

ON STRICHARTZ ESTIMATES FOR SCHRÖDINGER OPERATORS IN COMPACT MANIFOLDS WITH BOUNDARY

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1. INTRODUCTION

Let (M, g) be a Riemannian manifold of dimension $n \geq 2$. Strichartz estimates are a family of dispersive estimates on solutions $u(t, x) : [-T, T] \times M \rightarrow \mathbb{C}$ to the Schrödinger equation

$$(1.1) \quad D_t u + \Delta_g u = 0, \quad u(0, x) = f(x)$$

where Δ_g denotes the Laplace-Beltrami operator on (M, g) and $D_t = i^{-1}\partial_t$. In their most general form, local Strichartz estimates state that

$$(1.2) \quad \|u\|_{L^p([-T, T]; L^q(M))} \leq C \|f\|_{H^s(M)}$$

where H^s denotes the L^2 Sobolev space over M , and $2 \leq p, q \leq \infty$ satisfies

$$(1.3) \quad \frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \quad (n, p, q) \neq (2, 2, \infty).$$

Such estimates are well established in flat Euclidean space, where $M = \mathbb{R}^n$ and $g_{ij} = \delta_{ij}$. In that case $s = 0$, and one can take $T = \infty$; see for example Strichartz [11], Ginibre and Velo [5], Keel and Tao [6], and references therein. Estimates for the standard flat 2-torus were shown by Bourgain [2] to hold for any $s > 0$.

There is also considerable interest in developing these estimates for non-flat geometries, and also for compact domains. In the case where M is compact and $\partial M = \emptyset$, Burq, Gérard, and Tzvetkov [3] established (1.2) with $s = \frac{1}{p}$. Hence there is a loss of derivatives in their estimate when compared to the case of flat geometries.

A simple investigation of the Schrödinger evolution on spherical harmonics where $M = \mathbb{S}^n$ shows that some loss of derivatives must occur. For instance, with $n = 2$, by taking the initial data to be a highest weight spherical harmonic on \mathbb{S}^2 one concludes that the best possible local $L_x^2 \rightarrow L_{t,x}^4$ bounds would involve a loss of $1/8$ derivatives. This sharp estimate and related ones for Zoll surfaces were obtained in [3]. It is not known, however, whether the weaker estimates involving a loss of $\frac{1}{p}$ derivatives in [3] for general compact manifolds without boundary can be improved.

In the case where $\partial M \neq \emptyset$, one also considers Dirichlet or Neumann boundary conditions in addition to (1.1)

$$u(t, x)|_{\partial M} = 0 \text{ (Dirichlet)}, \quad N_x \cdot \nabla u(t, x)|_{\partial M} = 0 \text{ (Neumann)}$$

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where N_x denotes the unit normal vector field to ∂M . Here one expects a further loss of derivatives in the estimates. The Rayleigh whispering gallery modes over the unit disk in \mathbb{R}^2 provide examples of Dirichlet eigenfunctions which accumulate their energy near the boundary, contributing to high L^p norms. Applying the Schrödinger evolution to these eigenfunctions show that $s \geq \frac{1}{6}$ is necessary for Strichartz estimate with $p = q = 4$. Recently, Anton [1] showed that the estimates (1.2) hold on general manifolds with boundary provided $s > \frac{3}{2p}$. In addition, the arguments of [1] work equally well for a manifold M without boundary equipped with a metric g of Lipschitz regularity.

In this work, we improve on the current results for compact (M, g) where either $\partial M \neq \emptyset$, or $\partial M = \emptyset$ and g is Lipschitz, by showing that Strichartz estimates hold with a loss of fewer derivatives.

Theorem 1.1. *Let (M, g) be either a smooth compact Riemannian manifold with boundary, or a manifold without boundary equipped with a Lipschitz metric g . Then the following Strichartz estimate holds for any Strichartz pair (1.3)*

$$(1.4) \quad \|e^{it\Delta_g} f\|_{L^p([-T, T]; L^q(M))} \lesssim \|f\|_{H^{\frac{4}{3p}}(M)}$$

In the case where (M, g) is a boundaryless manifold with $g \in C^\infty$, the estimate of Burq-Gérard-Tzvetkov (1.2) with $s = \frac{1}{p}$, while not known to be sharp, is a natural result by the following heuristic argument. For a general compact manifold, there are no conjugate points for the geodesic flow at distance less than the injectivity radius of the manifold. Given a solution to the Schrödinger equation whose frequencies are concentrated at λ , energy propagates at speed $\approx \lambda$. Hence, a frequency λ solution should possess good dispersive properties at least until time $T_\lambda \approx \frac{1}{\lambda}$. We thus expect to be able to prove a Strichartz estimate with no loss of derivatives for such a solution over a time interval of size roughly $\frac{1}{\lambda}$. By considering a sum over such intervals we should obtain a Strichartz estimate over a time interval of unit size, only with a constant appearing on the right hand side which is a constant multiple of $\lambda^{\frac{1}{p}}$. This corresponds to $s = \frac{1}{p}$ in the estimate, and Littlewood-Paley theory yields the estimate for arbitrary solutions.

In the case where $\partial M \neq \emptyset$, the boundary conditions affect the flow of energy near the boundary. A key strategy involves reflecting the metric and the solution across the boundary, to obtain a Schrödinger equation on a manifold without boundary, but with a metric that has Lipschitz singularities along ∂M . Hence matters reduce to considering the Schrödinger evolution for Lipschitz metrics. In this case, when establishing estimates for solutions at frequency λ , one can replace the rough metric by a regularized metric which has conjugate points at distance roughly $\lambda^{-\frac{1}{3}}$ apart. Therefore, the solutions should possess good dispersive properties over a time interval of size roughly $\lambda^{-\frac{4}{3}}$. This now yields a Strichartz estimate over a time interval of unit size with a loss of $\frac{4}{3p}$ derivatives. Hence, for manifolds with boundary (1.4) appears to be the natural analog of the aforementioned estimates of [3] for the general boundaryless case.

Our proof of Theorem 1.1 follows the above heuristics. In section 2 the solution is localized spatially and a coordinate chart is used to work on \mathbb{R}^n ; a Littlewood-Paley decomposition then reduce matters to establishing Strichartz estimates for

components of the solutions dyadically localized in frequency. As alluded to above, we then seek to prove Strichartz estimates with no loss of derivatives over time intervals of size $\lambda^{-\frac{4}{3}}$ for components of the solution localized at frequency λ . This involves regularizing the metric by truncating its frequency to a scale dependent on λ . Rescaling the solution then reduces the problem to establishing Strichartz estimates for metrics with 2 bounded derivatives over small time intervals whose size also depends on the frequency. Section 3 uses a phase space transform to construct a parametrix for such Schrödinger operators, and section 4 concludes the paper by showing that the parametrix yields the desired estimates.

Notation. In what follows d will denote the gradient operator which maps scalar functions to vector fields and vector fields to matrix functions in the natural way. The expression $X \lesssim Y$ means that $X \leq CY$ for some C depending only on n and on the Lipschitz norm of the metric.

2. REDUCTIONS

We will establish Theorem 1.1 more generally for operators on M which take the following form in local coordinates

$$(2.1) \quad (Pf)(x) = \rho(x)^{-1} \sum_{i,j=1}^n \partial_i \left(\rho(x) g^{ij}(x) \partial_j f(x) \right).$$

Such an operator is self-adjoint in the measure $d\mu = \rho(x) dx$. Neumann conditions and the boundary normal are defined with respect to the metric g_{ij} .

We start by reducing the case of a manifold M with boundary and P smooth, to the case of a compact manifold M without boundary, with P having coefficients of Lipschitz regularity. For this, let \tilde{M} denote the double of M , identified along ∂M . We define a differentiable structure on \tilde{M} near ∂M using geodesic normal coordinates in g_{ij} , so $x_n > 0$ and $x_n < 0$ define the two copies of M . In these coordinates, $g^{ni} = 0$ for $i \neq n$, hence P contains no cross terms between ∂_n and ∂_i . The operator \tilde{P} with coefficients $g^{ij}(x', |x_n|)$ and $\rho(x', |x_n|)$ is thus symmetric under $x_n \rightarrow -x_n$, and extends the lift of P to \tilde{M} across ∂M to one with Lipschitz coefficients. Eigenspaces for \tilde{P} decompose into symmetric and antisymmetric functions; these correspond to extensions of eigenfunctions for P satisfying Dirichlet (resp. Neumann) conditions, and each eigenfunction is of regularity $C^{1,1}$ across the boundary. The Schrödinger flow for \tilde{P} is thus easily seen to extend that for P , and Strichartz estimates for P follow by establishing such estimates for \tilde{P} on \tilde{M} .

We assume henceforth that M is a compact manifold with smooth differentiable structure, on which an operator P of the form (2.1) is given, with coefficients of Lipschitz regularity. Define L^q -Sobolev spaces on M using the spectral resolution of P ,

$$\|f\|_{W^{s,q}(M)} = \|\langle D_P \rangle^s f\|_{L^q(M)}, \quad \langle D_P \rangle = (1 - P)^{\frac{1}{2}}.$$

When $q = 2$ we denote $W^{s,q}$ by H^s . By elliptic regularity (e.g. [4, Theorem 8.10] for $q = 2$, and [4, Theorem 9.11] or [14, §2.2] for other q) the spaces $W^{s,q}$ for $1 < q < \infty$ coincide with the Sobolev spaces defined using local coordinates, provided $0 \leq s \leq 2$.

Suppose that $u(t, x) = (e^{itP}f)(x)$. Then we need to establish

$$\|u\|_{L^p([-T, T]; W^{s, q}(M))} \lesssim \|f\|_{H^1(M)}, \quad s = 1 - \frac{4}{3p}.$$

Let $u = \sum_{k=0}^{\infty} u_k$ denote a Littlewood-Paley partition of u , where u_k for $k \geq 1$ is spectrally localized to $\langle D_P \rangle \approx 2^k$. Then, for $p, q \geq 2$,

$$\|\langle D_P \rangle^s u\|_{L_t^p L_x^q} \approx \|(\langle D_P \rangle^s u)_k\|_{L_t^p L_x^q \ell_k^2} \leq \|\langle D_P \rangle^s u_k\|_{\ell_k^2 L_t^p L_x^q},$$

and

$$\|f\|_{H^1(M)} \approx \|f_k\|_{\ell_k^2 H^1(M)},$$

hence it suffices to show, uniformly over k , that

$$\|u_k\|_{L^p([-T, T]; W^{s, q}(M))} \lesssim \|f_k\|_{H^1(M)}, \quad s = 1 - \frac{4}{3p}.$$

By taking a finite partition of unity, it suffices to prove that

$$(2.2) \quad \|\psi u_k\|_{L^p([-T, T]; W^{s, q}(\mathbb{R}^n))} \lesssim \|u_k\|_{L^\infty([-T, T]; H^1(M))}$$

for each smooth cutoff ψ supported in a suitably chosen coordinate chart. We will choose coordinate charts such that the image contains the unit ball, and

$$\|g^{ij} - \delta_{ij}\|_{Lip(B_1(0))} \leq c_0, \quad \|\rho - 1\|_{Lip(B_1(0))} \leq c_0,$$

for c_0 to be taken suitably small. (This may require multiplying ρ by a harmless constant). We take ψ supported in the unit ball, and assume g^{ij} and ρ are extended so that the above holds globally on \mathbb{R}^n .

Let $\{\beta_j(D)\}_{j \geq 0}$ be a Littlewood-Paley partition of unity on \mathbb{R}^n , and $v_j = \beta_j(D)\psi u_k$. We will prove that, for each j ,

$$(2.3) \quad \|\langle D \rangle^s v_j\|_{L_t^p L_x^q} \lesssim \|v_j\|_{L_t^\infty H_x^1} + 2^{-\frac{j}{3}} \|(D_t + P)v_j\|_{L_t^\infty L_x^2},$$

with all norms taken over $[-T, T] \times \mathbb{R}^n$, and $\langle D \rangle = (1 - \Delta)^{\frac{1}{2}}$.

This will imply (2.2), provided we dominate the sum over j of the right hand side by $\|u_k\|_{L^\infty([-T, T]; H^1(M))}$.

For a Lipschitz function a , $[\beta_j(D), a] : H^{s-1} \rightarrow H^s$ for $s = 0, 1$. Hence $[P, \beta_j(D)\psi] : H^1 \rightarrow L^2$, and it follows that

$$\|(D_t + P)v_j\|_{L_t^\infty L_x^2} \lesssim \|u_k\|_{L_t^\infty H^1(M)},$$

hence the second term is bounded by a geometric series. For the first term, note that

$$\begin{aligned} \|v_j\|_{L_t^\infty H_x^1} &\lesssim \min(2^j \|v_j\|_{L_t^\infty L_x^2}, 2^{-j} \|v_j\|_{L_t^\infty H_x^2}) \\ &\lesssim \min(2^j \|u_k\|_{L_t^\infty L^2(M)}, 2^{-j} \|u_k\|_{L_t^\infty H^2(M)}). \end{aligned}$$

The sum over j is dominated by

$$2^k \|u_k\|_{L_t^\infty L^2(M)} + 2^{-k} \|u_k\|_{L_t^\infty H^2(M)} \lesssim \|u_k\|_{L_t^\infty H^1(M)}.$$

Setting $\lambda = 2^j$, and denoting v_j by v_λ , the estimate (2.3) is equivalent to

$$\begin{aligned} \|v_\lambda\|_{L^p([-T, T]; L^q(\mathbb{R}^n))} \\ \lesssim \lambda^{\frac{4}{3p}} \|v_\lambda\|_{L^\infty([-T, T]; L^2(\mathbb{R}^n))} + \lambda^{\frac{4}{3p} - \frac{4}{3}} \|(D_t + P)v_\lambda\|_{L^\infty([-T, T]; L^2(\mathbb{R}^n))}. \end{aligned}$$

This, in turn, follows by showing that for any interval I_λ of length $\lambda^{-\frac{4}{3}}$, we have

$$(2.4) \quad \|v_\lambda\|_{L^p(I_\lambda; L^q(\mathbb{R}^n))} \lesssim \|v_\lambda\|_{L^\infty(I_\lambda; L^2(\mathbb{R}^n))} + \|(D_t + P)v_\lambda\|_{L^1(I_\lambda; L^2(\mathbb{R}^n))}.$$

We now regularize the coefficients of P by setting

$$g_\lambda^{ij} = S_{\lambda^{-2/3}}(g^{ij}), \quad \rho_\lambda = S_{\lambda^{-2/3}}(\rho),$$

where $S_{\lambda^{2/3}}$ denotes a truncation of a function to frequencies less than $\lambda^{\frac{2}{3}}$, and let P_λ denote the operator with coefficients g_λ^{ij} and ρ_λ . Since $|g_\lambda^{ij} - g^{ij}| \lesssim \lambda^{-\frac{2}{3}}$, and similarly for ρ , it follows that

$$\|(P - P_\lambda)v_\lambda\|_{L^1(I_\lambda; L^2(\mathbb{R}^n))} \lesssim \|v_\lambda\|_{L^\infty(I_\lambda; L^2(\mathbb{R}^n))},$$

and we may thus replace P by P_λ on the right hand side of (2.4) without changing the estimate.

Finally, we rescale the problem. Let $\mu = \lambda^{\frac{2}{3}}$, and define

$$u_\mu(t, x) = v_\lambda(\lambda^{-\frac{2}{3}}t, \lambda^{-\frac{1}{3}}x)$$

$$Q_\mu(x, D) = P_\lambda(\lambda^{-\frac{1}{3}}x, D).$$

The function $u_\mu(t, \cdot)$ is localized to frequencies of size μ , and the coefficients of Q_μ are localized to frequencies of size less than $\mu^{\frac{1}{2}}$. This implies the following estimates on the coefficients of Q_μ

$$|\partial_x^\alpha g_\lambda^{ij}(\lambda^{-\frac{1}{3}}x)| + |\partial_x^\alpha \rho_\lambda(\lambda^{-\frac{1}{3}}x)| \leq C_\alpha \mu^{\frac{1}{2} \max(0, |\alpha| - 2)}.$$

The interval I_λ scales to an interval of length μ^{-1} . We have thus reduced the proof of Theorem 1.1 to the following.

Theorem 2.1. *Suppose that $u_\mu(t, x)$ is localized to frequencies $|\xi| \in [\frac{1}{4}\mu, 4\mu]$ and solves*

$$(2.5) \quad \left(D_t + \sum_{1 \leq i, j \leq n} a_\mu^{ij}(x) \partial_{x_i} \partial_{x_j} + \sum_{1 \leq i \leq n} b_\mu^i(x) \partial_{x_i} \right) u_\mu = F_\mu$$

Assume also that the metric satisfies

$$\|a_\mu^{ij} - \delta_{ij}\|_{C^2} \ll 1, \quad \|b_\mu^i\|_{C^1} \lesssim 1,$$

$$\text{supp}(\widehat{a_\mu^{ij}}), \text{supp}(\widehat{b_\mu^i}) \subset B_{\mu^{1/2}}(0).$$

Then the following estimate holds

$$\|u_\mu\|_{L^p([0, \mu^{-1}]; L^q(\mathbb{R}^n))} \lesssim \|u_\mu\|_{L^\infty([0, \mu^{-1}]; L^2(\mathbb{R}^n))} + \|F_\mu\|_{L^1([0, \mu^{-1}]; L^2(\mathbb{R}^n))}.$$

3. THE PARAMETRIX

We will establish Theorem 2.1 using a short-time wave packet parametrix for the equation (2.5). Wave packet parametrices have been used to establish Strichartz estimates for Schrödinger equations in the work of Staffilani-Tataru [10] and Koch-Tataru [7]; see Tataru [13] in for an overview of the methods. The result we need, in fact, is included as a special case in Theorem 2.5 of [7]. The proof of the short time estimate Theorem 2.1 is comparatively simple, though, and therefore we include a self-contained proof here for the reader's benefit.

In this section, then, we use a wave packet transform to construct a parametrix for the operator

$$D_t + A(x, D) + B(x, D) = D_t + \sum_{1 \leq i, j \leq n} a_\mu^{ij}(x) \partial_{x_i} \partial_{x_j} + \sum_{1 \leq i \leq n} b_\mu^i(x) \partial_{x_i}$$

in (2.5) that will yield the Strichartz estimates. For convenience, we suppress the μ from both the operator and the solution. Let g be a radial Schwartz function over \mathbb{R}^n such that $\text{supp}(\widehat{g}) \subset B_1(0)$ and $\|g\|_{L^2} = (2\pi)^{-\frac{n}{2}}$. For $\mu \geq 1$, we define the operator $T_\mu : \mathcal{S}'(\mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^{2n})$ by

$$T_\mu f(x, \xi) = \mu^{\frac{n}{4}} \int e^{-i(\xi, z-x)} g(\mu^{\frac{1}{2}}(z-x)) f(z) dz.$$

T_μ enjoys the property that its adjoint as a map from $L_{x,\xi}^2(\mathbb{R}^{2n}) \rightarrow L_z^2(\mathbb{R}^n)$ also serves as a left inverse for T_μ , that is, $T_\mu^* T_\mu = I$. This implies that T_μ is an isometry

$$\|T_\mu f\|_{L_{x,\xi}^2(\mathbb{R}^{2n})} = \|f\|_{L_z^2(\mathbb{R}^n)}.$$

We conjugate $A(x, D)$ by T_μ and take a suitable approximation to the resulting operator. Specifically, define the following differential operator over (x, ξ)

$$\widetilde{A} = -id_\xi a(x, \xi) \cdot d_x + id_x a(x, \xi) \cdot d_\xi + a(x, \xi) - \xi \cdot d_\xi a(x, \xi).$$

By a standard argument from wave packet methods (see for example [12] or [13] where g is Gaussian, or Lemmas 3.1-3.3 in [8] for g as above) we have that if $\widetilde{\beta}_\mu$ is a Littlewood-Paley cutoff truncating to frequencies $|\xi| \approx \mu$ then

$$(3.1) \quad \|T_\mu A(\cdot, D) \widetilde{\beta}_\mu(D) - \widetilde{A} T_\mu \widetilde{\beta}_\mu(D)\|_{L_z^2 \rightarrow L_{x,\xi}^2} \lesssim \mu$$

This yields that, if $\tilde{u}(t, x, \xi) = (T_\mu u_\mu(t, \cdot))(x, \xi)$, then \tilde{u} solves the equation $(\partial_t + d_\xi a(x, \xi) \cdot d_x - d_x a(x, \xi) \cdot d_\xi + ia(x, \xi) - i\xi \cdot d_\xi a(x, \xi)) \tilde{u}(t, x, \xi) = \widetilde{G}(t, x, \xi)$, where \widetilde{G} satisfies

$$\int_0^{\mu^{-1}} \|\widetilde{G}(t, x, \xi)\|_{L_{x,\xi}^2} dt \lesssim \|u_\mu\|_{L^\infty([0, \mu^{-1}]; L^2)} + \|F_\mu\|_{L^1([0, \mu^{-1}]; L^2)}.$$

Given an integral curve $\gamma(r) \in \mathbb{R}_{x,\xi}^{2n}$ of the vector field

$$\partial_t + d_\xi a(x, \xi) \cdot d_x - d_x a(x, \xi) \cdot d_\xi$$

with $\gamma(t) = (x, \xi)$, we denote $\chi_{s,t}(x, \xi) = (x_{s,t}, \xi_{s,t}) = \gamma(s)$. Now define

$$\sigma(x, \xi) = a(x, \xi) - \xi \cdot d_\xi a(x, \xi), \quad \psi(t, x, \xi) = \int_0^t \sigma(\chi_{r,t}(x, \xi)) dr.$$

This allows us to write

$$\tilde{u}(t, x, \xi) = e^{-i\psi(t,x,\xi)} u_0(\chi_{0,t}(x, \xi)) + \int_0^t e^{-i\psi(t-r,x,\xi)} \widetilde{G}(r, \chi_{r,t}(x, \xi)) dr,$$

which expresses \tilde{u} as an integrable superposition over r of functions invariant under the flow of \widetilde{A} , truncated to $t > r$.

Since $u(t, x) = T_\mu^* \tilde{u}(t, x, \xi)$ it thus suffices to obtain estimates

$$\|\widetilde{\beta}_\mu(D) W_t f\|_{L_t^p L_x^q} \lesssim \|f\|_{L_{x,\xi}^2}$$

where W_t acts on functions $f(x, \xi)$ by the formula

$$(W_t f)(y) = T_\mu^* \left(e^{-i\psi(t, x, \xi)} f(\chi_{0, t}(\cdot)) \right)(y).$$

By a standard duality argument and an application of the endpoint estimates of Keel-Tao [6] this results from establishing

$$(3.2) \quad \|\tilde{\beta}_\mu W_t W_s^* \tilde{\beta}_\mu\|_{L^1 \rightarrow L^\infty} \lesssim |t - s|^{-\frac{n}{2}}$$

$$(3.3) \quad \|\tilde{\beta}_\mu W_t W_s^* \tilde{\beta}_\mu\|_{L^2 \rightarrow L^2} \lesssim 1$$

The inequality (3.3) follows from the fact that T_μ is an isometry and $\chi_{0, t}(x, \xi)$ is a symplectomorphism, hence preserves the measure $dx d\xi$. The inequality (3.2) is the focus of the next section.

4. THE DISPERSIVE ESTIMATE

In this section, we establish the inequality (3.2). We write the kernel $K(t, y, s, x)$ of $W_t W_s^*$ as

$$\mu^{\frac{n}{2}} \int e^{-i\langle \zeta, x-z \rangle - i \int_s^t \sigma(\chi_{r, t}(z, \zeta)) dr + i\langle \zeta_{t, s}, y - z_{t, s} \rangle} g(\mu^{\frac{1}{2}}(y - z_{t, s})) g(\mu^{\frac{1}{2}}(x - z)) dz d\zeta.$$

Recall that $\text{supp}(\hat{g}) \subset B_1(0)$. Since we are concerned with $W_t W_s^* \tilde{\beta}_\mu$, we can insert a cutoff $S_\mu(\zeta)$ into the integrand which is supported in a set $|\zeta| \approx \mu$. Note that the Hamiltonian vector field is independent of time, and hence $\chi_{t, s} = \chi_{t-s, 0}$. We drop the zero and abbreviate the latter transformation as $\chi_{t-s}(z, \zeta) = (z_{t-s}, \zeta_{t-s})$. It then suffices to consider $s = 0$, and we write the kernel $K(t, x, 0, y)$ as

$$\mu^{\frac{n}{2}} \int e^{-i\langle \zeta, x-z \rangle - i\psi(t, z, \zeta) + i\langle \zeta_t, y - z_t \rangle} g(\mu^{\frac{1}{2}}(y - z_t)) g(\mu^{\frac{1}{2}}(x - z)) S_\mu(\zeta) dz d\zeta.$$

We need to establish uniform bounds over x and y , $|K(t, x, 0, y)| \lesssim t^{-\frac{n}{2}}$. A straightforward estimate shows that

$$|K(t, x, 0, y)| \lesssim \mu^n$$

meaning that the dispersive estimate holds for $t \leq \mu^{-2}$. We thus assume $t \geq \mu^{-2}$ for the remainder of the section. Lastly we suppose that, in addition, $t \leq \varepsilon \mu^{-1}$ with ε chosen sufficiently small and independent of μ .

We first need derivative estimates on the transformation $\chi_t(z, \zeta)$.

Lemma 4.1. *Consider the solutions $(z_t(z, \zeta), \zeta_t(z, \zeta))$ to Hamilton's equations*

$$(4.1) \quad \partial_t z_t = d_\xi a_\mu(z, \zeta), \quad \partial_t \zeta_t = -d_x a_\mu(z, \zeta), \quad (z_0, \zeta_0) = (z, \zeta).$$

We then have the following estimates on the first partial derivatives of (z_t, ζ_t) when $|\zeta| \in [\frac{1}{4}\mu, 4\mu]$ and $|t| \leq \mu^{-1}$

$$(4.2) \quad \begin{aligned} |d_z z_t - I| &\lesssim \mu t & |d_\zeta z_t| &\lesssim t \\ |d_z \zeta_t| &\lesssim \mu^2 t & |d_\zeta \zeta_t - I| &\lesssim \mu t \end{aligned}$$

$$(4.3) \quad \left| d_\zeta z_t - \int_0^t (d_\zeta^2 a_\mu)(\chi_s(z, \zeta)) ds \right| \lesssim \mu t^2$$

The higher partial derivatives satisfy, for $j + k \geq 2$,

$$(4.4) \quad \mu |d_z^j d_\zeta^k z_t| + |d_z^j d_\zeta^k \zeta_t| \lesssim \mu^{2-k} t \langle \mu^{\frac{3}{2}} t \rangle^{j+k-1}.$$

Proof. If $|\zeta| \approx 1$, then we can write the Hamilton equations as:

$$(z_t, \zeta_t) = (z, \zeta) + \int_0^t v(z_s, \zeta_s) ds,$$

where the vector field v satisfies

$$|d_{z,\zeta}^k v| \lesssim \mu^{\frac{1}{2}(k-1)}, \quad k \geq 1.$$

Differentiating the equation and using induction yields the bound,

$$|d_{z,\zeta}^k(z_t, \zeta_t) - d_{z,\zeta}^k(z, \zeta)| \lesssim t \langle \mu^{\frac{1}{2}} t \rangle^{k-1}, \quad |t| \leq 1.$$

Estimates (4.2) and (4.4) now follow by the rescaling property

$$(z_t(z, \zeta), \zeta_t(z, \zeta)) = (z_{\mu t}(z, \mu^{-1}\zeta), \mu \zeta_{\mu t}(z, \mu^{-1}\zeta)).$$

Estimate (4.3) follows by differentiating Hamilton's equations as above and applying the bounds (4.2). \square

We take a partition of unity $\{\phi_m\}_{m \in \mathbb{Z}^n}$ over \mathbb{R}^n with $\phi_m(\zeta) = \phi(t^{\frac{1}{2}}(\zeta - t^{-\frac{1}{2}}m))$ for some ϕ smooth and compactly supported. We then write

$$K(t, y, 0, x) = \sum_{m \in \mathbb{Z}^n} K_m(t, y, x)$$

where $K_m(t, y, x)$ is defined by

$$\mu^{\frac{n}{2}} \int e^{-i\langle \zeta, x-z \rangle - i\psi(t, z, \zeta) + i\langle \zeta_t, y-z_t \rangle} g(\mu^{\frac{1}{2}}(y - z_t)) g(\mu^{\frac{1}{2}}(x - z)) \phi_m(\zeta) S_\mu(\zeta) dz d\zeta$$

The key estimate is that, for $\xi_m = t^{-\frac{1}{2}}m$,

$$(4.5) \quad |K_m(t, x, y)| \lesssim t^{-\frac{n}{2}} (1 + t^{-\frac{1}{2}}|y - x_t(x, \xi_m)|)^{-N}.$$

Estimate (4.3), and the fact that

$$\|d_\xi^2 a(x, \xi) - 2I\| = 2\|a^{ij} - \delta^{ij}\| \ll 1$$

yields for $l, m \in \mathbb{Z}^n$ and $t \leq \varepsilon \mu^{-1}$

$$|x_t(x, \xi_m) - x_t(x, \xi_l)| \approx t |\xi_m - \xi_l| = t^{\frac{1}{2}}|m - l|.$$

This now yields

$$\sum_{m \in \mathbb{Z}^n} |K_m(t, x, y)| \lesssim t^{-\frac{n}{2}} \sum_{m \in \mathbb{Z}^n} (1 + |m|)^{-N}.$$

Since the sum on the right converges for N large this establishes the dispersive estimate.

To prove (4.5), we start by noting that

$$\partial_{\zeta_i} \left(\int_0^t a(z_r, \zeta_r) - \zeta_r \cdot (d_\zeta a)(z_r, \zeta_r) dr \right) + \zeta_t \cdot \partial_{\zeta_i} z_t = 0.$$

The expression vanishes at $t = 0$ since $d_\zeta z_0 = 0$, and Hamilton's equations show that the derivative of the expression with respect to t vanishes.

As in Theorem 5.4 of Smith-Sogge [9], we now proceed by defining the differential operator

$$L = \frac{1 + it^{-1}(x - z - d_\zeta \zeta_t \cdot (y - z_t)) \cdot d_\zeta}{1 + t^{-1}|x - z - d_\zeta \zeta_t \cdot (y - z_t)|^2}.$$

By the observation above, L preserves the phase function in the definition of K_m . The estimates (4.2) and (4.4) show that, if p is any one of the functions $\phi_m(\zeta)$, $t^{-\frac{1}{2}}z_t$, $\mu^{\frac{1}{2}}z_t$, $S_\mu(\zeta)$, $\mu^{-\frac{1}{2}}t^{-\frac{1}{2}}\zeta_t$, then for $\mu^{-2} \leq t \leq \mu^{-1}$,

$$|(t^{-\frac{1}{2}}\partial_\zeta)^k p| \lesssim 1.$$

Integration by parts now yields the following upper bound on $K_m(t, x, y)$

$$\begin{aligned} \mu^{\frac{n}{2}} \int_{\mathbb{R}^n \times \text{supp}(\phi_m)} (1 + t^{-1}|x - z - d_\zeta \zeta_t \cdot (y - z_t)|^2)^{-N} \\ \times (1 + \mu^{\frac{1}{2}}|x - z|)^{-N} (1 + \mu^{\frac{1}{2}}|y - z_t|)^N dz d\zeta \end{aligned}$$

We conclude by showing that

$$(4.6) \quad t^{-\frac{1}{2}}|(x - z) - d_\zeta \zeta_t \cdot (x_t - z_t)| \lesssim 1 + \mu|x - z|^2$$

where x_t denotes $x_t(x, \xi_m)$. This implies that the integrand is dominated by

$$(1 + t^{-1}|d_\zeta \zeta_t \cdot (y - x_t)|^2)^{-N} (1 + \mu^{\frac{1}{2}}|x - z|)^{-N}.$$

Since $|d_\zeta \zeta_t - I| \lesssim \varepsilon$, this establishes the estimate (4.5), since the z decay and compact ζ support imply that the integral is essentially over a region in phase space of volume roughly $t^{-\frac{n}{2}}\mu^{-\frac{n}{2}}$.

To establish (4.6), we employ a Taylor expansion and (4.4) to obtain

$$\begin{aligned} t^{-\frac{1}{2}}|x_t - z_t - (d_z z_t)(x - z) - (d_\zeta z_t)(\xi_m - \zeta)| \\ \lesssim t^{\frac{1}{2}}\langle \mu^{\frac{3}{2}}t \rangle (\mu|x - z|^2 + |x - z||\xi_m - \zeta| + \mu^{-1}|\xi_m - \zeta|^2) \lesssim 1 + \mu|x - z|^2 \end{aligned}$$

where the last inequality uses the fact that $\mu^{-2} \leq t \leq \mu^{-1}$ and $|\xi_m - \zeta| \lesssim t^{-\frac{1}{2}}$. In addition, by (4.2)

$$t^{-\frac{1}{2}}|(d_\zeta z_t)(\xi_m - \zeta)| \lesssim t(t^{-\frac{1}{2}})^2 = 1.$$

Since $\chi_t(z, \zeta)$ is a symplectomorphism, we have

$$\partial_{\zeta_i} \zeta_t \cdot \partial_{z_j} z_t - \partial_{\zeta_i} z_t \cdot \partial_{z_j} \zeta_t = \delta_{ij}$$

where \cdot pairs the z_t and ζ_t indices. Lastly, by (4.2),

$$t^{-\frac{1}{2}}|d_\zeta z_t| |d_z \zeta_t| |x - z| \lesssim \mu^2 t^{\frac{3}{2}} |x - z| \leq \mu^{\frac{1}{2}} |x - z|.$$

These facts now combine to yield the estimate (4.6). \square

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