Foundations of arithmetic differential geometry

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Preface

Aim of the book. The aim of this book is to develop an arithmetic analogue of *classical differential geometry*; this analogue will be referred to as *arithmetic differential geometry*. In this new geometry the ring of integers $\mathbb{Z}$ will play the role of a ring of functions on an infinite dimensional manifold. The role of coordinate functions on this manifold will be played by the prime numbers $p \in \mathbb{Z}$. The role of partial derivatives of functions with respect to the coordinates will be played by the *Fermat quotients*, $\delta_p n := \frac{n-n^p}{p} \in \mathbb{Z}$, of integers $n \in \mathbb{Z}$ with respect to the primes p. The role of metrics (respectively 2-forms) will be played by symmetric (respectively anti-symmetric) matrices with coefficients in $\mathbb{Z}$. The role of connections (respectively curvature) attached to metrics or 2-forms will be played by certain adelic (respectively global) objects attached to matrices as above. One of the main conclusions of our theory will be that (the "manifold" corresponding to) $\mathbb{Z}$ is "intrinsically curved;" this curvature of $\mathbb{Z}$ (and higher versions of it) will be encoded into a $\mathbb{Q}$-Lie algebra $\mathfrak{hol}_{\mathbb{Q}}$, which we refer to as the *holonomy algebra*, and the study of this algebra is, essentially, the main task of the theory.

Needless to say, *arithmetic differential geometry* is still in its infancy. However, its foundations, which we present here, seem to form a solid platform upon which one could further build. Indeed, the main differential geometric concepts of this theory turn out to be related to classical number theoretic concepts (e.g., Christoffel symbols are related to Legendre symbols); existence and uniqueness results for the main objects (such as the arithmetic analogues of Ehresmann, Chern, Levi-Cività, and Lax connections) are being proved; the problem of defining curvature (which in arithmetic turns out to be non-trivial) is solved in some important cases (via our method of analytic continuation between primes and, alternatively, via algebraization by correspondences); and some basic vanishing/non-vanishing theorems are being proved for various types of curvature. It is hoped that all of the above will create a momentum for further investigation and further discovery.

Immediate context. A starting point for this circle of ideas can be found in our paper [**21**] where we showed how one can replace derivation operators by Fermat quotient operators in the construction of the classical jet spaces of Lie and Cartan; the new objects thus obtained were referred to as *arithmetic jet spaces*. With these objects at hand it was natural to try to develop an arithmetic analogue of *differential calculus*, in particular of *differential equations* and *differential geometry*. A theory of *arithmetic differential equations* was developed in a series of papers [**21**]-[**40**], [**5**] and was partly summarized in our monograph [**33**] (cf. also the survey papers [**41, 89**]); this theory led to a series of applications to invariant theory [**25, 5, 26, 33**], congruences between modular forms [**25, 35, 36**], and Diophantine geometry of Abelian and Shimura varieties [**22, 34**]. On the other hand an *arithmetic differential geometry* was developed in a series of papers [**40**]-[**45**], [**6**]; the present book follows, and further develops, the theory in this latter series of papers.

We should note that our book [**33**] on *arithmetic differential equations* and the present book on *arithmetic differential geometry*, although both based on the same conceptual framework introduced in [**21**], are concerned with rather different objects. In particular the two books are independent of each other and the overlap between them is minimal. Indeed the book [**33**] was mainly focussed on the arithmetic differential equations naturally occurring in the context of Abelian and Shimura varieties. By contrast, the present book naturally concentrates on the arithmetic differential equations related to the classical groups

$$GL_n, SL_n, SO_n, Sp_n,$$

and their corresponding symmetric spaces.

Of course, the world of Abelian/Shimura varieties and the world of classical groups, although not directly related within *abstract algebraic geometry*, are closely related through *analytic* concepts such as *uniformization* and *representation theory*. The prototypical example of this relation is the identification

$$M_1(\mathbb{C}) \simeq SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})/SO_2(\mathbb{R})$$

where M_1 is the coarse moduli space of Abelian varieties

$$\mathbb{C}/\text{lattice}$$

of dimension one; the curve M_1 is, of course, one of the simplest Shimura varieties. It is conceivable that the analytic relation between the two worlds referred to above could be implemented via certain *arithmetic differential equations/correspondences*; in particular it is conceivable that the subject matter of the present book and that of our previous book [**33**] might be related in ways that go beyond what can be seen at this point. A suggestion for such a possible relation comes from the theory of *δ-Hodge structures* developed in [**20**] and from a possible arithmetic analogue of the latter that could emerge from [**5, 33**].

There are at least two major differential geometric themes that are missing from the present book: *geodesics* and the *Laplacian*. It is unclear at this point what the arithmetic analogues of geodesics could be. On the other hand, for first steps towards an arithmetic Laplacian (and, in particular, for a concept of curvature based on it), we refer to [**32**]. But note that the flavor of [**32**] is quite different from that of the present book. Again, it is conceivable that the arithmetic Laplacian theory in [**32**] could be related, in ways not visible at this point, with our theory here.

Larger context. By what has been said so far this book is devoted to unveiling a new type of “geometric” structures on $\mathbb{Z}$. This is, of course, in line with the classical effort of using the analogy between numbers and functions to the advantage of number theory; this effort played a key role in the development of number theory throughout the 20th century up to the present day, as reflected in the work of Dedekind, Minkovski, Hilbert, Artin, Weil, Iwasawa, Grothendieck, Manin, Drinfeld, Parshin, Arakelov (to name just a few of the early contributors).

However, as we shall explain in the Introduction, the theory of the present book differs in essential ways from this classical theory (which is based on the analogy between number fields and function fields of *one* variable) and also from more recent geometric theories of "the discrete" such as: arithmetic topology [**93**]; the theory around the p-curvature conjecture of Grothendieck [**79**]; the Ihara differential [**72**]; the Fontaine-Colmez calculus [**58**]; discrete differential geometry [**8**]; and the geometries over the field with one element, $\mathbb{F}_1$, as developed by Soulé [**104**], Connes-Consani [**52**], Kurokawa et.al. [**84**], and Haran [**64**]. The non-vanishing curvature in our theory also prevents our arithmetic differential geometry from directly fitting into Borger's λ-ring framework [**11**] for $\mathbb{F}_1$; indeed, roughly speaking, λ-ring structure leads to zero curvature. For each individual prime, though, our theory is consistent with Borger's philosophy of $\mathbb{F}_1$; cf. [**11, 89**] and the Introduction to [**33**]. In spite of the differences between these theories and ours we expect interesting analogies and interactions.

This being said what is, after all, the position of our theory among more established mathematical theories? The answer we would like to suggest is that our curvature of $\mathbb{Z}$, encoded in the Lie algebra $\mathfrak{hol}_{\mathbb{Q}}$, could be viewed as an infinitesimal counterpart of the absolute Galois group $\Gamma_{\mathbb{Q}} := Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ of $\mathbb{Q}$: our Lie algebra $\mathfrak{hol}_{\mathbb{Q}}$ should be to the absolute Galois group $\Gamma_{\mathbb{Q}}$ what the identity component Hol^0 of the holonomy group Hol is to the monodromy group Hol/Hol^0 in classical differential geometry. As such our $\mathfrak{hol}_{\mathbb{Q}}$ could be viewed as an object of study in its own right, as fundamental, perhaps, as the absolute Galois group $\Gamma_{\mathbb{Q}}$ itself. A "unification" of $\mathfrak{hol}_{\mathbb{Q}}$ and $\Gamma_{\mathbb{Q}}$ may be expected in the same way in which Hol^0 and Hol/Hol^0 are "unified" by Hol. Such a unification of $\mathfrak{hol}_{\mathbb{Q}}$ with $\Gamma_{\mathbb{Q}}$ might involve interesting Galois representations in the same way in which the unification of $\Gamma_{\mathbb{Q}}$ with the geometric fundamental group via the arithmetic fundamental group in Grothendieck's theory yields interesting Galois representations [**71, 53**].

Organization of the book. The book starts with an Introduction in which we give an outline of our theory and we briefly compare our theory with other theories.

In Chapter 1 we briefly review the basic algebra and superalgebra concepts that we are going to use throughout the book. This is standard material which the reader might nevertheless want to quickly review for the sake of notation and terminology.

Chapter 2 will be devoted to revisiting classical differential geometry from an algebraic standpoint. We will be using the language of differential algebra [**82, 101**], i.e., the language of rings with derivations. We call the attention upon the fact that some of the classical differential geometric concepts will be presented in a somewhat non-conventional way in order to facilitate and motivate the transition to the arithmetic setting. This chapter plays no role in our book other than that of a motivational and referential framework; the reader may choose to skip this chapter and then go back to it as needed or just in order to compare the arithmetic theory with (the algebraic version of) the classical one.

Chapter 3 is where the exposition of our theory properly begins; here we present the basic notions of arithmetic differential geometry and, in particular, we introduce our arithmetic analogues of connection and curvature. The theory will be presented in the framework of arbitrary (group) schemes.

Chapter 4 specializes the theory in Chapter 3 to the case of the group scheme GL_n; here we prove, in particular, our main existence results for Ehresmann, Chern, Levi-Cività, and Lax connections respectively.

Chapters 5, 6, 7, 8 are devoted to the in-depth analysis of these connections; in particular we prove here the existence of the analytic continuation between primes necessary to define curvature for these connections and we give our vanishing/non-vanishing results for these curvatures. In Chapter 5, we also take first steps towards a corresponding (arithmetic differential) Galois theory.

The last Chapter 9 lists some of the natural problems, both technical and conceptual, that one faces in the further development of the theory.

Chapters 1, 2, 3, 4 should be read in a sequence (with Chapter 1 merely skimmed through and Chapter 2 possibly skipped and consulted as needed); Chapters 5, 6, 7, 8 depend on Chapters 1, 3, 4 but are essentially independent of each other. Chapter 9 can be read right after the Introduction.

Cross references are organized in a series of sequences as follows. Sections are numbered in one sequence and will be referred to as "section $x.y$." Definitions, Theorems, Propositions, Lemmas, Remarks, and Examples are numbered in a separate sequence and are referred to as "Theorem $x.y$, Example $x.y$," etc. Finally equations are numbered in yet another separate sequence and are referred to simply as "$x.y$." For all three sequences x denotes the number of the chapter.

Readership and prerequisites. The present book addresses graduate students and researchers interested in algebra, number theory, differential geometry, and the analogies between these fields. The only prerequisites are some familiarity with commutative algebra (cf., e.g., Matsumura's book [**92**]) and with foundational scheme theoretic algebraic geometry (e.g., the first two chapters of Hartshorne's book [**66**]). The text also contains a series of remarks that assume familiarity with basic concepts of classical differential geometry (as presented in [**81**], for instance); but these remarks are not essential for the understanding of the book and can actually be skipped.

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Contents

Introduction

By *classical differential geometry* we will understand, in this book, the theory of differentiable manifolds equipped with various "geometric structures" such as metrics, connections, curvature, etc. Our standard reference for this theory is [**81**]. However we shall only be interested in the algebraic aspects of this theory so we will allow ourselves below to reintroduce some of its concepts using the language of differential algebra [**82, 101**] (i.e., the language of rings equipped with derivations) and we will still refer to this "algebraized" version of differential geometry as *classical differential geometry*. On the other hand our purpose, in this book, is to develop an arithmetic analogue of classical differential geometry, which we will refer to as *arithmetic differential geometry*; in this latter theory derivation operators acting on functions will be replaced by Fermat quotient type operators acting on numbers, in the spirit of [**21, 33**]. In this Introduction we present an outline of our theory and we provide a quick comparison of our theory with other theories.

0.1. Outline of the theory

0.1.1. Classical versus arithmetic differentiation. In the classical theory of smooth (i.e., differentiable) manifolds one considers, for any m-dimensional smooth manifold M, the ring of smooth real valued functions $C^\infty(M,\mathbb{R})$. For the purposes of this Introduction it is enough to think of M as being the Euclidean space $M=\mathbb{R}^m$. In this book the arithmetic analogue of the manifold $\mathbb{R}^m$ will be the scheme *Spec* $\mathbb{Z}$. In order to stress our analogy between functions and numbers it is convenient to further fix a subring

$$A \subset C^\infty(\mathbb{R}^m,\mathbb{R}) \tag{0.1}$$

that is stable under partial differentiation and to frame the classical differential geometric definitions in terms of this ring. The arithmetic analogue of A in 0.1 will then be the ring of integers $\mathbb{Z}$ or, more generally, rings of fractions A of rings of integers in an abelian extension of $\mathbb{Q}$. (The theory can be easily extended to the case when $\mathbb{Q}$ is replaced by an arbitrary number field; we will not consider this more general situation in the book.) A prototypical example of such a ring is the ring

$$A=\mathbb{Z}[1/M,\zeta_N], \tag{0.2}$$

where M is an even integer, N is a positive integer, and ζ_N is a primitive N-th root of unity. Inverting M allows, as usual, to discard a set of "bad primes" (the divisors of M); adjoining ζ_N amounts to adjoining "new constants" to $\mathbb{Z}$.

Most of the times the ring A in 0.1 can be thought of as equal to $C^\infty(\mathbb{R}^m, \mathbb{R})$. But it will be sometimes useful to think of the ring A in 0.1 as consisting of analytic functions; indeed we will sometimes view analytic/algebraic functions as corresponding, in our theory, to global arithmetic objects in which case C^∞ objects will correspond, in our theory, to adelic objects. We will adopt these various viewpoints according to the various situations at hand.

The analogue of the set of coordinate functions

$$\mathcal{U} = \{\xi_1, \xi_2, ..., \xi_m\} \subset C^\infty(\mathbb{R}^m, \mathbb{R}) \tag{0.3}$$

on $\mathbb{R}^m$ will be a (possibly infinite) set of primes,

$$\mathcal{V} = \{p_1, p_2, p_3, ...\} \subset \mathbb{Z} \tag{0.4}$$

not dividing MN. We denote by $m = |\mathcal{V}| \leq \infty$ the cardinality of $\mathcal{V}$. (In case $\mathbb{Q}$ is replaced by an arbitrary number field the set $\mathcal{V}$ needs to be replaced by a set of finite places of that field.)

Next one considers the partial derivative operators on $A \subset C^\infty(\mathbb{R}^m, \mathbb{R})$,

$$\delta_i := \delta_i^A : A \to A, \quad \delta_i f := \frac{\partial f}{\partial \xi_i}, \quad i \in \{1, ..., m\}. \tag{0.5}$$

Following [**21**] we propose to take, as an analogue of the operators 0.5, the operators δ_p on $A = \mathbb{Z}[1/M, \zeta_N]$ defined by

$$\delta_p := \delta_p^A : A \to A, \quad \delta_p(a) = \frac{\phi_p(a) - a^p}{p}, \quad p \in \mathcal{V}, \tag{0.6}$$

where $\phi_p := \phi_p^A : A \to A$ is the unique ring automorphism sending ζ_N into ζ_N^p.

More generally recall that a *derivation* on a ring B is an additive map $B \to B$ that satisfies the Leibniz rule. This concept has, as an arithmetic analogue, the concept of p-*derivation* defined as follows; cf. [**21, 33, 76**].

DEFINITION 0.1. Assume B is a ring and assume, for simplicity, that p is a non-zero divisor in B; then a p-*derivation* on B is a set theoretic map

$$\delta_p := \delta_p^B : B \to B \tag{0.7}$$

with the property that the map

$$\phi_p := \phi_p^B : B \to B \tag{0.8}$$

defined by

$$\phi_p(b) := b^p + p\delta_p b \tag{0.9}$$

is a ring homomorphism.

We will always denote by ϕ_p the ring homomorphism 0.8 attached to a p-derivation δ_p as in 0.7 via the formula 0.9 and we shall refer to ϕ_p as the *Frobenius lift* attached to δ_p; note that ϕ_p induces the p-power Frobenius on B/pB.

0.1.2. Classical differential geometry revisited. Classical differential geometric concepts are often introduced by considering frame bundles $P \to M$ of rank n vector bundles over m-dimensional manifolds M. For our purposes we take here $M = \mathbb{R}^m$. Such a P is a principal homogeneous space for the group $GL_n(\mathbb{R})$; and if the vector bundle is trivial (which, for simplicity, we assume in what follows) then P is identified with $M \times GL_n(\mathbb{R})$. Note that the rank n of the vector bundle and the dimension m of M in this picture are unrelated.

We want to review the classical concept of connection in P; we shall do it in a somewhat non-standard way so that the transition to arithmetic becomes more transparent. Indeed consider an $n \times n$ matrix $x = (x_{ij})$ of indeterminates and consider the ring of polynomials in these n^2 indeterminates over some $A \subset C^{\infty}(\mathbb{R}^m, \mathbb{R})$ as in 0.1, with the determinant inverted,

$$B = A[x, \det(x)^{-1}]. \tag{0.10}$$

Note that B is naturally a subring of the ring $C^{\infty}(P, \mathbb{R})$. Denote by $\mathrm{End}(B)$ the Lie ring of $\mathbb{Z}$-module endomorphisms of B. Also consider $G = GL_n = Spec\ B$ as a scheme over A.

Definition 0.2. A *connection* on P (or on G, or on B) is a tuple $\delta = (\delta_i)$ of derivations

$$\delta_i := \delta_i^B : B \to B, \quad i \in \{1, ..., m\}, \tag{0.11}$$

extending the derivations 0.5. The *curvature* of δ is the matrix (φ_{ij}) of commutators $\varphi_{ij} \in \mathrm{End}(B)$,

$$\varphi_{ij} := [\delta_i, \delta_j] : B \to B, \quad i, j \in \{1, ..., m\}. \tag{0.12}$$

The *holonomy ring* of δ is the $\mathbb{Z}$-span $\mathfrak{hol}$ in $\mathrm{End}(B)$ of the commutators

$$[\delta_{i_1}, [\delta_{i_2}, [...., [\delta_{i_{s-1}}, \delta_{i_s}]...]]], \tag{0.13}$$

where $s \geq 2$; it is a Lie subring of $\mathrm{End}(B)$.

In particular one can consider the *trivial connection* $\delta_0 = (\delta_{0i})$ defined by

$$\delta_{0i} x = 0. \tag{0.14}$$

Here $\delta_{0i} x = (\delta_{0i} x_{kl})$ is the matrix with entries $\delta_{0i} x_{kl}$; this, and similar, notation will be constantly used in the sequel. A connection is called *flat* if its curvature vanishes: $\varphi_{ij} = 0$ for all $i, j = 1, ..., m$. For instance δ_0 is flat. For a flat δ the holonomy ring $\mathfrak{hol}$ vanishes: $\mathfrak{hol} = 0$.

More generally let us define a *connection* on an arbitrary A-algebra B as a tuple of derivations 0.11 extending the derivations 0.5; one can then define the *curvature* by the same formula 0.12.

Note that, in classical differential geometry, a prominent role is played by connections on vector bundles; the framework of vector bundles is, however, too "linear" to have a useful arithmetic analogue: our arithmetic theory will be essentially a "non-linear" theory in which vector bundles (more generally modules) need to be replaced by principal bundles (more generally by rings).

There are various types of connections that we shall be interested in and for which we will seek arithmetic analogues; they will be referred to as:

(1) Ehresmann connections,
(2) Chern connections,
(3) Levi-Cività connections,
(4) Fedosov connections,
(5) Lax connections,
(6) Hamiltonian connections,
(7) Cartan connections,
(8) Riccati connections,
(9) Weierstrass connections,
(10) Painlevé connections.

In some cases our terminology above is not entirely standard. For instance what we call *Fedosov connection* is usually called *symplectic connection* [**57**]; we chose a different name in order to avoid confusion with the symplectic paradigm underlying the Hamiltonian case; our choice of name is based on the role played by these connections in Fedosov's work [**57**]. Also there are many connections in classical differential geometry which are known under the name of *Cartan connection*; the Cartan connections that we will be considering are usually introduced as *Cartan distributions* [**1**], p. 133, and "live" on jet bundles; they are unrelated, for instance, to the Maurer-Cartan connections.

In what follows we discuss the connections (1) through (7) in some detail. For these connections we assume A as in 0.1; for the connections (1) through (5) we assume, in addition, that B is as in 0.10; for connections (6) and (7) B will be defined when we get to discuss these connections. Connections (8), (9), (10) will not be discussed in this Introduction but will appear in the body of the book; these are connections on curves (for (8), (9)) or surfaces (for (10)) appearing in relation to the classical theory of differential equations with no movable singularities. The Painlevé equations attached to (10) are closely related to the Hamiltonian connections (6). Both (9) and (10) lead to elliptic curves.

Definition 0.3. A connection (δ_i) on B in 0.10 is an *Ehresmann connection* if it satisfies one of the following two equivalent conditions:

1a) There exist $n \times n$ matrices A_i with coefficients in A such that

$$\delta_i x = A_i x \tag{0.15}$$

1b) The following diagrams are commutative:

$$\begin{array}{ccc} B & \xrightarrow{\mu} & B \otimes_A B \\ \delta_i \downarrow & & \downarrow \delta_i \otimes 1 + 1 \otimes \delta_{0i} \\ B & \xrightarrow{\mu} & B \otimes B \end{array} \tag{0.16}$$

Here μ is the comultiplication. Condition 1a can be referred to as saying that (δ_i) is *right linear*. Condition 1b can be referred to as saying that δ is *right invariant*. (As we shall see the arithmetic analogues of conditions 1a and 1b will cease to be equivalent.) If (δ_i) is an Ehresmann connection the curvature satisfies $\varphi_{ij}(x) = F_{ij}x$ where F_{ij} is the matrix given by the classical formula

$$F_{ij} := \delta_i A_j - \delta_j A_i - [A_i, A_j]. \tag{0.17}$$

Also there is a Galois theory attached to flat Ehresmann connections, the Picard-Vessiot theory. Indeed, for a flat Ehresmann connection $\delta = (\delta_i)$ consider the *logarithmic derivative map*, $l\delta : GL_n(A) \to \mathfrak{gl}_n(A)^m$, with coordinates

$$l\delta_i(u) = \delta_i u \cdot u^{-1}, \tag{0.18}$$

where $\mathfrak{gl}_n$ is the Lie algebra of GL_n. The fibers of the map $l\delta$ are solution sets of systems of linear equations

$$\delta_i u = A_i \cdot u. \tag{0.19}$$

And if one replaces A by a ring of complex analytic functions then Galois groups can be classically attached to such systems; these groups are algebraic subgroups of $GL_n(\mathbb{C})$ measuring the algebraic relations among the solutions to the corresponding systems.

We next discuss Chern connections. We will introduce two types of Chern connections: *real Chern connections* and *complex Chern connections*. Complex Chern connections will be modeled on the classical hermitian connections on hermitian vector bundles [**63, 80**] which play a central role in complex geometry; real Chern connections are a real analogue of the complex ones and do not seem to play an important role in differential geometry (although see Duistermaat's paper [**56**] for a special case of these connections). Both real and complex Chern connections will have arithmetic analogues; the arithmetic analogue of real Chern connections will be more subtle (and hence more interesting) than the arithmetic analogue of the complex Chern connection so we will mostly concentrate on the former than on the latter.

To introduce these concepts let us consider again the rings A as in 0.1 and B as in 0.10, and let $q \in GL_n(A)$ be an $n \times n$ invertible matrix with coefficients in A which is either symmetric or anti-symmetric,

$$q^t = \pm q; \tag{0.20}$$

so the t superscript means here *transposition*. Of course, a symmetric q as above is viewed as defining a (semi-Riemannian) metric on the trivial vector bundle

$$M \times \mathbb{R}^n \to M, \tag{0.21}$$

on $M = \mathbb{R}^m$ while an anti-symmetric q is viewed as defining a 2-form on 0.21; here metrics/2-forms on vector bundles mean smoothly varying non-degenerate symmetric/anti-symmetric bilinear forms on the fibers of the bundle. With B be as in 0.10 let $G = GL_n = Spec\ B$ be viewed as a scheme over A and consider the maps of schemes over A,

$$\mathcal{H}_q : G \to G, \quad \mathcal{B}_q : G \times G \to G, \tag{0.22}$$

defined on points by

$$\mathcal{H}_q(x) = x^t q x, \quad \mathcal{B}_q(x, y) = x^t q y. \tag{0.23}$$

We continue to denote by the same letters the corresponding maps of rings $B \to B$ and $B \to B \otimes_A B$. Consider the *trivial* connection $\delta_0 = (\delta_{0i})$ on G defined by $\delta_{0i} x = 0$. Then one has the following easy results:

THEOREM 0.4. *There is a unique connection $\delta = (\delta_i)$ on G such that the following diagrams are commutative:*

$$\begin{array}{ccc} B & \stackrel{\delta_i}{\longleftarrow} & B \\ \mathcal{H}_q \uparrow & & \uparrow \mathcal{H}_q \\ B & \stackrel{\delta_{0i}}{\longleftarrow} & B \end{array} \qquad \begin{array}{ccc} B & \stackrel{\delta_{0i}\otimes 1 + 1 \otimes \delta_i}{\longleftarrow} & B \otimes_A B \\ \delta_i \otimes 1 + 1 \otimes \delta_{0i} \uparrow & & \uparrow \mathcal{B}_q \\ B \otimes_A B & \stackrel{\mathcal{B}_q}{\longleftarrow} & B \end{array} \tag{0.24}$$

THEOREM 0.5. *There is a unique connection $\delta = (\delta_i)$ on G such that the following diagrams are commutative:*

$$\begin{array}{ccc} B & \stackrel{\delta_{0i}\otimes 1 + 1 \otimes \delta_i}{\longleftarrow} & B \otimes_A B \\ \mathcal{H}_q \uparrow & & \uparrow \mathcal{B}_q \\ B & \stackrel{\delta_{0i}}{\longleftarrow} & B. \end{array} \tag{0.25}$$

DEFINITION 0.6. A connection δ is $\mathcal{H}_q$*-horizontal* (respectively $\mathcal{B}_q$*-symmetric*) with respect to δ_0 if the left (respectively right) diagrams in 0.24 are commutative for all i. The unique connection that is $\mathcal{H}_q$*-horizontal* and $\mathcal{B}_q$*-symmetric* with respect to δ_0 (cf. Theorem 0.4) is called the *real Chern connection* attached to q.

DEFINITION 0.7. A connection δ is $(\mathcal{H}_q, \mathcal{B}_q)$-*horizontal* with respect to δ_0 if the diagrams 0.25 are commutative for all i. The unique connection that is $(\mathcal{H}_q, \mathcal{B}_q)$-*horizontal* with respect to δ_0 (cf. Theorem 0.5) is called the *complex Chern connection* attached to q.

Real and complex Chern connections turn out to be automatically Ehresmann connections. The arithmetic analogues of the real and complex Chern connections will not be "Ehresmann" in general.

The definitions just given may look non-standard. To see the analogy between of our (real and complex) Chern connections and the classical hermitian Chern connections in [**63, 80**] we need to look at the explicit formulas for these connections; so we need to introduce more notation as follows.

For any connection (δ_i) on B in 0.10 set

$$\begin{aligned} A_i &= A_i(x) := \delta_i x \cdot x^{-1} \in \mathfrak{gl}_n(B), \\ \Gamma^k_{ij} &= -A_{ikj} = (k,j) \text{ entry of } (-A_i). \end{aligned} \tag{0.26}$$

The quantities Γ^k_{ij} will be referred to as *Christoffel symbols of the* 2*nd kind* of the connection. Also, for q as in 0.20, set

$$\Gamma_{ijk} := \Gamma^l_{ij} q_{lk}, \tag{0.27}$$

which we refer to as the *Christoffel symbols of the* 1*st kind*. Here we use Einstein summation.

It is trivial to check that δ is $\mathcal{H}_q$-horizontal if and only if the followng condition is satisfied:

$$\delta_i q_{jk} = \Gamma_{ijk} \pm \Gamma_{ikj}. \tag{0.28}$$

Classically if 0.28 holds one says that q is *parallel* with respect to δ.

Similarly it is trivial to check that δ is $\mathcal{B}_q$-symmetric if and only if the following condition is satisfied:

$$\Gamma_{ijk} = \pm \Gamma_{ikj}. \tag{0.29}$$

This is, as we shall see, a condition different from the classical condition of *symmetry* for a connection on the tangent bundle.

In 0.28 and 0.29 the upper (respectively lower) sign correspond to the upper (respectively lower) sign in 0.20.

Using 0.28 and 0.29 it is trivial to see that the real Chern connection attached to q exists and is unique, being given by

$$\Gamma_{ijk} = \frac{1}{2} \delta_i q_{jk}. \tag{0.30}$$

This proves Theorem 0.4. Note that, as promised earlier, the Chern connection is an Ehresmann connection, i.e. A_i belong to $\mathfrak{gl}_n(A)$ rather than $\mathfrak{gl}_n(B)$.

By the way the real Chern connection has the following compatibility with the special linear group SL_n: if $\delta_i(\det(q)) = 0$ then the real Chern connection $\delta_i : B \to B$ attached to q sends the ideal of SL_n into itself and hence induces a "connection on SL_n." (This compatibility with SL_n will fail to hold in the arithmetic case.)

Similarly it is easy to see that δ is $(\mathcal{H}_q, \mathcal{B}_q)$-horizontal with respect to δ_0 if and only if

$$\Gamma_{ijk} = \delta_i q_{jk}. \tag{0.31}$$

Hence the complex Chern connection exists and is unique, being given by 0.31, which proves Theorem 0.5. Comparing 0.30 and 0.31 we see that the Christoffel symbols of the real and complex Chern connections coincide up to a $\frac{1}{2}$ factor and in this sense they are analogous to each other; on the other hand formula 0.31 is analogous to the formula for the classical hermitian Chern connection in [**63, 80**].

We next discuss Levi-Cività connections. Assume A as in 0.1, B as in 0.10, and assume that $n = m$. (Note we also implicitly assume here that a bijection is given between the set indexing the derivations δ_i and the set indexing the rows and columns of the matrix $x = (x_{ij})$; such a bijection plays the role of what is classically called a *soldering*.)

DEFINITION 0.8. The connection (δ_i) is *symmetric* or *torsion free* if

$$\Gamma_{ijk} = \Gamma_{jik}. \tag{0.32}$$

Note the difference between the condition 0.32 defining *symmetry* and the condition 0.29 defining $\mathcal{B}_q$*-symmetry*: the two types of symmetry involve different pairs of indices. To avoid any confusion we will use the term *torsion free* rather than *symmetric* in what follows. The fundamental theorem of Riemannian geometry is, in this setting, the following (completely elementary) statement:

THEOREM 0.9. *Let $q \in GL_n(A)$, $q^t = q$. Then there is a unique connection δ that is $\mathcal{H}_q$-horizontal (i.e., satisfies equation 0.28 with the + sign) and is torsion free (i.e., it satisfies equation 0.32); it is given by*

$$\Gamma_{kij} = \frac{1}{2}\left(\delta_k q_{ij} + \delta_i q_{jk} - \delta_j q_{ki}\right) \tag{0.33}$$

DEFINITION 0.10. The connection in Theorem 0.9 is called the *Levi-Cività* connection attached to q.

Note that, in particular, the Levi-Cività connection is an Ehresmann connection, i.e. A_i belong to $\mathfrak{gl}_n(A)$ rather than $\mathfrak{gl}_n(B)$. (The latter will fail to hold in the arithmetic case.) The Levi-Cività connection is, in a precise sense to be discussed later, "dual" to the real Chern connection and is generally different from the real Chern connection; it coincides with the real Chern connection if and only if

$$\delta_i q_{jk} = \delta_j q_{ik} \tag{0.34}$$

in which case q is called *Hessian* (the "real" analogue of *Kähler*).

We next discuss Fedosov connections. To explain these we start, again, with A as in 0.1 and B as in 0.10.

DEFINITION 0.11. Let $q \in GL_n(A)$, $q^t = -q$, so n is even. A *Fedosov connection* relative to q is a connection δ that is $\mathcal{H}_q$-horizontal (i.e., satisfies equation 0.28 with the $-$ sign) and is torsion free (i.e. it satisfies 0.32).

One trivially checks that a Fedosov connection relative to q exists if and only if q is *symplectic* in the sense that it satisfies

$$\delta_i q_{jk} + \delta_j q_{ki} + \delta_k q_{ij} = 0. \tag{0.35}$$

Fedosov connections are not necessarily Ehresmann. And, for a given symplectic matrix q, Fedosov connections that are Ehresmann exist but are not unique; one Fedosov connection which is an Ehresmann connection is given by

$$\Gamma_{ijk} = \frac{1}{3}\left(\delta_i q_{jk} + \delta_j q_{ik}\right). \tag{0.36}$$

We next discuss Lax connections. Let A be as in 0.1 and B as in 0.10.

DEFINITION 0.12. A connection (δ_i) on B is called a *Lax connection* if it satisfies

$$\delta_i x = [A_i(x), x] := A_i(x)x - xA_i(x) \tag{0.37}$$

for some $n \times n$ matrix $A_i(x)$ with coefficients in B.

Note that, unlike Chern and Levi-Cività connections, Lax connections are *not* a subclass of the Ehresmann connections. For a Lax connection, the following diagrams are commutative:

$$\begin{array}{ccc} B & \stackrel{\delta_i}{\longleftarrow} & B \\ \mathcal{P}\uparrow & & \uparrow\mathcal{P} \\ A[z] & \stackrel{\delta_{0i}}{\longleftarrow} & A[z] \end{array} \tag{0.38}$$

where $A[z] = A[z_1, ..., z_n]$ is a ring of polynomials in the variables z_j, δ_{0i} are the unique derivations extending the corresponding derivations on A with $\delta_{0i} z_j = 0$, and $\mathcal{P}$ is the A-algebra homomorphism with $\mathcal{P}(z_j) = \mathcal{P}_j(x)$,

$$\det(s \cdot 1 - x) = \sum_{j=0}^{n} (-1)^j \mathcal{P}_j(x) s^{n-j}. \tag{0.39}$$

The commutativity of 0.38 expresses the fact that the Lax connections describe "isospectral flows" on GL_n.

By the way the "real" theory summarized above has a "complex" analogue (and hence a "$(1,1)$"-analogue) for which we refer to the body of the book. Suffices to say here that, for the complex theory, one may start with $M = \mathbb{C}^m$ and with a subring

$$A \subset C^\infty(M, \mathbb{C}) \tag{0.40}$$

of the ring of smooth complex valued functions on M which is stable under the derivations

$$\delta_i := \frac{\partial}{\partial z_i}, \quad \delta_{\bar{i}} := \frac{\partial}{\partial \bar{z}_i}, \quad i = 1, ..., m, \tag{0.41}$$

where $z_1, ..., z_m$ are the complex coordinates on M. A connection on $G = Spec\ B$, $B = A[x, \det(x)^{-1}]$, is then an m-tuple of derivations $\delta_i : B \to B$ extending the derivations $\delta_i : A \to A$. Consider the unique derivations $\delta_{\bar{i}} : B \to B$, extending the derivations $\delta_{\bar{i}} : A \to A$, such that $\delta_{\bar{i}} x = 0$. Then one defines the $(1,1)$-curvature of $\delta = (\delta_i)$ as the matrix $(\varphi_{i\bar{j}})$ with entries the A-derivations

$$\varphi_{i\bar{j}} := [\delta_i, \delta_{\bar{j}}] : B \to B. \tag{0.42}$$

The theory proceeds from here.

So far we discussed connections on the ring B in 0.10. In what follows we informally discuss Hamiltonian and Cartan connections; for a precise discussion we refer to the main text. These are connections on rings B other than 0.10.

To discuss Hamiltonian connections consider the ring

$$B = A[x], \tag{0.43}$$

with A as in 0.1 and $x = \{x_1, ..., x_d\}$ a d-tuple of variables and consider a non-degenerate "vertical" 2-form

$$\omega := \omega^{ij} \cdot dx_i \wedge dx_j, \tag{0.44}$$

$\omega^{ij} \in B$, with ω *symplectic* (i.e. closed) "in the vertical directions"

$$\partial^i := \partial/\partial x_i$$

in the sense that

$$\partial^k \omega^{ij} + \partial^j \omega^{ki} + \partial^i \omega^{jk} = 0. \tag{0.45}$$

DEFINITION 0.13. A *Hamiltonian* connection with respect to ω is a connection on a B as in 0.43 for which the corresponding "Lie derivatives" annihilate ω:

$$0 = \delta_k \omega := (\delta_k \omega^{ij}) \cdot dx_i \wedge dx_j + \omega^{ij} \cdot d(\delta_k x_i) \wedge dx_j + \omega^{ij} \cdot dx_i \wedge d(\delta_k x_j). \tag{0.46}$$

Note the contrast with the case of Fedosov connections which are attached to a matrix q that is symplectic "in the horizontal directions"

$$\delta_j = \partial/\partial \xi_j;$$

cf. 0.35. Hamiltonian connections with respect to symplectic forms naturally appear, by the way, in the background of some of the basic differential equations of mathematical physics, in particular in the background of the Painlevé VI equations. This kind of Hamiltonian connections turn out to have an arithmetic analogue [**40**] which will be discussed in the body of our book but not in this Introduction.

On the other hand one can consider *Hamiltonian connections* on a B as in 0.43 with respect to a *Poisson structure*. The Poisson story will be reviewed in the main body of the book. The two concepts of Hamiltonian connections (relative to symplectic forms and relative to Poisson structures) are related in case

$$\delta_k \omega^{ij} = 0.$$

An example of Poisson structure is provided by *Lie-Poisson structures* on Lie algebras in which case the corresponding Hamiltonian connections typically lead to Lax connections. A classical example of equations arising from a Lie-Poisson structure are the *Euler equations* for the rigid body which will be reviewed in the body of the book; the Euler equations are related to Lax equations in at least two different ways (via 3×3 and 2×2 matrices respectively). Euler equations will have an arithmetic analogue, although this arithmetic analogue will not be a priori related to our arithmetic analogues of Lax equations.

Finally we mention *Cartan connections*. In the same way we considered principal bundles and attached to them algebras B as in 0.10 one can consider "infinite jet bundles" (which we do not define here) and attach to them polynomial algebras,

$$B = A[x_j^{(\alpha)};\ \alpha \in \mathbb{Z}^m_{\geq 0}, j = 1, ..., d], \tag{0.47}$$

where A is as in 0.1 and $x_j^{(\alpha)}$ are indeterminates; these algebras come equipped with a natural flat connection

$$\delta_i := \frac{\partial}{\partial \xi_i} + \sum_j \sum_\alpha x_j^{(\alpha + e_i)} \frac{\partial}{\partial x_j^{(\alpha)}}, \tag{0.48}$$

where e_i is the canonical basis of $\mathbb{Z}^m$.

DEFINITION 0.14. The flat connection (δ_i) on the ring B in 0.47 defined by 0.48 is called the *Cartan connection*.

Other names for (δ_i) above are: the *total derivative* or the *Cartan distribution*. The Cartan connections have an arithmetic analogue which was thoroughly studied in [**33, 32**], will be reviewed in the main text of the present book in case $m = 1$, and plays a central role in the whole theory.

0.1.3. Arithmetic differential geometry. Roughly speaking arithmetic differential geometry is obtained from classical differential geometry by replacing classical differentiation (derivations) with arithmetic differentiation (p-derivations). This is often a convoluted process that we now explain.

We start with the analogue of connections in the real case. The first step is to consider the ring

$$B = A[x, \det(x)^{-1}]$$

defined as in 0.10, with $x = (x_{ij})$ an $n \times n$ matrix of indeterminates, but where A is given now by

$$A = \mathbb{Z}[1/M, \zeta_N]$$

as in 0.2. We again consider the group scheme over A,

$$G = GL_n := Spec\ B.$$

A first attempt to define arithmetic analogues of connections would be to consider families of p-derivations

$$\delta_p : B \to B, \quad p \in \mathcal{V},$$

extending the p-derivations 0.6; one would then proceed by considering their commutators on B (or, if necessary, expressions derived from these commutators). But the point is that the examples of "arithmetic analogues of connections" we will encounter in practice will almost never lead to p-derivations $B \to B$! What we shall be led to is, rather, an adelic concept we next introduce. (Our guiding "principle" here is that, as mentioned before, C^∞ geometric objects should correspond to adelic objects in arithmetic while analytic/algebraic geometric objects correspond to global objects in arithmetic.)

To introduce our adelic concept let us consider, for each $p \in \mathcal{V}$, the p-adic completion of B:

$$B^{\widehat{p}} := \lim_{\leftarrow} B/p^n B. \tag{0.49}$$

Then we make the following:

DEFINITION 0.15. An *adelic connection* on $G = GL_n$ is a family (δ_p) of p-derivations

$$\delta_p := \delta_p^B : B^{\widehat{p}} \to B^{\widehat{p}}, \quad p \in \mathcal{V}, \tag{0.50}$$

extending the p-derivations in 0.6.

If $\phi_p : B^{\widehat{p}} \to B^{\widehat{p}}$ are the Frobenius lifts attached to δ_p and $G^{\widehat{p}} = Spf\ B^{\widehat{p}}$ is the p-adic completion of $G = GL_n = Spec\ B$ then we still denote by $\phi_p : G^{\widehat{p}} \to G^{\widehat{p}}$ the induced morphisms of p-adic formal schemes.

Next we will explore analogues of the various types of connections encountered in classical differential geometry: Ehresmann, Chern, Levi-Cività, Fedosov, and Lax. (The stories of Hamiltonian connections and Cartan connections will be discussed in the main body of the book and will be skipped here.)

In what follows we need an analogue of the trivial connection in 0.14. It will be given by the adelic connection (δ_{0p}) defined by

$$\delta_{0p}x = 0. \tag{0.51}$$

The associated Frobenius lifts will be denoted by (ϕ_{0p}); they satisfy

$$\phi_{0p}(x) = x^{(p)} \tag{0.52}$$

where $x^{(p)}$ is the matrix (x_{ij}^p). We call $\delta_0 = (\delta_{0p})$ the *trivial* adelic connection.

To introduce arithmetic analogues of Ehresmann connections one starts by noting that, for $n \geq 2$, there are *no* adelic connections $\delta = (\delta_p)$ whose attached Frobenius lifts (ϕ_p) make the following diagrams commute:

$$\begin{array}{ccc} G^{\widehat{p}} \times G^{\widehat{p}} & \xrightarrow{\mu} & G^{\widehat{p}} \\ \phi_p \times \phi_{0p} \downarrow & & \downarrow \phi_p \\ G^{\widehat{p}} \times G^{\widehat{p}} & \xrightarrow{\mu} & G^{\widehat{p}} \end{array} \tag{0.53}$$

Since 0.53 is an analogue of 0.16 one can view this as saying that there are no adelic connections that are analogues of right invariant connections. This is an elementary observation; we will in fact prove a less elementary result:

THEOREM 0.16. *For $n \geq 2$ and $p \nmid n$ there are no adelic connections (δ_p) and (δ_{1p}) whose attached Frobenius lifts (ϕ_p) and (ϕ_{1p}) make the following diagrams commute:*

$$\begin{array}{ccc} G^{\widehat{p}} \times G^{\widehat{p}} & \xrightarrow{\mu} & G^{\widehat{p}} \\ \phi_p \times \phi_{1p} \downarrow & & \downarrow \phi_p \\ G^{\widehat{p}} \times G^{\widehat{p}} & \xrightarrow{\mu} & G^{\widehat{p}} \end{array} \tag{0.54}$$

There is a useful property, weaker than the commutativity of 0.53, namely an invariance property with respect to the action on G by right translation of the group

$$N := (\text{normalizer of the diagonal maximal torus } T \text{ of } G). \tag{0.55}$$

Indeed we will say that an adelic connection (δ_p) with associated Frobenius lifts (ϕ_p) is *right invariant with respect to N* if the following diagrams are commutative:

$$\begin{array}{ccc} G^{\widehat{p}} \times N^{\widehat{p}} & \xrightarrow{\mu} & G^{\widehat{p}} \\ \phi_p \times \phi_{0p} \downarrow & & \downarrow \phi_p \\ G^{\widehat{p}} \times N^{\widehat{p}} & \xrightarrow{\mu} & G^{\widehat{p}} \end{array} \tag{0.56}$$

This latter property has its own merits but is too weak to function appropriately as a defining property of Ehresmann connections in arithmetic. Instead, we will consider an appropriate analogue of "linearity," 0.15. What we will do will be to replace the Lie algebra $\mathfrak{gl}_n$ by an arithmetic analogue of it, $\mathfrak{gl}_{n,\delta_p}$, and then we will introduce an arithmetic analogue of the logarithmic derivative. This new framework will naturally lead us to the following:

DEFINITION 0.17. An adelic connection (δ_p) is an *Ehresmann connection* if

$$\delta_p x = \alpha_p \cdot x^{(p)}, \tag{0.57}$$

where α_p are matrices with coefficients in A.

By the way, clearly, Ehresmann connections are right invariant with respect to N. We will attach Galois groups to such Ehresmann connections and develop the basics of their theory. A natural expectation is that these Galois groups belong to the group $N(A)^{\delta}$ of all matrices in $N(A)$ whose entries are roots of unity or 0. This expectation is not always realized but we will prove that something close to it is realized for (α_p) "sufficiently general." The above expectation is justified by the fact that, according to the general philosophy of the field with one element $\mathbb{F}_1$, the union of the $N(A)^{\delta}$'s, as A varies, plays the role of "$GL_n(\mathbb{F}_1^a)$," where $\mathbb{F}_1^a$ is the "algebraic closure of $\mathbb{F}_1$."

Next we explain our arithmetic analogue of Chern connections.

Let $q \in GL_n(A)$ with $q^t = \pm q$. Attached to q we have, again, maps

$$\mathcal{H}_q : G \to G, \quad \mathcal{B}_q : G \times G \to G; \quad \mathcal{H}_q(x) = x^t q x, \quad \mathcal{B}_q(x,y) = x^t q y.$$

We continue to denote by $\mathcal{H}_q, \mathcal{B}_q$ the maps induced on the p-adic completions $G^{\widehat{p}}$ and $G^{\widehat{p}} \times G^{\widehat{p}}$. Consider again the trivial adelic connection $\delta_0 = (\delta_{0p})$ on G (so $\delta_{0p} x = 0$) and denote by (ϕ_{0p}) the attached Frobenius lifts (so $\phi_{0p}(x) = x^{(p)}$). Then we will prove the following:

THEOREM 0.18. *For any $q \in GL_n(A)$ with $q^t = \pm q$ there exists a unique adelic connection $\delta = (\delta_p)$ whose attached Frobenius lifts (ϕ_p) make the following diagrams commute:*

$$\begin{array}{ccc} G^{\widehat{p}} & \stackrel{\phi_p}{\longrightarrow} & G^{\widehat{p}} \\ \mathcal{H}_q \downarrow & & \downarrow \mathcal{H}_q \\ G^{\widehat{p}} & \stackrel{\phi_{0p}}{\longrightarrow} & G^{\widehat{p}} \end{array} \qquad \begin{array}{ccc} G^{\widehat{p}} & \stackrel{\phi_{0p}\times\phi_p}{\longrightarrow} & G^{\widehat{p}} \times G^{\widehat{p}} \\ \phi_p \times \phi_{0p} \downarrow & & \downarrow \mathcal{B}_q \\ G^{\widehat{p}} \times G^{\widehat{p}} & \stackrel{\mathcal{B}_q}{\longrightarrow} & G^{\widehat{p}} \end{array} \tag{0.58}$$

DEFINITION 0.19. An adelic connection $\delta = (\delta_p)$ is *$\mathcal{H}_q$-horizontal* (respectively *$\mathcal{B}_q$-symmetric*) with respect to $\delta_0 = (\delta_{0p})$ if the left (respectively right) diagram in 0.58 is commutative. The unique connection which is *$\mathcal{H}_q$-horizontal* and *$\mathcal{B}_q$-symmetric* with respect to δ_0 (cf. Theorem 0.18) is called the *real Chern connection* attached to q.

The real Chern connection introduced above is an arithmetic analogue of the real Chern connection in classical differential geometry. Unlike in the case of classical differential geometry our adelic Chern connections will *not* be special cases of Ehresmann connections (although they *will* be right invariant with respect to N).

On the other hand we will have the following easy:

THEOREM 0.20. *For any $q \in GL_n(A)$ with $q^t = \pm q$ there is a unique adelic connection (δ_p) on GL_n whose attached Frobenius lifts (ϕ_p) make the following diagrams commute:*

$$\begin{array}{ccc} G^{\widehat{p}} & \stackrel{\phi_{0p}\times\phi_p}{\longrightarrow} & G^{\widehat{p}} \times G^{\widehat{p}} \\ \mathcal{H}_q \downarrow & & \downarrow \mathcal{B}_q \\ G^{\widehat{p}} & \stackrel{\phi_{0p}}{\longrightarrow} & G^{\widehat{p}} \end{array} \tag{0.59}$$

DEFINITION 0.21. An adelic connection $\delta = (\delta_p)$ is *$(\mathcal{H}_q, \mathcal{B}_q)$-horizontal* with respect to $\delta_0 = (\delta_{0p})$ if the diagram 0.59 is commutative. The unique connection which is *$(\mathcal{H}_q, \mathcal{B}_q)$-horizontal* with respect to δ_0 (cf. Theorem 0.20) is called the *complex Chern connection* attached to q.

The concept of real Chern connection introduced above is, from an arithmetic point of view, more subtle than the concept of complex Chern connection. This can be seen already in the case of GL_1. Indeed let

$$q \in GL_1(A) = A^{\times}, \quad A = \mathbb{Z}[1/M].$$

In this case it turns out that the "Christoffel symbols" defining the real Chern connection are related to the Legendre symbol; explicitly the Frobenius lift $\phi_p : G^{\widehat{p}} \to G^{\widehat{p}}$ corresponding to the real Chern connection attached to q is induced by the homomorphism

$$\phi_p : \mathbb{Z}_p[x, x^{-1}]^{\widehat{p}} \to \mathbb{Z}_p[x, x^{-1}]^{\widehat{p}} \tag{0.60}$$

satisfying

$$\phi_p(x) = q^{(p-1)/2} \left(\frac{q}{p}\right) x^p, \tag{0.61}$$

where $\left(\frac{q}{p}\right)$ is the Legendre symbol of $q \in A^{\times} \subset \mathbb{Z}_{(p)}$. On the other hand the Frobenius lift $\phi_p : G^{\widehat{p}} \to G^{\widehat{p}}$ corresponding to the complex Chern connection attached to q is simply defined by a homomorphism 0.60 satisfying

$$\phi_p(x) = q^{p-1} x^p. \tag{0.62}$$

In view of the above we will mainly concentrate, in this book, on the study of the real Chern connection while leaving the simpler case of the complex Chern connection to the reader. Note by the way that the quantity $\phi(x)/x^p$ in 0.62 is the square of the quantity $\phi(x)/x^p$ in 0.61; this has an analogue for GL_n with n arbitrary and is analogous to the fact that the Christoffel symbols of the classical complex Chern connection are twice the Christoffel symbols of the classical real Chern connection; cf. 0.30 and 0.31.

We shall also introduce, in our book, adelic connections that are analogues of Levi-Cività connections. They are already relevant in case $\mathcal{V}$ consists of one prime only. So assume, in the theorem below, that $\mathcal{V} = \{p\}$ consists of one prime p.

THEOREM 0.22. *For any symmetric $q \in GL_n(A)$ with $q^t = q$ there is a unique n-tuple $(\delta_{1p}, ..., \delta_{np})$ of adelic connections on $G = GL_n$ with attached Frobenius lifts $(\phi_{1p}, ..., \phi_{np})$, such that the following diagrams are commutative for $i = 1, ..., n$,*

$$\begin{array}{ccc} G^{\widehat{p}} & \xrightarrow{\phi_{ip}} & G^{\widehat{p}} \\ \mathcal{H}_q \downarrow & & \downarrow \mathcal{H}_q \\ G^{\widehat{p}} & \xrightarrow{\phi_{0p}} & G^{\widehat{p}} \end{array} \tag{0.63}$$

and such that, for all $i, j = 1, ..., n$, we have:

$$\delta_{ip} x_{kj} = \delta_{jp} x_{ki}. \tag{0.64}$$

DEFINITION 0.23. The tuple $(\delta_{1p}, ..., \delta_{np})$ in Theorem 0.22 is called the *Levi-Cività connection* attached to q,

The condition 0.63 says of course that for each i, (δ_{ip}) is $\mathcal{H}_q$-horizontal with respect to (δ_{0p}) and hence is analogous to the condition of parallelism 0.28 (with the + sign). The condition 0.64 is analogous to the condition of torsion freeness 0.32. This justifies our Definition 0.23 of the Levi-Cività connection. But note that, unlike in the case of classical differential geometry, our adelic Levi-Cività connections are *not* special cases of Ehresmann connections.

The adelic Levi-Cività connection and the adelic real Chern connection attached to q will be related by certain congruences mod p that are reminiscent of the relation between the two connections in classical differential geometry.

The one prime paradigm of the Levi-Cività connection above can be viewed as corresponding to the classical Levi-Cività connections of *univariate* metrics, by which we understand metrics $g = \sum g_{ij} d\xi_i d\xi_j$ satisfying

$$\delta_k g_{ij} = \delta_l g_{ij} \tag{0.65}$$

for all i, j, k, l. After a linear change of coordinates the coefficients of such metrics can be made to depend on only one of the coordinates (which justifies the term *univariate*).

Continuing to assume $\mathcal{V} = \{p\}$ one can also attempt to develop an arithmetic analogue of Fedosov connections as follows. Consider an anti-symmetric $q \in GL_n(A)$, $q^t = -q$. Let us say that an n-tuple of $(\delta_{1p}, ..., \delta_{np})$ of adelic connections on $G = GL_n$ is a *Fedosov connection* relative to q if the attached Frobenius lifts $(\phi_{1p}, ..., \phi_{np})$ make the diagrams 0.63 commutative and, in addition, the equalities 0.64 hold. We will prove that for $n = 2$ and any anti-symmetric q Fedosov connections relative to q exist. However, in contrast with the Levi-Cività story, we will prove that for $n \geq 4$ there is no Fedosov connection relative to the split q, for instance.

Finally there are adelic connections that are analogous to Lax connections. In fact there are two such analogues which we call *isospectral* and *isocharacteristic* Lax connections. They offer two rather different arithmetic analogues of isospectral flows in the space of matrices. Indeed the isocharacteristic property essentially says that a certain characteristic polynomial has "δ-constant" coefficients whereas the isospectrality property essentially says that the characteristic polynomial has "δ-constant" roots. (Here *δ-constant* means "killed by all δ_p" which amounts to "being a root of unity or 0".) In usual calculus the two properties are equivalent but in our arithmetic calculus these two properties are quite different. By the way isospectral and isocharacteristic Lax connections will not be defined on the whole of G but rather on certain Zariski open sets G^* and G^{**} of G respectively. Let us give some details of this in what follows. Let $G^* \subset G$ be the open set of *regular* matrices, consider the diagonal maximal torus T in G, and let $T^* = G^* \cap T$; *regular* means here "with distinct diagonal entries." We will prove:

THEOREM 0.24. *There exists a unique adelic connection $\delta = (\delta_p)$ on G^* with attached Frobenius lifts (ϕ_p) such that each ϕ_p makes the following diagram commute:*

$$\begin{array}{ccc} (T^*)^{\widehat{p}} \times G^{\widehat{p}} & \stackrel{\phi_{0p} \times \phi_{0p}}{\longrightarrow} & (T^*)^{\widehat{p}} \times G^{\widehat{p}} \\ \mathcal{C} \downarrow & & \downarrow \mathcal{C} \\ (G^*)^{\widehat{p}} & \stackrel{\phi_p}{\longrightarrow} & (G^*)^{\widehat{p}} \end{array} \tag{0.66}$$

where $\mathcal{C}(t, x) := x^{-1} t x$.

DEFINITION 0.25. The adelic connection δ in Theorem 0.24 is called the *canonical isospectral Lax connection.*

More generally let (δ_p) be the canonical isospectral Lax connection, with attached Frobenius lifts (ϕ_p), $\phi_p(x) = \Phi_p$, i.e., $\Phi_p = (\Phi_{pij}(x))$ where $\Phi_{pij}(x) = \phi_p(x_{ij})$. Let $\alpha_p(x)$ be $n \times n$ matrices with entries in $\mathcal{O}(G^*)^{\widehat{p}}$. Then, setting $\epsilon_p(x) = 1 + p\alpha_p(x)$, one can consider the *isospectral Lax connection* attached to (α_p), defined by the family of Frobenius lifts $(\phi_p^{(\epsilon)})$,

$$\phi_p^{(\epsilon)}(x) = \Phi_p^{(\epsilon)}(x) = \epsilon_p(x) \cdot \Phi_p(x) \cdot \epsilon_p(x)^{-1}.$$

The latter has the following property that justifies the term *isospectral*. Let

$$u_p = a_p^{-1} b_p a_p, \quad a_p \in GL_n(A^{\widehat{p}}), \quad b_p = diag(b_{p1}, ..., b_{pn}) \in T^*(A^{\widehat{p}})$$

be such that $\phi_p^{(\epsilon)}(u) = \Phi_p^{(\epsilon)}(u)$. Then

$$\delta_p b_{pi} = 0, \quad i = 1, ..., n.$$

So the eigenvalues b_{pi} of u_p are δ_p-constant.

On the other we will prove:

THEOREM 0.26. *There is an adelic connection $\delta = (\delta_p)$ on an open set $G^{**} \subset G$ with attached Frobenius lifts (ϕ_p), such that each ϕ_p makes the following diagram commutative:*

$$\begin{array}{ccc} (G^{**})^{\widehat{p}} & \xrightarrow{\phi_p} & (G^{**})^{\widehat{p}} \\ \mathcal{P} \downarrow & & \downarrow \mathcal{P} \\ (\mathbb{A}^n)^{\widehat{p}} & \xrightarrow{\phi_{0p}} & (\mathbb{A}^n)^{\widehat{p}}, \end{array} \tag{0.67}$$

where $\mathbb{A}^n = Spec\ A[z]$.

The diagram 0.67 should be viewed as analogous to diagram 0.38. The adelic connection δ is, of course, not unique.

DEFINITION 0.27. An adelic connection δ as in Theorem 0.26 is called an *isocharacteristic Lax connection*.

Among isocharacteristic Lax connections there will be a *canonical* one.

For any isocharacteristic Lax connection (δ_p), if $\phi_p(x) = \Phi_p(x)$ and $u_p \in G^{**}(A^{\widehat{p}})$ satisfies $\phi_p(u) = \Phi_p(u)$ we have that

$$\delta_p(\mathcal{P}_j(u)) = 0, \quad j = 1, ..., n.$$

So the coefficients of the characteristic polynomial of u_p are δ_p-constant.

Note that the canonical Frobenius lifts ϕ_p in the two diagrams 0.66 and 0.67 will *not* coincide on the intersection $G^* \cap G^{**}$; so the isospectral and isocharacteristic stories are really different.

Next we would like to explore the curvature of adelic connections.

Consider first the case of Ehresmann connections, 0.57. Since $\alpha_p \in A$ for all p our Frobenius lifts $\phi_p : B^{\widehat{p}} \to B^{\widehat{p}}$ induce Frobenius lifts $\phi_p : A[x] \to A[x]$ and hence one can consider the "divided" commutators

$$\varphi_{pp'} := \frac{1}{pp'}[\phi_p, \phi_{p'}] : A[x] \to A[x], \quad p, p' \in \mathcal{V}. \tag{0.68}$$

The family $(\varphi_{pp'})$ will be referred to as the *curvature* of the Ehresmann connection (δ_p).

The situation for general adelic connections (including the cases of real Chern and Lax connections) will be quite different. Indeed, in defining curvature we face the following dilemma: our p-derivations δ_p in 0.50 do not act on the same ring, so there is no a priori way of considering their commutators and, hence, it does not seem possible to define, in this way, the notion of curvature. It will turn out, however, that some of our adelic connections will satisfy a remarkable property which we call *globality along the identity* (more generally along various subvarieties); this property will allow us to define curvature via commutators.

DEFINITION 0.28. Consider the matrix $T = x - 1$, where 1 is the identity matrix. An adelic connection $\delta = (\delta_p)$ on GL_n, with attached family of Frobenius lifts (ϕ_p), is *global along* 1 if, for all p, $\phi_p : B^{\widehat{p}} \to B^{\widehat{p}}$ sends the ideal of 1 into itself and, moreover, the induced homomorphism $\phi_p : A^{\widehat{p}}[[T]] \to A^{\widehat{p}}[[T]]$ sends the ring $A[[T]]$ into itself. If the above holds then the *curvature* of (δ_p) is defined as the family of "divided" commutators $(\varphi_{pp'})$,

$$\varphi_{pp'} := \frac{1}{pp'}[\phi_p, \phi_{p'}] : A[[T]] \to A[[T]], \tag{0.69}$$

where $p, p' \in \mathcal{V}$. Let $\text{End}(A[[T]])$ denote the Lie ring of $\mathbb{Z}$-module endomorphisms of $A[[T]]$. Then define the *holonomy ring* $\mathfrak{hol}$ of δ as the $\mathbb{Z}$-linear span in $\text{End}(A[[T]])$ of all the *Lie monomials*

$$[\phi_{p_1}, [\phi_{p_2}, ..., [\phi_{p_{s-1}}, \phi_{p_s}]...]] : A[[T]] \to A[[T]]$$

where $s \geq 2$, $p_i \in \mathcal{V}$. Similarly define the *holonomy* $\mathbb{Q}$*-algebra* $\mathfrak{hol}_{\mathbb{Q}}$ of δ as the $\mathbb{Q}$-linear span of $\mathfrak{hol}$ in $\text{End}(A[[T]]) \otimes \mathbb{Q}$. Finally define the *completed holonomy ring*,

$$\widehat{\mathfrak{hol}} = \lim_{\leftarrow} \mathfrak{hol}_n,$$

where $\mathfrak{hol}_n$ is the image of the map

$$\mathfrak{hol} \to \text{End}(A[[T]]/(T)^n). \tag{0.70}$$

The various maps referred to above can be traced on the following diagram:

$$\begin{array}{ccccccccc} B^{\widehat{p}} & \subset & A^{\widehat{p}}[[T]] & \supset & A[[T]] & \subset & A^{\widehat{p'}}[[T]] & \supset & B^{\widehat{p'}} \\ \phi_p \downarrow & & \phi_p \downarrow & & \phi_p \downarrow\downarrow \phi_{p'} & & \downarrow \phi_{p'} & & \downarrow \phi_{p'} \\ B^{\widehat{p}} & \subset & A^{\widehat{p}}[[T]] & \supset & A[[T]] & \subset & A^{\widehat{p'}}[[T]] & \supset & B^{\widehat{p'}} \end{array}$$

The idea of comparing p-adic phenomena for different p's by "moving along the identity section" was introduced in [**32**] where it was referred to as *analytic continuation between primes*. Analytic continuation is taken here in the sense of Zariski [**108**], Preface, pp. xii-xiii, who was the first to use completions of varieties along subvarieties as a substitute for classical analytic continuation over the complex numbers. The technique of analytic continuation is also used, in the form of *formal patching*, in inverse Galois theory [**65**]. Note that in our context we are patching data defined on "tubular neighborhoods"

$$Spf\ B^{\widehat{p}} \quad \text{and} \quad Spf\ A[[T]]$$

of two *closed* subsets

$$Spec\ B/pB \quad \text{and} \quad Spec\ B/(T)$$

of the scheme *Spec B*; the data are required to coincide on the "tubular neighborhood"

$$Spf\ A^{\widehat{p}}[[T]]$$

of the intersection

$$Spec\ B/(p,T) = (Spec\ B/pB) \cap (Spec\ B/(T)).$$

This is in contrast with the use of patching in Galois theory [**65**] where one patches data defined on two *open* sets covering a formal scheme.

Of course, the trivial adelic connection $\delta_0 = (\delta_{0p})$, $\delta_{0p}x = 0$, is global along 1 so it induces ring endomorphisms $\phi_{0p} : A[[T]] \to A[[T]]$,

$$\phi_{0p}(T) = (1+T)^{(p)} - 1.$$

We may morally view δ_0 as an analogue of a flat connection in real geometry (where $A \subset C^{\infty}(\mathbb{R}^m, \mathbb{R})$). Alternatively we may view δ_0 as an arithmetic analogue of the derivations $\delta_{\overline{i}} = \partial/\partial \overline{z}_i$ on $A[x, \det(x)^{-1}]$ which kill x, where $A \subset C^{\infty}(\mathbb{C}^m, \mathbb{C})$. Following this second analogy we may consider an arbitrary adelic connection $\delta = (\delta_p)$ global along 1, with attached Frobenius lifts (ϕ_p), and introduce the following:

DEFINITION 0.29. The $(1,1)$-*curvature* of δ is the matrix of "divided commutators" $(\varphi_{p\overline{p}'})$

$$\varphi_{p\overline{p}'} := \frac{1}{pp'}[\phi_p, \phi_{0p'}] : A[[T]] \to A[[T]], \quad p \neq p', \tag{0.71}$$

$$\varphi_{p\overline{p}} := \frac{1}{p}[\phi_p, \phi_{0p}] : A[[T]] \to A[[T]]. \tag{0.72}$$

Let us discuss next the curvature of real Chern connections. (The case of complex Chern connections is easier and will be skipped here.) We will prove:

THEOREM 0.30. *Let $q \in GL_n(A)$ with $q^t = \pm q$. If all the entries of q are roots of unity or* 0 *then the real Chern connection δ attached to q is global along* 1*; in particular δ has a well defined curvature and $(1,1)$-curvature.*

So we may address the question of computing the curvature and $(1,1)$-curvature of real Chern connections for various q's whose entries are 0 or roots of unity. A special case of such q's is given by the following:

DEFINITION 0.31. A matrix $q \in GL_n(A)$ is *split* if it is one of the following:

$$\begin{pmatrix} 0 & 1_r \\ -1_r & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1_r \\ 1_r & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1_r \\ 0 & 1_r & 0 \end{pmatrix}, \tag{0.73}$$

where 1_r is the $r \times r$ identity matrix and $n = 2r, 2r, 2r+1$ respectively.

We will prove:

THEOREM 0.32. *Let q be split and let $(\varphi_{pp'})$ and $(\varphi_{p\overline{p}'})$ be the curvature and the $(1,1)$-curvature of the real Chern connection on G attached to q. Then the following hold:*

1) Assume $n \geq 4$. Then for all $p \neq p'$ we have $\varphi_{pp'} \neq 0$.

2) Assume $n = 2r \geq 2$. Then for all p, p' we have $\varphi_{pp'}(T) \equiv 0$ mod $(T)^3$.

3) Assume $n = 2$ and $q^t = -q$. Then for all p, p' we have $\varphi_{pp'} = 0$.

4) Assume $n \geq 2$. Then for all p, p' we have $\varphi_{p\overline{p}'} \neq 0$.

5) Assume $n = 1$. Then for all p, p' we have $\varphi_{pp'} = \varphi_{p\overline{p}'} = 0$.

Assertion 1 morally says that *Spec* $\mathbb{Z}$ is "curved," while assertion 2 morally says that *Spec* $\mathbb{Z}$ is only "mildly curved." Assertions 1 and 2 will imply, in particular, assertion 1 in the following:

THEOREM 0.33. *Assume q split and $n \geq 4$ is even. Then, for the real Chern connection attached to q, the following hold:*

1) $\widehat{\mathfrak{hol}}$ is non-zero and pronilpotent.

2) $\mathfrak{hol}_{\mathbb{Q}}$ is not spanned over $\mathbb{Q}$ by the components of the curvature.

Assertion 1 is in stark contrast with the fact, to be explained in the body of the book, that *holonomy Lie algebras* arising from Galois theory are never nilpotent unless they vanish. Assertion 2 should be viewed as a statement suggesting that the flavor of our arithmetic situation is rather different from that of classical *locally symmetric* spaces; indeed, for the latter, the Lie algebra of holonomy *is* spanned by the components of the curvature.

Note that the above theorem says nothing about the vanishing of the curvature $\varphi_{pp'}$ in case $n = 2, 3$ and $q^t = q$; our method of proof does not seem to apply to these cases.

By the way if $q \in GL_{2r}$ is symmetric and split then the real Chern connection $\delta_p : B^{\widehat{p}} \to B^{\widehat{p}}$ attached to q *does not* send the ideal of SL_n into itself; this is in contrast with the situation encountered in classical differential geometry. To remedy this situation we will construct connections on GL_n that *do* send the ideal of SL_n into itself; there are many such connections and the "simplest" one will be called the *special linear connection*.

Other curvatures can be introduced and vanishing/non-vanishing results for them will be proved; they will be referred to as 3*-curvature, first Chern form, first Chern* $(1,1)$*-form, mean curvature, scalar curvature*, etc. We will also introduce *Chern connections* attached to hermitian metrics and results will be proved for their curvature. A similar theory can be developed for complex Chern connections.

Similar results will also be proved for the curvature of Lax connections. As we shall see the open sets G^* and G^{**} where isospectral and isocharacteristic Lax connections are defined (cf. 0.66 and 0.67) do not contain the identity of the group $G = GL_n$ hence curvature cannot be defined by analytic continuation along the identity; however these open sets will contain certain torsion points of the diagonal maximal torus of G and we will use analytic continuation along such torsion points to define curvature and $(1,1)$-curvature. We will then prove the non-vanishing of the $(1,1)$-curvature of isocharacteristic Lax connection for $n = 2$. The (canonical) isospectral Lax connection has, immediately from its definition, a vanishing curvature.

Note that the concept of curvature discussed above was based on what we called *analytic continuation between primes*; this was the key to making Frobenius lifts corresponding to different primes act on a same ring and note that it only works as stated for adelic connections that are global along 1. This restricts the applicability of our method to "metrics" q with components roots of unity or 0. One can generalize our method to include q's with more general entries by replacing the condition of being global along 1 with the condition of "being global along certain tori;" this will be discussed in the body of the book. However, there is a different approach towards making Frobenius lifts comparable; this approach is based on *algebraizing Frobenius lifts via correspondences* and works for adelic connections that are not necessarily global along 1 (or along a torus). The price to pay for allowing this generality is that endomorphisms (of $A[[T]]$) are replaced by correspondences (on GL_n). Let us explain this alternative road to curvature in what follows.

We will prove the following:

THEOREM 0.34. *Let $\delta=(\delta_p)$ be the real Chern connection on $G=GL_n$ attached to a matrix $q\in GL_n(A)$ with $q^t=\pm q$. Then there exist maps of A-schemes $\pi_p:Y_p\to G$ and $\varphi_p:Y_p\to G$ such that for each p,*
1) π_p is affine and étale;
2) the p-adic completion of π_p, $\pi_p^{\widehat{p}}:Y_p^{\widehat{p}}\to G^{\widehat{p}}$, is an isomorphism;
3) we have an equality of maps, $\varphi_p^{\widehat{p}}=\phi_p\circ\pi_p^{\widehat{p}}:Y_p^{\widehat{p}}\to G^{\widehat{p}}$.

The triples

$$\Gamma_p:=(Y_p,\pi_p,\varphi_p)$$

can be viewed as *correspondences* on G which provide "algebraizations" of our Frobenius lifts ϕ_p.

DEFINITION 0.35. The family (Γ_p) in Theorem 0.34 is called a *correspondence structure* for (δ_p).

This structure is not unique but does have some "uniqueness features." On the other hand any correspondence Γ_p acts on the field E of rational functions of $G=GL_n$ by the formula $\Gamma_p^*:E\to E$,

$$\Gamma_p^*(z):=\mathrm{Tr}_{\pi_p}(\varphi_p^*(z)),\quad z\in E, \tag{0.74}$$

where

$$\mathrm{Tr}_{\pi_p}:F_p\to E$$

is the trace of the extension

$$\pi_p^*:E\to F_p:=Y_p\otimes_G E$$

and

$$\varphi_p^*:E\to F_p$$

is induced by φ_p.

DEFINITION 0.36. The *curvature* of (Γ_p) is the matrix $(\varphi_{pp'}^*)$ with entries the additive homomorphisms

$$\varphi_{pp'}^*:=\frac{1}{pp'}[\Gamma_p^*,\Gamma_{p'}^*]:E\to E,\quad p,p'\in\mathcal{V}. \tag{0.75}$$

Note that, in this way, we have defined a concept of curvatur for real Chern connections attached to arbitrary q's (that do not necessarily have entries zeroes or roots of unity). There is a $(1,1)$-version of the above as follows. Indeed the trivial adelic connection $\delta_0=(\delta_{0p})$ has a canonical correspondence structure (Γ_{0p}) given by

$$\Gamma_{0p}=(G,\pi_{0p},\varphi_{0p}),$$

where π_{0p} is the identity, and $\varphi_{0p}(x)=x^{(p)}$.

Definition 0.37. The $(1,1)$-*curvature* of (Γ_p) is the family $(\varphi^*_{p\overline{p}'})$ where $\varphi^*_{p\overline{p}'}$ is the additive endomorphism

$$\varphi^*_{p\overline{p}'} := \frac{1}{pp'}[\Gamma^*_{0p'}, \Gamma^*_p] : E \to E \quad \text{for } p \neq p', \tag{0.76}$$

$$\varphi^*_{p\overline{p}} := \frac{1}{p}[\Gamma^*_{0p}, \Gamma^*_p] : E \to E. \tag{0.77}$$

Then we will prove the following:

theorem 0.38. *Let $q \in GL_2(A)$ be split. Then, for the real Chern connection attached to q, the following hold:*

1) Assume $q^t = -q$. Then for all p, p' we have $\varphi^*_{pp'} = 0$ and $\varphi^*_{p\overline{p}'} \neq 0$.

2) Assume $q^t = q$. Then for all p, p' we have $\varphi^*_{p\overline{p}'} \neq 0$.

Once again our results say nothing about curvature in case $n = 2$ and $q^t = q$; our method of proof does not seem to apply to this case.

Later in the body of the book the curvature and $(1,1)$-curvature of (Γ_p) introduced above will be called *upper $*$-curvatures* and we will also introduce the concepts of *lower $*$-curvature* and *lower $*$-$(1,1)$-curvature* coming from actions on "cycles."

Finally note that one can define curvature for the Levi-Cività connection.

Definition 0.39. Let $(\delta_{1p}, ..., \delta_{np})$ be the Levi-Cività connection attached to a symmetric $q \in GL_n(A)$. The *curvature* is defined as the family (φ^{ij}_p), indexed by $i, j = 1, ..., n$ given by the divided commutators

$$\varphi^{ij}_p := \frac{1}{p}[\phi_{ip}, \phi_{jp}] : \mathcal{O}(G^{\widehat{p}}) \to \mathcal{O}(G^{\widehat{p}}). \tag{0.78}$$

This is a "vertical" curvature (indexed by the index set of the columns and rows of x) rather than a "horizontal" curvature, in the style of the previously introduced curvatures (which are indexed by primes). We will prove non-vanishing results for these curvatures. For instance we have:

theorem 0.40. *Let i, j, k, l be fixed indices between 1 and n. Then, for the Levi-Cività connection, the following hold:*

1) Assume $\delta_p q_{jk} + \delta_p q_{il} \not\equiv \delta_p q_{ik} + \delta_p q_{jl}$ mod p. Then $\varphi^{ij}_p, \varphi^{kl}_p \not\equiv 0$ mod p.

2) Assume $n = 2r$ and q is split. Then $\varphi^{ij}_p \not\equiv 0$ mod p for $i \neq j$.

In case the entries of q are roots of unity or 0 we will prove that, for each i, the adelic connection (δ_{ip}) appearing in the Levi-Cività connection is global along 1 so we will be able to define a *mixed curvature* $(\varphi^{ij}_{pp'})$ indexed by $i, j = 1, ..., n$ and $p, p' \in \mathcal{V}$ given by the divided commutators

$$\varphi^{ij}_{pp'} := \frac{1}{pp'}[\phi_{ip}, \phi_{jp'}] : A[[T]] \to A[[T]], \quad p \neq p', \tag{0.79}$$

$$\varphi^{ij}_{pp} := \frac{1}{p}[\phi_{ip}, \phi_{jp}] : A[[T]] \to A[[T]]. \tag{0.80}$$

For a fixed p and the Fedosov connection $(\delta_{1p}, \delta_{2p})$ relative to any anti-symmetric $q \in GL_2(A)$ the formula 0.78 defines, again, a *curvature*; we will prove that this curvature does not vanish in general even if q is split.

0.2. Comparison with other theories

0.2.1. Three perspectives. A number of analogies between primes and geometric objects have been proposed. Here are three of them:

A) Primes are analogous to points on a Riemann surface.
B) Primes are analogous to knots in a 3-dimensional manifold.
C) Primes are analogous to directions in an infinite dimensional manifold.

The viewpoint A is classical, it has a complex analytic flavor, and goes back to Dedekind, Hilbert, etc. The framework of Grothendieck, Arakelov, etc., also fits into viewpoint A. According to this viewpoint the ring of integers $\mathbb{Z}$, or more generally rings of integers in number fields, can be viewed as analogues of rings of functions on Riemann surfaces or affine algebraic curves; these are objects of complex dimension 1 (or real dimension 2). Genera of number fields are classically defined and finite, as in the case of Riemann surfaces. There is a related viewpoint according to which $\mathbb{Z}$ is the analogue of an algebraic curve of infinite genus; cf. e.g., [**52**].

The viewpoint B has a topological flavor and originates in suggestions of Mazur, Manin, Kapranov, and others. According to this viewpoint $Spec\,\mathbb{Z}$ should be viewed as an analogue of a 3-dimensional manifold, while the embeddings $Spec\ \mathbb{F}_p \to Spec\ \mathbb{Z}$ should be viewed as analogues of embeddings of circles. The Legendre symbol is then an analogue of linking numbers. This analogy goes rather deep [**93**].

Our book and previous papers by the author adopt the viewpoint C and have a differential geometric flavor.

There is a possibility that our theory has connections with viewpoint B, as shown, for instance, by the presence of the Legendre symbol in our real Chern connections. Indeed the underlying Galois theory of reciprocity is an analogue of the monodromy in the 3-dimensional picture; in the same way our arithmetic curvature theory could be an analogue of the identity component of a natural "holonomy" in the 3-dimensional picture.

0.2.2. Field with one element. There are other approaches that adopt the viewpoint C. For instance Haran's theory of the field with one element, $\mathbb{F}_1$, cf. [**64**], (and previous $\mathbb{F}_1$ flavored work of Kurokawa and others [**84**]) considers the operators

$$\frac{\partial}{\partial p} : \mathbb{Z} \to \mathbb{Z}, \quad \frac{\partial a}{\partial p} := v_p(a)\frac{a}{p},$$

where $v_p(a)$ is the p-adic valuation of a. These operators have a flavor that is rather different from that of Fermat quotients, though, and it seems unlikely that Haran's theory and ours are directly related. Even more remote from our theory are the $\mathbb{F}_1$ theories of Soulé [**104**] and Connes-Consani [**52**] which do not directly provide a way to differentiate integers.

Borger's philosophy of $\mathbb{F}_1$, cf. [**9**], is, in some sense, perpendicular to the above mentioned approaches to $\mathbb{F}_1$ and, in the "case of one prime" is consistent with our approach: roughly speaking, in the case of one prime, Borger's theory [**9, 11**] can be viewed as an algebraization of our analytic theory in [**21, 33**]. In the case of more (all) primes Borger's $\mathbb{F}_1$ theory can also be viewed as a viewpoint consistent with C above: indeed Borger's beautiful suggestion is to take λ-*structures* (in the sense of Grothendieck) as *descent data* from $\mathbb{Z}$ to $\mathbb{F}_1$. Recall that a λ-*structure* on a scheme X flat over $\mathbb{Z}$ is the same as a commuting family (ϕ_p) of Frobenius lifts $\phi_p : X \to X$. So our theory would fit into "λ-geometry" as long as:

1) the Frobenius lifts are defined on the schemes X themselves (rather than on the various p-adic completions $X^{\widehat{p}}$) and

2) the Frobenius lifts commute.

However conditions 1 and 2 are almost never satisfied in our theory: the failure of condition 2 is precisely the origin of our curvature, while finding substitutes for condition 1 requires taking various convoluted paths (such as analytic continuation between primes or algebraization by correspondences). So in practice our approach places us, most of the times, outside the paradigm of λ-geometry.

0.2.3. Ihara's differential. Next we would like to point out what we think is an important difference between our viewpoint here and the viewpoint proposed by Ihara in [**72**]. Our approach, in its simplest form, proposes to see the operator

$$\delta = \delta_p : \mathbb{Z} \to \mathbb{Z},\ a \mapsto \delta a = \frac{a - a^p}{p},$$

where p is a fixed prime, as an analogue of a derivation with respect to p. In [**72**] Ihara proposed to see the map

$$d : \mathbb{Z} \to \prod_p \mathbb{F}_p,\ a \mapsto \left(\frac{a - a^p}{p}\ mod\ p \right) \tag{0.81}$$

as an analogue of differentiation for integers and he proposed a series of conjectures concerning the "zeroes" of the differential of an integer. These conjectures are still completely open; they are in the spirit of the approach A listed above, in the sense that counting zeroes of 1-forms is a Riemann surface concept. But what we see as the main difference between Ihara's viewpoint and ours is that *we do not consider the reduction mod p of the Fermat quotients but the Fermat quotients themselves.* This allows the possibility of considering compositions between our δ_p's which leads to the possibility of considering arithmetic analogues of differential equations, curvature, etc.

0.2.4. Fontaine-Colmez calculus. The Fontaine-Colmez theory of p-adic periods [**58**] also speaks of a *differential calculus with numbers.* Their calculus is perpendicular to ours in the following precise sense. For a fixed prime p our calculus based on the Fermat quotient operator δ_p should be viewed as a a differential calculus in the "unramified direction, "i.e. the "direction" given by the extension

$$\mathbb{Q} \subset \bigcup_{p \nmid N} \mathbb{Q}(\zeta_N),$$

whereas the Fontaine-Colmez calculus should be viewed as a differential calculus in the "totally ramified direction," i.e., roughly, in the "direction" given by the extension

$$\mathbb{Q} \subset \bigcup_n \mathbb{Q}(\zeta_{p^n}).$$

The Fontaine-Colmez theory is based on the *usual* Kähler differentials of totally ramified extensions and hence, unlike ours, it is about *usual* derivations. It is not unlikely, however, that, for a fixed p, a theory unifying the unramified and the totally ramified cases, involving two "perpendicular" directions, could be developed leading to arithmetic partial differential equations in two variables. Signs of possibility of such a theory can be found in our papers [**30, 31**].

0.2.5. Grothendieck's p-curvature. It is worth pointing out that the study of our *curvature* here resembles the study of the *p-curvature* appearing in the arithmetic theory of differential equations that has been developed around the Grothendieck conjecture (cf., e.g., [**79**]). Both curvatures measure the lack of commutation of certain operators and both of theories rely on technical matrix computations. However we should also point out that the nature of the above mentioned operators in the two theories is quite different. Indeed our curvature here involves the "p-differentiation" of numbers with respect to primes p (in other words it is about d/dp) whereas the theory in [**79**] (and related papers) is about usual differentiation d/dt with respect to a variable t of power series in t with arithmetically interesting coefficients. In spite of these differences the two types of curvatures could interact; a model for such an interaction between d/dp and d/dt is in the papers [**30, 31**]. A similar remark can be made about the difference between our approach and that in [**15**] which, again, is about usual Kähler differentials and hence about usual derivations.

0.2.6. Discrete geometry as Euclidean geometry. Finally we would like to point out that the theory in this book is a priori unrelated to topics such as *the geometry of numbers* [**47**] on the one hand and *discrete differential geometry* [**8**] on the other. Indeed in both these geometries what is being studied are discrete configurations of points in the Euclidean space $\mathbb{R}^m$; in the geometry of numbers the configurations of points typically represent rings of algebraic numbers while in discrete differential geometry the configurations of points approximate smooth submanifolds of the Euclidean space. This framework is, therefore, that of the classical geometry of Euclidean space, based on $\mathbb{R}$-coordinates, and *not* that of an analogue of this geometry, based on "prime coordinates." It may very well happen, however, that (one or both of) the above topics are a natural home for some (yet to be discovered) Archimedian counterpart of our (finite) adelic theory.

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