor involved in this lifting. Let M_0 denote the dense open subspace of $M_{l_1,l_2,\mathbf{w}}$ defined by the condition $z_1z_2\neq 0$, and let Z_i be the submanifolds of $M_{\mathbf{l},\mathbf{w}}$ defined by setting $z_{i+1}=0$ with $i=1,2 \mod 2$. This gives a stratification

$$M_{l_1,l_2,\mathbf{w}} = M_0 \sqcup Z_1 \sqcup Z_2. \tag{52}$$

It is easy to see that

Lemma 5.1. *For each pair of relatively prime positive integers v_1, v_2 the dense open submanifold M_0 is the total space of an S^1-bundle over S_{n0} and $Z_i = \pi_{\mathbf{v}}^{-1}(D_i)$ is independent of $\mathbf{v}$ and n.*

This lemma says that although the quotient spaces of different Reeb vector fields in the Sasaki cone may be quite different even topologically, their lifted geometry on $M_{l_1,l_2,\mathbf{w}}$ is similar.

Now Theorem 3.8 shows that the quotient space of $M_{l_1,l_2,\mathbf{w}}$ by the circle action generated by the quasi-regular Reeb vector field $\xi_{\mathbf{v}}$ is a ruled projective algebraic orbifold given as the log pair (S_n,Δ); however, although there is a specific Sasakian structure $\mathcal{S}_{\mathbf{v}}$ on $M_{l_1,l_2,\mathbf{w}}$ the theorem does not specify the Kähler structure on (S_n,Δ). It is now our purpose to do so, and relate it to $\mathcal{S}_{\mathbf{v}}$.

Proposition 5.2. *Let v_1, v_2 be relatively prime positive integers, and consider the Sasakian structure $\mathcal{S}_{\mathbf{v}} = (\xi_{\mathbf{v}}, \eta_{\mathbf{v}}, \Phi, g_{\mathbf{v}})$ on $M_{l_1,l_2,\mathbf{w}}$. Then the induced Kähler structure (g_B,ω_B) on (S_n,Δ) satisfies*

$$g^T = \frac{l_2}{4\pi}\frac{\pi_{\mathbf{v}}^* g}{m v_1 v_2} = \frac{\pi_{\mathbf{v}}^* g_B}{m v_1 v_2}, \qquad d\eta_{\mathbf{v}} = \frac{l_2}{4\pi}\frac{\pi_{\mathbf{v}}^*\omega}{m v_1 v_2} = \frac{\pi_{\mathbf{v}}^*\omega_B}{m v_1 v_2} = \omega^T$$

where $g^T = d\eta \circ (\mathbb{1} \otimes \Phi)$.

Proof. For any $\xi_{\mathbf{v}} \in \mathfrak{t}^+_{\mathbf{w}}$ define the quadratic form $q_{\mathbf{v}} = v_1|z_1|^2 + v_2|z_2|^2$. Then the Sasakian structure $\mathcal{S}_{\mathbf{v}}$ is related to reducible Sasakian structure $\mathcal{S}_{\mathbf{w}}$ by $q_{\mathbf{v}}\eta_{\mathbf{v}} = q_{\mathbf{w}}\eta_{\mathbf{w}}$. This gives the relation between the transverse Kähler forms,

$$d\eta_{\mathbf{v}}|_{\mathcal{D}} = q_{\mathbf{v}}^{-1} q_{\mathbf{w}} d\eta_{\mathbf{w}}|_{\mathcal{D}}. \tag{53}$$

Now choose coordinates on S^3 so that $q_{\mathbf{v}} = v_1|z_1|^2 + v_2|z_2|^2 = 2\kappa$ with $\kappa \in \mathbb{R}^+$. Let $\tilde{\mathfrak{z}} : M_{l_1,l_2,\mathbf{w}} \longrightarrow [-1,1]$ be the moment map of the lifted circle action of the moment map $\mathfrak{z}$. Then Lemma 5.1 implies

$$\tilde{\mathfrak{z}} = \frac{\kappa - v_2|z_2|^2}{\kappa} = \frac{v_1|z_1|^2 - \kappa}{\kappa}$$

which gives

$$|z_1|^2 = (\kappa\tilde{\mathfrak{z}} + \kappa)/v_1 \qquad |z_2|^2 = (\kappa - \kappa\tilde{\mathfrak{z}})/v_2.$$

This gives, using Equation (50)

$$q_{\mathbf{v}}^{-1} q_{\mathbf{w}} = \frac{(w_1 v_2 - w_2 v_1)\tilde{\mathfrak{z}} + w_1 v_2 + w_2 v_1}{2 v_1 v_2} = \frac{w_1 v_2 - w_2 v_1}{2 v_1 v_2}(\tilde{\mathfrak{z}} + r^{-1}). \tag{54}$$

Now by Equation (51) the Kähler form $(4\pi/l_2)\omega_B$ is in the admissible class $\Omega_{\mathbf{r}}$ and we choose it to be admissible. So $(4\pi/l_2)\omega_B = \omega$. Thus, pulling back and using Equation (39) we have by identifying N with the zero section of $S_n = \mathbb{P}(\mathbb{1} \oplus L_n)$,

$$\omega_B|_N = \frac{l_2 n}{2}(r^{-1} + \mathfrak{z})\omega_N. \tag{55}$$

Thus, to find the scale factor we write

$$\pi^*_{\mathbf{v}}\omega_B = b d\eta_{\mathbf{v}} \tag{56}$$

for some positive constant b. Now from the commutative Diagram (36) we have $\pi^*_{\mathbf{w}}\omega_N = \pi^*_{\mathbf{v}}\omega_N$. To find b it is enough to compare coifficients of the pullback of ω_N on both sides of Equation (56). Using Equations (53)-(55) together with the equation for n in Theorem 3.8 gives $b = m v_1 v_2$. □

Remark 5.3. The factor mv_1v_2 can be thought of as arising from the multiple cover argument in Diagram 30 which occurs on Sasakian level as well.

This proposition allows us to consider the Sasakian structure $\mathcal{S}_{\mathbf{v}}$ as an *admissible Sasakian structure*. We simply view Θ as a function of the lifted moment map $\tilde{\mathfrak{z}}$. This function $\Theta(\tilde{\mathfrak{z}})$ satisfies the positivity conditions and boundary conditions of Equation (40). We then get a Sasaki metric in the usual way, namely $g_{\mathbf{v}} = g^T + \eta_{\mathbf{v}} \otimes \eta_{\mathbf{v}}$ together with its full Sasakian structure $\mathcal{S}_{\mathbf{v}}=(\xi_{\mathbf{v}},\eta_{\mathbf{v}},\Phi_{\mathbf{v}},g_{\mathbf{v}})$. Although this construction was done for a pair of relatively prime positive integers v_1,v_2 we can extend this to the entire ray by applying a transverse homothety $(\xi,\eta)\mapsto(a^{-1}\xi,a\eta)$ which implies the following scaling of the admissible data:

$$\theta\mapsto a^{-1}\theta,\qquad \Theta\mapsto a\Theta,\qquad m_i\mapsto a^{-1}m_i,$$

and m is scale invariant. This defines the Sasaki admissible data for all quasi-regular Reeb vector fields.

We now wish to extend the concept of admissible Sasaki data to the irregular case. For this we consider the components v_1,v_2 of $\mathbf{v}$ to be any positive real numbers. We shall assume that the function Θ of Section 4 is chosen such that $m\Theta$ is independent of m and varies smoothly with v_1,v_2. As we shall see later this is the case for any (quasi-regular) extremal Sasakian structure. We need

Lemma 5.4. *Let $v_1,v_2\in\mathbb{R}^+$. Then the family of transverse Kähler metrics and forms of Proposition 5.2 vary smoothly with $\mathbf{v}=(v_1,v_2)$.*

Proof. It is convenient to rewrite the transverse metric g^T and Kähler form ω^T of Proposition 5.2 on the dense open set M_0 in the form

$$g^T=\frac{l_2}{4\pi}\Big(\frac{2\pi l_1(w_1v_2-w_2v_1)}{l_2v_1v_2}(r^{-1}+\tilde{\mathfrak{z}})\pi^*_{\mathbf{v}}g_N+\frac{d\tilde{\mathfrak{z}}^2}{\tilde\Theta(\tilde{\mathfrak{z}})}+\tilde\Theta(\tilde{\mathfrak{z}})\tilde\theta^2\Big) \tag{57}$$

$$\omega^T=\frac{l_2}{4\pi}\Big(\frac{2\pi l_1(w_1v_2-w_2v_1)}{l_2v_1v_2}(r^{-1}+\tilde{\mathfrak{z}})\pi^*_{\mathbf{v}}\omega_N+d\tilde{\mathfrak{z}}\wedge\tilde\theta\Big)$$

where $\tilde\Theta=mv_1v_2\pi^*_{\mathbf{v}}\Theta$ and $\tilde\theta=\frac{\pi^*_{\mathbf{v}}\theta}{mv_1v_2}$. Note that $\tilde\Theta$ satisfies the boundary conditions $\tilde\Theta(\pm1)=0$, $\tilde\Theta'(-1)=2v_1$ and $\tilde\Theta'(1)=-2v_2$. We will ignore the term $l_2/4\pi$ and consider the terms in brackets as our admissible data.

We claim that we can interpret Equation (57) as a family of transverse Kähler metrics and forms that varies smoothly with $\mathbf{v}$. First from the commutative Diagram (36) we see that $\pi^*_{\mathbf{v}}g_N=\pi^*_{\mathbf{w}}g_N$, so the term $\pi^*_{\mathbf{v}}g_N$ is independent of $\mathbf{v}$.

So to show that the family is smooth on M_0 we only need to show $\tilde\theta$ is a family of 1-forms on M_0 that varies smoothly with $\mathbf{v}$. Since on M_0 we have coordinates induced by z_1,z_2 such that

$$v_1|z_1|^2=\kappa+\epsilon,\qquad v_2|z_2|^2=\kappa-\epsilon$$

where $-\kappa<\epsilon<\kappa$. This trivializes M_0 as $M_0\approx T^2\times(-\kappa,\kappa)\times N$ as well as $S_{n0}\approx S^1\times(-\kappa,\kappa)\times N$. Now the Hamiltonian vector field K vanishes nowhere on S_{n0} and lifts to a vector field on M_0. Choosing $\kappa=v_1$ we see that this vector field is H_1 with moment map $\tilde{\mathfrak{z}}$ and satisfies $\pi_{\mathbf{v}*}H_1 = mv_1v_2K$ (cf. Remark 5.3). Since $\tilde{\theta}$ is a pullback we have $\tilde{\theta}(\xi_{\mathbf{v}}) = 0$ implying that $\tilde{\theta}(H_2) = -v_1/v_2$. Moreover, since both H_1 and H_2 are nowhere vanishing on M_0 we have coordinates φ_1 and φ_2 such that $\tilde{\theta} = d\varphi_1 - \frac{v_1}{v_2}d\varphi_2 + A$ where the A is a 1-form on N satisfying

$$dA = \frac{2\pi l_1(w_1v_2 - w_2v_1)}{l_2v_1v_2}\pi_{\mathbf{v}}^*\omega_N.$$

Since $\pi_{\mathbf{v}}^*\omega_N$ is independent of $\mathbf{v}$, this shows that $\tilde{\theta}$ depends smoothly on $\mathbf{v}$ on M_0.

As in the Kähler case the admissible quasi-regular Sasaki structures smoothly extend to the boundary $Z_1 \sqcup Z_2$ with the indicated boundary conditions. Moreover, any irregular Sasakian structure $\mathcal{S}_{\mathbf{v}}$ in the $\mathbf{w}$-Sasaki cone can be represented as a limit of quasi-regular structures by Theorem 7.1.10 of [BG08] which from the above can be taken to be admissible. Hence, by continuity the irregular admissible structures on M_0 extend to the boundary as well. □

Remark 5.5. Beginning from a Kähler class Ω_r of Equation (41) we obtain an admissible Kähler form within the Kähler class by performing a deformation of the form $\omega \mapsto \omega + i\partial\bar{\partial}\varphi$ where the function φ is invariant under the Hamiltonian circle action. This is equivalent on the Sasaki level to a deformation of the contact structure of the form $\eta \mapsto \eta + \zeta$ where ζ is a basic 1-form that is invariant under the lifted Hamiltonian circle action. Once this is done for a fixed $\mathbf{v}$ we see from the discussion above that it holds for all $\mathbf{v}$.

Remark 5.6. It is convenient to consider the space of rays of the $\mathbf{w}$-Sasaki cone. We let $\mathfrak{R}_{\mathbf{w}}$ denote the *space of rays* in $\mathfrak{t}_{\mathbf{w}}^+$ and the ray defined by the vector $\mathbf{v}$ by $\bar{\mathbf{v}}$. By mapping a ray $\bar{\mathbf{v}} \in \mathfrak{t}_{\mathbf{w}}^+$ to its slope v_2/v_1 gives a homeomorphism of $\mathfrak{R}_{\mathbf{w}}$ with the open interval $(0,\infty)$. It follows from Equation 7.3.12 of [BG08] that under the transverse homothety $(\xi,\eta) \mapsto (a^{-1}\xi, a\eta)$ extremality as well as constant scalar curvature are preserved. Thus, being extremal or CSC is a property of rays and descends to $\mathfrak{R}_{\mathbf{w}}$. Let $\mathfrak{R}_{\mathbf{w}}^{\mathfrak{rat}}$ denote the subset of *rational rays*, that is, those rays with rational slope. By Theorem 7.1.10 of [BG08], $\mathfrak{R}_{\mathbf{w}}^{\mathfrak{rat}}$ is dense in $\mathfrak{R}_{\mathbf{w}}$. Moreover, for every rational ray there is a unique pair of relatively prime positive integers v_1, v_2. So by Theorem 3.8 there is unique log pair (S_n, Δ) associated to the ray $\mathfrak{r} \in \mathfrak{R}_{\mathbf{w}}^{\mathfrak{rat}}$.

5.2. **Applying the Admissible Sasaki Data.** For a choice of co-prime integers $(v_1, v_2) \neq (w_1, w_2)$ and the values of m_i, n, and r given as above, we recall that the metric (39) is extremal when $\Theta(\mathfrak{z})$, satisfying the boundary conditions (40), is such that when $\Theta(\mathfrak{z}) = F(\mathfrak{z})/\mathfrak{p}(\mathfrak{z})$, then $F(\mathfrak{z})$ satisfies the ODE (49). The constants α and β are uniquely determined from this ODE and the boundary conditions.

Now we are setting $s_{N_n} = A/n = \frac{As}{l_1(w_1v_2-w_2v_1)}$, where, by Remark 2.3, $A \leq d_N+1$. In any case, A depends solely on (N, g_N, ω_N). (If ω_N is Kähler-Einstein, A is just $\mathcal{I}_N$ as introduced in Remark 2.3). As a consequence, since $m = l_2/s$, ms_{N_n} depends only on the join data and the choice of (v_1, v_2). Thus the function $m\Theta(\mathfrak{z})$ is independent of m and varies smoothly with v_1, v_2. This is precisely the assumption we need to be able to use Lemma 5.4, and so moving forward any pair (v_1, v_2) such that $v_1, v_2 \in \mathbb{R}^+$ has a well-defined "extremal" $\tilde{\Theta}(\tilde{\mathfrak{z}})$ resulting in the existence of an admissible extremal Sasakian metric whenever $\tilde{\Theta}(\tilde{\mathfrak{z}}) > 0$ for $-1 < \tilde{\mathfrak{z}} < 1$.

Notice that together with Propostion 4.3, this proves that when the scalar curvature s_N of (N, ω_N) is non-negative, then each ray in $\mathfrak{R}_{\mathbf{w}}$ can be represented by extremal Sasaki metrics. Consequently this proves Theorem 1.2.

Assuming $w_1 > w_2$, the existence of a quasi-regular CSC ray in the $\mathbf{w}$-Sasaki cone $\mathfrak{t}^+_{\mathbf{w}}$ corresponds to showing that for a choice of $(v_1, v_2) \neq (w_1, w_2)$ and the values of m_i, n, and r given as above, the equation system (45), (46), and (48) is solved. Notice that with $s_{N_n} = A/n = \frac{As}{l_1(w_1v_2-w_2v_1)}$ the value s (equivalently m) predictably cancels from the equation system (45), (46), and (48). In fact, we have (assuming $(v_1, v_2) \neq (w_1, w_2)$), that the system is equivalent to the equation $f(b) = 0$ for $b > 0$, where $b = \frac{v_2}{v_1} \neq \frac{w_2}{w_1}$ is the slope alluded to in Remark 5.6 and

(58)
$$\begin{aligned}
f(b) \;&=\; w_1^{2(d_N+1)} b^{2d_N+3}(Al_2 + l_1(d_N+1)w_2 - b(d_N+1)l_1w_1) \\
&-\; w_1^{d_N+2} w_2^{d_N} b^{d_N+3}((d_N+1)(A(d_N+1)l_2 - l_1((d_N+1)w_1 + (d_N+2)w_2))) \\
&+\; w_1^{d_N+1} w_2^{d_N+1} b^{d_N+2}(2Ad_N(d_N+2)l_2 - (d_N+1)(2d_N+3)l_1(w_1+w_2)) \\
&-\; w_1^{d_N} w_2^{d_N+2} b^{d_N+1}(d_N+1)(A(d_N+1)l_2 - l_1((d_N+2)w_1 + (d_N+1)w_2)) \\
&+\; w_2^{2(d_N+1)}(b(Al_2 + l_1(d_N+1)w_1) - (d_N+1)l_1w_2).
\end{aligned}$$

When a solution $b \in \mathbb{Q}^+$, we have a quasi-regular CSC metric and, since CSC is just a special case of extremal, when $b \in \mathbb{R}^+ \setminus \mathbb{Q}^+$, we have an irregular CSC metric.

Since $f(b)$ is a polynomial which is formally defined for any real value of b and

$$f(\frac{w_2}{w_1}) = f'(\frac{w_2}{w_1}) = f''(\frac{w_2}{w_1}) = 0$$

while

$$f'''(\frac{w_2}{w_1}) = 3(d_N+1)(d_N+2)l_1 w_1^{d_N} w_2^{d_N}(w_1 - w_2) > 0$$

and

$$\lim_{b\to+\infty} f(b) = -\infty,$$

we see that there is at least one solution $b \in (\frac{w_2}{w_1}, +\infty)$ to $f(b) = 0$. This completes the proof of Theorem 1.1. $\square$

Remark 5.7. In the special case when ω_N is Kähler-Einstein, $\mathcal{I}_N > 0$ and $l_1 = \frac{\mathcal{I}_N}{\gcd(w_1+w_2,\mathcal{I}_N)}$ and $l_2 = \frac{w_1+w_2}{\gcd(w_1+w_2,\mathcal{I}_N)}$ we saw in [BTF13b] that we have the Sasaki-Einstein ray discovered by the physicists [GMSW04]. While the majority of these are irregular we do know some of them to be quasi-regular (see e.g. Example 5.5 in [BTF13b]). As the examples in the next subsection will illustrate we can also produce many non-Einstein quasi-regular CSC rays in all dimensions.

5.3. **A Special Case:** $N = \mathbb{CP}^p$**.** Recall Example 3.2 in which case $A = \mathcal{I}_N = p+1$. Now if we let (l_1, l_2) be any relatively prime pair of positive integers except $(\frac{p+1}{\gcd(|\mathbf{w}|,p+1)}, \frac{|\mathbf{w}|}{\gcd(|\mathbf{w}|,p+1)})$ we know that the CSC ray from the proof of Theorem 1.1 is not Sasaki-Einstein (see Lemma 3.1 in [BTF13b]), and by Equation (10) $c_1(\mathcal{D}_{l_1,l_2,\mathbf{w}}) \neq 0$. Again, for the majority of choices of (w_1, w_2), the CSC ray discovered will be irregular. However we can produce quite a lot of quasi-regular CSC rays as the example below shows.

This case is studied in much more depth in [BTF14b]. In particular, it is shown there that if p, l_1 and $\mathbf{w}$ are fixed, there is only a finite number of diffeomorphism types among the manifolds $M_{l_1,l_2,\mathbf{w}}$. So for each $p > 1, l_1, w_1, w_2$ with $w_1 > w_2$, there exists a smooth $2p+3$-dimensional manifold $M_{l_1,\mathbf{w}}$ which admits a countably infinite number of contact structures of Sasaki type each with a compatible Sasaki metric of constant scalar curvature.

Example 5.8. For example with $A = p+1$, $l_1 = \frac{p}{\gcd(|\mathbf{w}|,p)}$, and $l_2 = \frac{|\mathbf{w}|}{\gcd(|\mathbf{w}|,p)}$ we set $t = \frac{w_2}{w_1}$ and $b = kt$ in $f(b)$ above. Then the equation $f(b) = 0$ is equivalent to

$$t = \frac{p - (p+1)k + k^{p+1}}{k(1 - (p+1)k^p + pk^{p+1})}.$$

It is a straightforward calculus exercise to check that for $k>1$ we get a solution $0<t<1$ as predicted in the previous section (so $b=kt>t=\frac{w_2}{w_1}$ and $w_1>w_2$). In particular, if we pick a rational $k>1$ we get a rational t. This value of t will determine (w_1,w_2) and then with (v_1,v_2) such that $v_2/v_1=b=kt$ we have our CSC quasi-regular Sasaki metric.

Example 5.9. Let us assume that $p=2$ (hence $A=3$), $l_1=1$ and $\mathbf{w}=(3,2)$. So to have smooth 7-manifolds $M^7_{1,l_2,(3,2)}$ we must have $\gcd(l_2,6)=1$. Then $f(b)=3(2-3b)^3g(b)$, where

$$g(b)=81b^5-27(l_2-4)b^4-54(l_2-2)b^3+36(l_2-1)b^2+8(l_2-6)b-16.$$

Now $g(2/3)=-32<0$ and $\lim_{b->+\infty} g(b)=+\infty$, justifying the solution to $g(b)=0$ in the interval $(2/3,+\infty)$ as already established. Notice however that $g(0)=-16<0$ and $g(1/3)=(13l_2-115)/3$. So for any $l_2\geq 9$ with $\gcd(l_2,6)=1$, we have two additional solutions to $g(b)=0$ in the interval $(0,2/3)$. Furthermore, one can check that the other two solutions are negative, so there are 3 rays of CSC Sasaki metrics in the $\mathbf{w}$-cone. It can also be checked that for $l_2=1,5,7$, there is only one solution to $g(b)=0$. We thus have

Proposition 5.10. *For each $l_2\geq 9$ relatively prime to 6 there are three distinct constant scalar curvature rays in the $\mathbf{w}$-Sasaki cone of the toric contact 7-manifold $(M^7_{1,l_2,(3,2)},\mathcal{D}_{1,l_2,(3,2)})$.*

It also follows from our results in [BTF14b] that infinitely many of the manifolds $M^7_{1,l_2,(3,2)}$ are diffeomorphic. Thus, there exists an infinite subsequence $s_j\subset\{l_2\}$ of the integers $l_2\geq 9$ giving distinct contact structures $\mathcal{D}_{s_j}$ of Sasaki type occurring on the same 7-manifold all containing three rays of CSC Sasaki metrics in their $\mathbf{w}$-Sasaki cone.

Example 5.11. Wang-Ziller manifolds. In the calculus analysis we have done on $f(b)$ so far, we have assumed that $w_1>w_2$. For arguments sake let us assume that $w_1=w_2=1$ in which case our manifolds $M_{l_1,l_2,(1,1)}=M^{1,p}_{l_2,l_1}$, a Wang-Ziller manifold [WZ90]. If we assume that $N=\mathbb{CP}^2$ and pick $l_1=1$, we know from Proposition 2.3 of Wang and Ziller that $M^{1,2}_{l_2,1}$ is $S^2\times S^5$ when l_2 is even and the non-trivial S^5-bundle over S^2, which we denote by $S^2\tilde{\times}S^5$, when l_2 is odd. So there are exactly these two diffeomorphism types. Moreover, we know that we have at least one CSC ray, namely the regular ray in the S^1-bundle over the product $N\times\mathbb{CP}^1$. This case corresponds to $b=1$, although $f(b)$ has no geometric meaning for $b=1$. However, we also get that

$f(b) = -3(b-1)^4 g(b)$ with

$$g(b) = (1 + (3 - l_2)b - 4b^2 l_2 + 6b^2 + (3 - l_2)b^3 + b^4).$$

Now we observe that $g(0) = 1 > 0$, $g(1) = 2(7 - 3l_2)$, and $\lim_{b\to+\infty} g(b) = +\infty$. So for $l_2 \geq 3$ $f(b)$ has at least 2 roots not equal to 1 ; one in the interval $(0,1)$ and one in the interval $(1,+\infty)$. Thus we have at least three CSC rays in this case as well.

Now the Wang-Ziller manifolds are toric, in fact, they are homogeneous, and in our case $M^{1,2}_{l_2,1}$ have a four-dimensional Sasaki cone, and when $l_2 \geq 3$ the $\mathbf{w}$-Sasaki cone (i.e. the 2-dimensional Sasaki cone associated with S^3) has three CSC rays, one regular and the other two irregular or quasi-regular. Notice also in our case the first Chern class of the contact bundle is $c_1(\mathcal{D}_{1,l_2}) = (3l_2 - 2)\gamma$. Summarizing we have

Theorem 5.12. *The 7-manifolds $S^2 \times S^5$ and $S^2 \tilde{\times} S^5$ admit countably infinite inequivalent toric contact structures $\mathcal{D}_{1,l_2}$ of Reeb type with l_2 even for the former and l_2 odd for the latter. Furthermore, when $l_2 \geq 3$ these contact structures admit three distinct rays of Sasaki metrics with constant scalar curvature in $\mathfrak{t}^+_{\mathbf{w}}$.*

As l_2 varies the contact structures are clearly inequivalent as contact structures, not just as toric contact structures.

Remark 5.13. In this Wang-Ziller case two of the three CSC metrics are actually equivalent under a transformation in the Weyl group $\mathbb{Z}_2$ acting on the unreduced $\mathbf{w}$-cone $\mathfrak{t}^+_{\mathbf{w}}$. This transformation sends a root to its reciprocal. Thus, there are only two CSC Sasaki metrics in the moduli space κ. See the proof of Theorem 1.3 in [Leg11] for another approach to this phenomenon.

5.4. **Multiple CSC Rays.** The multiple CSC rays in Proposition 5.10 and Theorem 5.12 illustrate a somewhat common phenomenon that was first illustrated in the case of quadrilateral toric structures (S^3-bundles over S^2) by Legendre [Leg11]. Consider $f(b)$ in (58). As already stated, any positive solution $b \neq \frac{w_2}{w_1}$ to the equation $f(b) = 0$ corresponds to a CSC ray in the $\mathbf{w}$-Sasaki cone. So far we know that, assuming $w_1 > w_2$, there is at least one solution in the interval $(\frac{w_2}{w_1}, +\infty)$. Since

$$f(\frac{w_2}{w_1}) = f'(\frac{w_2}{w_1}) = f''(\frac{w_2}{w_1}) = 0$$

while

$$f'''(\frac{w_2}{w_1}) = 3(d_N+1)(d_N+2)l_1 w_1^{d_N} w_2^{d_N}(w_1 - w_2) > 0$$

we know that for $b < \frac{w_2}{w_1}$ sufficiently close to $\frac{w_2}{w_1}$, we have $f(b) < 0$. Further it is easy to see that $f(0) < 0$. Now we notice that $f(\frac{w_2}{2w_1})$ is a linear function of l_2 with slope equal to

$$\frac{Aw_2^{2d_N+3}}{2^{2d+3}w_1}\left[1 + 2^{d_N}\left(2^{d_N+2} - (d_N^2 + 2d_N + 5)\right)\right].$$

When $A > 0$, which is equivalent to the scalar curvature s_N of (N, ω_N) being positive, then this slope is positive and thus for sufficiently large value of l_2 we have that $f(\frac{w_2}{2w_1}) > 0$. In that case we have at least two more roots; one in the interval $(0, \frac{w_2}{2w_1})$ and one in the interval $(\frac{w_2}{2w_1}, \frac{w_2}{w_1})$. As Example 5.11 illustrates, even if $w_1 = w_2 = 1$, we can have several CSC rays in the **w**-Sasaki cone. This proves Theorem 1.3.

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Charles P. Boyer, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, NM 87131.
E-mail addresses: cboyer@math.unm.edu

Christina W. Tønnesen-Friedman, Department of Mathematics, Union College, Schenectady, New York 12308, USA
E-mail addresses: tonnesec@union.edu