## Research Statement

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### 1. Introduction

My research seeks to provide a rigorous mathematical foundation for understanding the equations which model wave phenomena, such as those which arise in acoustics and optics. Much of my work from 2015 to the present has dealt with investigating the mass concentration, and more generally phase space concentration, properties of so-called *standing waves* or *vibrational modes*. One problem of particular interest is to establish upper bounds on the $L^p$ norms of these modes in the high frequency limit. These norms are sensitive to the phase space concentration properties of these modes, a theme which is of considerable interest in disciplines such as quantum chaos and semiclassical physics.

One historical reference point for this research lies in the experiments of Ernst Chladni (1756-1827). He would dust a metal plate with sand, then vibrate the plate with a violin bow, causing the plate to resonate at a fundamental frequency, much like a guitar or violin string does at certain harmonics. This is an early example of a standing wave/vibrational mode, exhibiting periodic motion in time, much like a vibrating string. It was observed that the sand would accumulate along certain lines, creating intricate patterns which we now call *Chladni figures*. The accumulation of sand along these lines indicated that along the 2 dimensional plate, there were 1 dimensional arcs where the plate did not vibrate, tracing out what are often called the *nodal lines* or *nodal sets*. It is not hard to find videos on the internet which display this phenomena where dusted plates of different sizes and shapes vibrate at certain frequencies which are fundamental to the plate.

Figure 1 shows the Chladni figures that result from vibrating a square plate at different fundamental frequencies. The frequency of vibration increases moving left to right in the table

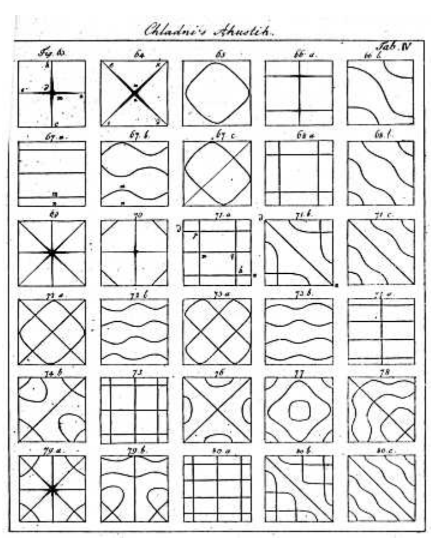


Figure 1. Chladni figures on a square plate, `commons.wikimedia.org/wiki/File:Clafig1.jpg`. Taken from the book *Die Akustik* by Ernst Chladni

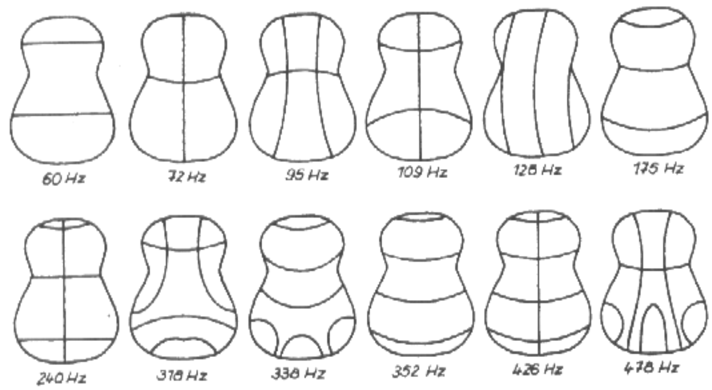


FIGURE 2. Chladni figures on a guitar shaped plate, uploaded to Wikipedia by Denis Diderot `commons.wikimedia.org/w/index.php?curid=8987701`

of figures. Examining the patterns here already leads to a number of interesting mathematical questions; we mention just a few here. For one, the total lengths of the nodal lines appear to increase as the frequency increases, is this indeed the case? If so, what is the rate at which this total length increases? Second, these nodal lines confine a number of connected components often called *nodal domains*. Does the quantity of these domains increase with the frequency, and if so, can anything be said about the rate at which they increase? Third, to what extent are these patterns predictable? Is it possible that they follow some sort of self-similar pattern? Finally, how does the shape of the plate determine the nodal lines that result? For example, replacing the square plate in Figure 1 by the guitar shaped plate in Figure 2 changes the structure of the nodal lines and the figures they form. Such questions have fascinated physicists and mathematicians for centuries, and in spite of the progress that has been made on such problems, there is a great deal that remains mysterious.

The equation which models these vibrational modes is an example of a time independent partial differential equation, namely the Helmholtz equation, which we discuss further in §2. The Helmholtz equation has a close relative in the wave equation, a time dependent equation which is ubiquitous in describing various types of waves and vibrations. The relation lies in the consideration of solutions to the wave equation which are periodic in time. The Helmholtz equation is also closely related to the time-independent Schrödinger equation in quantum mechanics where one is interested in the same type of reduction to stationary waves, typically introducing a potential to the equation in the process. The solutions to the Helmholtz equation will be functions which describe the displacement of each point on the plate relative to a non-vibrating plate. Hence the nodal sets discussed above are the zero sets of the solutions. It is also noted that the Helmholtz equation can be posed in a variety of geometries and dimensions, which together with possible boundary conditions will model a host of standing wave phenomena.

All this said, my work has centered more around estimating the "peaks" that develop outside of the nodal set rather than the nodal set itself (though there are works showing these two ideas are related e.g. [SZ11], [CM11]). As illustrated in Figure 3, the amplitude of these vibrational modes will peak in certain areas. In other words, the solutions to the Helmholtz equation achieve maxima and minima and one can seek to estimate how large these extrema are. This

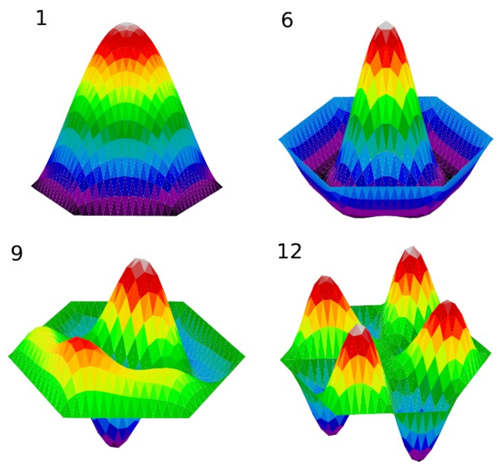


FIGURE 3. Amplitudes of standing waves, figure from `www.faculty.umassd.edu/j.wang/vp/`

in turn raises a similar set of questions to those posed above: do the amplitudes increase as the frequency increases? If so, what is the rate at which it increases? Moreover, how do the peaks formed depend on the shape of the plate?

The most straightforward way to measure the strength of these amplitudes is to consider the maxima and minima of the solutions to the Helmholtz equation, or similarly the maxima of the absolute value of the solutions. However, another means is to consider how the solutions behave in $L^p$ *norms*. These norms involve taking an integral of the $p$-th power of the solution. A thorough study of solutions to the Helmholtz equation and their significance often cannot ignore the case $p = 2$, and for various reasons, one often normalizes these solutions so that their square integral is 1 (i.e. unit norm in $L^2$), and we will assume this is the case throughout the discussion. Thus when $p > 2$, an $L^p$ norm will be sensitive to the extreme values of the solution. In contrast to simply estimating the extrema, these norms are also sensitive to the structure of the region where the extrema are obtained or nearly obtained. Consequently, understanding $L^p$ norms involves understanding where the mass of a solution is concentrated.

This latter theme is closely related to investigations emanating from the analysis of the (time-independent) Schrödinger equation concerning *scarring* phenomenon, which we loosely describe as the tendency of a solution to concentrate its mass along a lower dimensional set. Heuristics from the quantum-classical correspondence suggest that the mass of the stationary wave should in some sense be invariant under the classical dynamics associated to the system in the high frequency limit under examination here. Consequently this sort of scarring may occur when the classical dynamics associated to the system possesses a closed periodic orbit or other invariant manifold. However, understanding when this scarring does or does not occur, and the profiles that are associated to it is a very subtle question. Figure 4 shows a temperature map of a few solutions to the Helmholtz equation on a lemon shaped plate so that the darker regions indicate a larger amplitude. Next to each temperature map, periodic orbits of the underlying classical billiard dynamics are shown in red, accentuating that the amplitude is the largest along these orbits. By classical dynamics, in this setting and in general, we mean the paths of least action

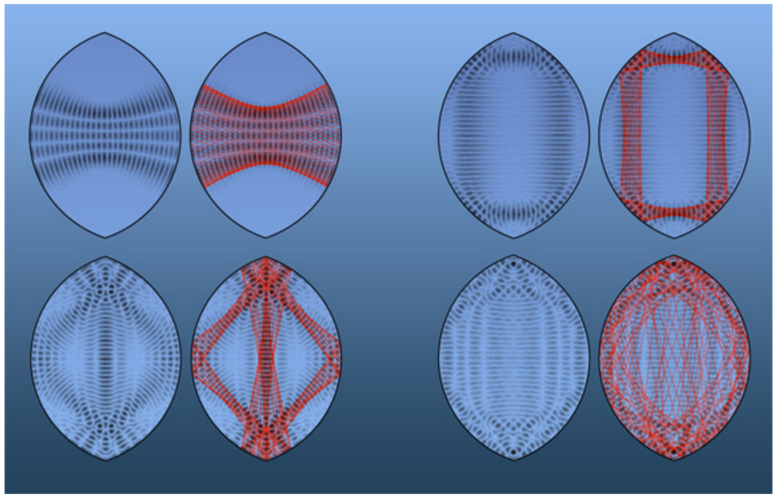

FIGURE 4. Scarring in a lemon shaped plate, figure from `http://www.tcm.phy.cam.ac.uk/~bds10/research.html`

for a classical particle on the plate. They can be considered as *generalized geodesics*, which reflect off the plate's walls according to Snell's law.

Studying $L^p$ norms is natural from the standpoint of classical Fourier and harmonic analysis. Indeed, some portion of the recent interest in understanding them stems from interest in analogous problems in Fourier analysis such as the Fourier restriction problem. Here one seeks to understand the mapping properties of restricting the Fourier transform to a sphere or other hypersurface in $L^p$ spaces, closely related to the idea of spectral restriction implicit in these problems. $L^p$ norms on solutions to the Helmholtz equation have been applied to establish more general families of regularity estimates for wave and Schrödinger equations such as the Strichartz estimates and their variants, using that eigenfunctions form the building blocks of solutions to these equations (see e.g. [BuGT04, BuGT05, BuLP08]). In addition, results of Sogge [Sog87, Sog02] use such bounds in establishing the convergence of Bochner-Riesz means associated to the orthonormal sequence of eigenfunctions.

1.1. **Strichartz inequalities.** While the bulk of my attention over the past three years or so has been focused on vibrational modes, the circle of problems described above has a close relative in my prior work on so called *Strichartz estimates* or *Strichartz inequalities*. Here time-dependent wave and Schrödinger equations are of interest rather than the time-independent case described above. In the 1970's Robert Strichartz established a family of regularity estimates for the solutions to the constant coefficient wave and free Schrödinger equations on Euclidean space in relation to the aforementioned Fourier restriction problem; such inequalities and their variants now bear his name. These estimates consider $L^p$ integrals of solutions to these equations in which the integration occurs both in the spatial and time variables. However, as observed above, $L^p$ norms are sensitive to waves with large amplitudes, so these inequalities reflect a *dispersive effect* of the evolution, which spreads out waves that may initially be very peaked in certain regions (though in the case of the classical wave equation, it is perhaps more precisely a *partial* dispersion). Strichartz estimates have proven to be very significant in the analysis of nonlinear versions of the wave and Schrödinger equations as they show that the dispersion of linearized approximations can limit or overwhelm instabilities that may result from the nonlinearity.

There is now a reasonably thorough theory for the Strichartz estimates for constant coefficient wave and free Schrödinger equations. However, when the coefficients are not constant, possibly describing equations in an anisotropic media, much less is known. The same is true if one includes a potential term in the equation or boundary conditions. In the setting of the wave equation, this can be motivated by appealing to Fermat's principle, which suggests waves propagate along paths of least action. For constant coefficient equations, these are just straight lines. However, the effect of changing the equation in one of the above fashions means that the paths of least action are now curves, or reflected rays, which not only complicates their analysis considerably but also may inhibit dispersion due to the formation of caustics and a refocusing of energy.

A substantial portion of my work prior to 2015 involved establishing Strichartz estimates for wave and Schrödinger boundary value problems, considering the validity of these estimates in this setting. Gaining an understanding of the Strichartz estimates here is significant for developing the theory of nonlinear versions of these equations. My earliest work considered closely related problems involving equations with coefficients of limited differentiability. These problem areas are of continuing interest to me, and I anticipate to pursue further investigations along these lines even in spite of a recent period of dormancy.

At first glance this line of work may appear to be much different from the problems on standing waves described above, and indeed there are some differences in the audiences interested in the impact of this work. However, the two research tracks have common threads in that they both look to understand how the classical dynamics (paths of least action) associated to a system influence how waves can concentrate and what this means for their $L^p$ norms. Indeed, §3 further emphasizes the interplay between the two lines of work.

**Organization of this statement.** The preceding introduction was meant to give a colloquial introduction to my research, at least to the extent possible. The remainder of this statement is intended to give a more formal description of my work. The next section begins in §2.1 with an introduction to the global aspects of the Helmholtz equation for boundaryless Riemannian manifolds and the $L^p$ bounds of interest. The remaining subsections §2.2–2.4 then summarize my contributions here, discussing my works in this problem area either individually or in groups. Similarly, the final section begins in §3.1 with an introduction to boundary value problems for wave, Schrödinger, and Helmholtz equations and the $L^p$ estimates of interest on their solutions. Closely related problems involving wave propagation in rough media are also introduced here. The subsections §3.2–3.7 then once again summarize my contributions in these problem areas.

While the results here represent significant progress in this body of work, it is also fair to say it is the tip of the iceberg in rather vast problem areas where there is much to be understood. Hence ongoing investigations and future directions are not emphasized here, though rest assured there is no dearth of inquiries to be pursued.

## 2. Global harmonic analysis of eigenfunctions on a Riemannian manifold

2.1. **Introduction.** Let $(M, g)$ be a compact $C^\infty$ Riemannian manifold of dimension $d \geq 2$. One of many constructions intrinsic to $(M, g)$ is the *Laplace-Beltrami operator* $\Delta_g$ (*Laplacian* for short), defined by first constructing the divergence and gradient operations to such a manifold, then composing the two. The Laplacian defines a second order elliptic operator on $M$ which is invariant under coordinate changes and is self-adjoint with respect to an intrinsic Riemannian measure denoted by $dV_g$. The Riemannian measure allows us to define $L^p$ norms for *continuous* functions $f : M \to \mathbb{C}$ for $1 \le p \le \infty$

$$\|f\|_{L^p} := \begin{cases} \left(\int_M |f(x)|^p \, dV_g\right)^{\frac{1}{p}}, & 1 \le p < \infty \\ \max_{x\in M} |f(x)|, & p = \infty. \end{cases}$$

When $1 \le p < \infty$, the $L^p$ spaces (or more precisely $L^p(M)$ spaces) can be defined in the usual fashion from measure theory or equivalently as the completion of the continuous functions on $M$ with respect to this norm. It is well known $L^2$ is then a Hilbert space with inner product given by $\langle f, g\rangle = \int_M f\overline{g}\, dV_g$.

The Helmholtz equation on $(M, g)$ reads as $(\Delta_g + \lambda^2)\varphi_\lambda = 0$, showing that its solutions are *eigenfunctions* of $\Delta_g$, with eigenvalue $-\lambda^2$, much like one defines eigenvectors of a linear transformation in linear algebra. The eigenvalues of $\Delta_g$ are labeled as $-\lambda^2$ in the interest of consistency with the wave formalism of the equation, so that $\lambda$ gives the frequency of vibration and is an eigenvalue of $\sqrt{-\Delta_g}$. When the boundary of $M$ is nonempty ($\partial M \neq \emptyset$), we consider the imposition of either homogeneous Dirichlet or Neumann boundary conditions:

$$(1) \qquad \varphi_{\lambda_j}(x)\big|_{x\in\partial M} = 0 \quad \text{(Dirichlet)}, \qquad \text{or} \qquad \frac{\partial\varphi_{\lambda_j}}{\partial\vec{n}}(x)\big|_{x\in\partial M} = 0 \quad \text{(Neumann)},$$

with $\partial/\partial\vec{n}$ denoting a unit vector field which is normal (orthogonal) to the boundary. After selecting one of the two boundary conditions if necessary, the compactness of $M$ means that there is a sequence $\lambda_j \nearrow \infty$, $\lambda_j \ge 0$, for which the Helmholtz equation admits a $C^\infty$ solution to $(\Delta_g + \lambda_j^2)\varphi_{\lambda_j} = 0$. These $\lambda_j$ can be thought of as the fundamental frequencies associated to the geometry, generalizing those for the Chladni plates and vibrating strings discussed above. Moreover, there exists a corresponding sequence of eigenfunctions $\{\varphi_{\lambda_j}\}_{j=1}^\infty$ which form an orthonormal basis for $L^2$. The dimension of the eigenspaces are always finite, but can be larger than 1, so we have implicitly counted multiplicities in the sequence $\{\lambda_j\}_{j=1}^\infty$. In what follows, we use $\varphi_\lambda$ to denote any $L^2$ normalized function satisfying the Helmholtz equation, namely

$$(\Delta_g + \lambda^2)\varphi_\lambda = 0, \qquad \|\varphi_\lambda\|_{L^2} = 1,$$

so that in particular, $\varphi_\lambda$ may be a linear combination of the $\varphi_{\lambda_j}$ within a fixed eigenspace.

When $(M, g)$ is a flat manifold, the Laplace-Beltrami operator and Riemannian measure agree with the usual Laplacian from calculus and Lebesgue measure respectively, meaning this class of examples with $\partial M \neq \emptyset$ is inclusive of the Chladni plates discussed above. However, in the near term, we assume the boundary of $M$ is empty, and the other case will be discussed in §3.

When $p \in (2, \infty]$, the largest $L^p$ norm of an $L^2$ normalized eigenfunction within an eigenspace may increase with $\lambda$. A natural upper bound to seek is one of the form

$$(2) \qquad \|\varphi_\lambda\|_{L^p(M)} \le C_p \lambda^{\delta(p)}, \qquad \lambda > 0$$

where $\delta(p), C_p > 0$ depend only on $p$ and $(M, g)$. Bounds of the form $\|\varphi_\lambda\|_{L^p(M)} = O(f(\lambda))$ or $o(f(\lambda))$ for some nondecreasing function $f : (0, \infty) \to (0, \infty)$ are also of interest. A seminal work on $L^p$ bounds is that of Sogge [Sog88], who showed the estimates (2) with

$$(3) \qquad \delta(p) = \max\left(\frac{d-1}{2}\left(\frac{1}{2} - \frac{1}{p}\right), \frac{d-1}{2} - \frac{d}{p}\right) = \begin{cases} \frac{d-1}{2}\left(\frac{1}{2} - \frac{1}{p}\right), & 2 < p \le \frac{2(d+1)}{d-1}, \\ \frac{d-1}{2} - \frac{d}{p}, & \frac{2(d+1)}{d-1} \le p \le \infty, \end{cases}$$

are satisfied on *any* $C^\infty$ compact Riemannian manifold without boundary of dimension $d \geq 2$. We therefore refer to these bounds as being *universal* throughout this class of manifolds.

These estimates are sharp for eigenfunctions on the canonical sphere with constant curvature 1 (denoted as ($\mathbb{S}^d$,can)) in that for each $p \in (2, \infty]$, there exist subsequences of eigenfunctions which show that the power of $\lambda_j$ in (2) given by (3) cannot be replaced by any smaller power (see [Sog86]). In particular, the zonal spherical harmonics are a family of eigenfunctions which assume values $\approx \lambda^{\frac{d-1}{2}}$ often enough in a ball $B(x_0, \lambda^{-1})$ and decay outside this ball, thus saturating the upper bounds with exponent (3) when $\frac{2(d+1)}{d-1} \leq p \leq \infty$ in that the subsequence satisfies $\|\varphi_{\lambda_{j_k}}\|_{L^p(M)} \geq c_p \lambda_{j_k}^{\delta(p)}$ for some $c_p > 0$. When $2 < p \leq \frac{2(d+1)}{d-1}$, the bounds (2) with exponent (3) are saturated by the highest weight spherical harmonics. They assume a value of $\lambda^{\frac{d-1}{4}}$ in a $\lambda^{-\frac{1}{2}}$ tubular neighborhood of a single closed geodesic (in this case a great circle) and are rapidly decreasing outside that neighborhood.

While the upper bounds with exponent (3) are sharp for the spherical harmonics on ($\mathbb{S}^d$,can), they are not (or are not expected to be) sharp for most other Riemannian geometries. As alluded to above, in the high frequency limit $\lambda \to \infty$ it is expected that the mass of eigenfunctions should be invariant under the geodesic flow (and indeed, there are rigorous theorems reflecting this). The geodesics on the round sphere are all $2\pi$-periodic, and are stable in that small perturbations in position or velocity yields geodesics which stay nearby for all times. This gives rise to certain types of concentration behavior such as eigenfunctions concentrating in small $\lambda^{-1/2}$ neighborhoods of a closed geodesic. Such behavior is in contrast to the flat torus, which possesses geodesics that are dense on $\mathbb{T}^d$ and negatively curved manifolds, whose geodesic flow is ergodic. Concrete evidence of the influence of geometry in the behavior of $L^p$ norms can be found in a result of Zygmund [Zyg74], showing that in stark contrast to the sphere, the $L^4$ norms of ($L^2$-normalized) eigenfunctions on $\mathbb{T}^2$ are uniformly bounded. Further improvements on the exponent in (3) for $\mathbb{T}^d$ with other values of $p, d$ can be found in [Bou93, Bou13, BD13, BD15]. As will be discussed below, other improvements hold on more general nonpositively curved manifolds. For example, the results in [IS95] give similar improvements on the exponent in (3) on special arithmetic hyperbolic quotients.

The preceding discussion highlights the one above that $L^p$ norms, for a fixed $p \in (2, \infty]$, provide a means of measuring the size and concentration of eigenfunctions. If the $L^p$ norm of an eigenfunction $\varphi_\lambda$ is in some sense large, it suggests that its mass is highly concentrated in small, $\lambda$-dependent regions, resulting in sizable amplitudes. On the other hand, smaller $L^p$ norms suggest that the mass is in some sense "spread out" throughout $M$. The pursuit of $L^p$ norms thus illuminates how the characteristics of the geodesic flow influence such phenomena.

One route to proving the bounds (2) with exponent (3) (as articulated in [Sog93]) is to consider the operator $\chi(\lambda - \sqrt{-\Delta_g})$ for a positive, Schwartz class function $\chi$ with $\widehat{\chi}(t) = 0$ if $|t| \notin (\epsilon/2, \epsilon)$ and $\chi(0) = 1$. Hence $\chi(\lambda - \sqrt{-\Delta_g})\varphi_\lambda = \varphi_\lambda$ and the operator can be realized as an integral

$$\chi(\lambda - \sqrt{-\Delta_g}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\lambda - it\sqrt{-\Delta_g}} \widehat{\chi}(t)\, dt. \tag{4}$$

Taking $\epsilon > 0$ small, the half wave propagator $e^{-it\sqrt{-\Delta_g}}$ is a local Fourier integral operator (FIO) and stationary phase then shows that up to negligible error, the integral kernel $\chi(\lambda - \sqrt{-\Delta_g})$ is

$$\lambda^{\frac{d-1}{2}} e^{i\lambda d_g(x,y)} a_\lambda(x, y), \qquad \operatorname{supp}(a_\lambda) \subset \{(x, y) : d_g(x, y) \in (\epsilon/2, \epsilon)\}. \tag{5}$$

Here $d_g(x,y)$ is the Riemannian distance function and $a_\lambda \in C^\infty$. This expression is perhaps not surprising since waves propagate at unit speed, the wave fronts from a source $y$ are singular along geodesic spheres $t = d_g(x,y)$, meaning that leading contribution from integration in time is determined by the distance function. The phase function in (5) constitutes a *Carleson-Sjölin phase*, as defined and discussed in [Sog93]. Oscillatory integral operators with such kernels have been studied in classical Fourier analysis as they arise in convergence of Bochner-Riesz means and Fourier restriction problems. The $L^2 \to L^p$ bounds on such operators were established in [CS72, Hör73, Ste86] and the results therein complete the proof of Sogge's result (2), (3).

This discussion underscores that Sogge's bounds (2) with exponent (3) fall under the domain of *local* harmonic analysis in that they follow from local properties of the wave group. However, such a local method neglects the *global* dynamics of the wave kernel and geodesic flow. Indeed, ideally $\chi(\lambda - \sqrt{-\Delta_g})$ would be replaced by different operator $\tilde{\chi}_\lambda(\sqrt{-\Delta_g})$, where $\mathrm{supp}(\tilde{\chi}_\lambda)$ is narrow enough so that it only intersects $\mathrm{spec}(\sqrt{-\Delta_g})$ at $\lambda$. However, this would mean that the Fourier transform of $\tilde{\chi}_\lambda$ is no longer compactly supported, meaning one would have to understand the structure of the half-wave evolution for all $t \in \mathbb{R}$. For most manifolds, this remains elusive. As it stands, even though $\chi(\lambda - \sqrt{-\Delta_g})\varphi_\lambda = \varphi_\lambda$, the operator serves as an approximation of sorts to $\mathbf{1}_{[\lambda,\lambda+1]}(\sqrt{-\Delta_g})$, the operator projecting onto eigenspaces of $\sqrt{-\Delta_g}$ with $\lambda_j \in [\lambda, \lambda+1]$. The range of both of these operators are typically considerably larger than a projection onto an eigenspace, containing a range of eigenspaces. At best, obtaining $L^2 \to L^p$ bounds in this fashion will deliver the sharp bounds on the operators $\chi(\lambda - \sqrt{-\Delta_g})$, $\mathbf{1}_{[\lambda,\lambda+1]}(\sqrt{-\Delta_g})$, which may contain too many linear combinations of eigenfunctions from different eigenspaces in order to deliver the best possible $L^p$ bounds on the $\varphi_\lambda$. Indeed, Weyl's law indicates that there are $\approx \lambda^{d-1}$ frequencies $\lambda_j$ in the interval $[\lambda, \lambda+1]$. Moreover, on any manifold there are approximate eigenfunctions in the range of $\mathbf{1}_{[\lambda,\lambda+1]}(\sqrt{-\Delta_g})$ which saturate the bounds (2),(3) (see [Sog93]).

The reason the canonical sphere is exceptional, in that this process gives the optimal bounds for eigenfunctions in this case, is that $\mathrm{spec}(\sqrt{-\Delta_g}) = \{\sqrt{k(k+d-1)} : k = 0,1,2,3,\dots\}$ behaves almost in an arithmetic fashion and hence $\chi(\lambda - \sqrt{-\Delta_g})$ is a sufficiently close approximation to projection onto an eigenspace. This turns out to be closely related to the $2\pi$-periodicity of the geodesics, implying that $e^{-it\sqrt{-\Delta_g}}$ and $e^{-i(t+2\pi)\sqrt{-\Delta_g}}$ are FIOs with the same canonical relation, see [DG75]. However, this sort of periodicity of the geodesic flow is in some sense rare and the selective class of manifolds satisfying this are often called *Zoll manifolds*.

2.2. **Kakeya-Nikodym norms and nonconcentration of mass in tubular neighborhoods: review of** [BlSo15b, BlSo14, BlSo15c, BlSo15a]**.** Many of my recent contributions in this problem area have been in understanding the interplay between $L^p$ norms and the *Kakeya-Nikodym* norm for eigenfunctions defined by

$$\|\varphi_\lambda\|^2_{KN(\lambda)} := \sup_{\gamma \in \Pi} \int_{\mathcal{T}_{\lambda^{-1/2}}(\gamma)} |\varphi_\lambda(x)|^2 \, dV_g, \tag{6}$$

where $\Pi$ denotes the space of unit length geodesic segments on $M$, and

$$\mathcal{T}_{\lambda^{-1/2}}(\gamma) := \{x \in M : d_g(x,\gamma) \le \lambda^{-1/2}\}$$

denotes the $\lambda^{-\frac{1}{2}}$-neighborhood of a geodesic segment $\gamma$. Tubular neighborhoods of this size are natural to work with as $\lambda^{-\frac{1}{2}}$ is the finest scale that respects both invariance under the geodesic flow and the uncertainty principle. These norms were first considered by Sogge in [Sog11] (and implicitly in a related work of Bourgain [Bou09]) who showed that when $d = 2$, for any $\epsilon > 0$, there exists a constant $C_{\epsilon,p}$ such that

$$\|\varphi_\lambda\|_{L^p(M)} \leq \epsilon\lambda^{\delta(p)} + C_{\epsilon,p}\lambda^{\delta(p)}\|\varphi_\lambda\|_{KN(\lambda)}^{1-\frac{2}{p}}, \qquad 4 \leq p < 6. \tag{7}$$

This bound gives a criteria for showing that when $4 \leq p < 6$, the universal $L^p$ bounds of Sogge from (2), (3) can be improved from $O(\lambda^{\delta(p)})$ to $o(\lambda^{\delta(p)})$. After interpolation with the trivial $L^2$ bound this yields the same improvement for all $2 < p < 6$. For if one can show that $\lim_{\lambda\to\infty} \|\varphi_\lambda\|_{KN(\lambda)} = 0$, then a standard "$\epsilon/2$" argument shows that $\|\varphi_\lambda\|_{L^p} = o(\lambda^{\delta(p)})$. Shortly after [Sog11], Sogge and Zelditch [SZ14] showed that when $(M, g)$ is a surface with nonpositive curvatures, then indeed the limit of the Kakeya-Nikodym norms tend to 0.

It should be stressed that (7) makes use of the fact that $p < 6 = \frac{2(d+1)}{d-1}\big|_{d=2}$. When $d = 2$ and $p < 6$ the bound (2) on $(\mathbb{S}^d, \mathrm{can})$ is saturated by the highest weight harmonics which concentrate in a $\lambda^{-\frac{1}{2}}$ neighborhood of a geodesic. In particular, for this family of spherical harmonics, $\limsup_{\lambda\to\infty} \|\varphi_\lambda\|_{KN(\lambda)} > 0$. So in order to improve $L^p$ bounds for such values of $p$, and more generally when $2 < p < \frac{2(d+1)}{d-1}$ and $d \geq 3$, one expects this should involve disproving mass concentration along geodesic segments.

A bound of the form (7) can be thought of as creating a bridge between $L^p$ norms and *global* harmonic analysis, where the behavior of the wave kernel over larger time scales is incorporated. Historically, such methods are more amenable to analysis in $L^2$-based norms rather than other $L^p$ spaces. As mentioned above, the global dynamics of the geodesic flow expect to determine if eigenfunctions concentrate in the neighborhoods $\mathcal{T}_{\lambda^{-1/2}}(\gamma)$. While this is of course possible on the canonical sphere, in other geometries such as manifolds with ergodic geodesic flow it is expected that the $\lim_{\lambda\to\infty} \|\varphi_\lambda\|_{KN(\lambda)} = 0$ would indeed be satisfied. One might even conjecture this should hold in any manifold not possessing an elliptic (stable) closed geodesic.

The proof of (7) involves local harmonic analysis in that it uses the operator $\chi(\lambda - \sqrt{-\Delta_g})$ and the resulting expression for its kernel (5). In particular, it takes a bilinear decomposition of the Carleson-Sjölin integral, splitting into parallel and transversal contributions. The former contributes to the Kakeya-Nikodym norm while the latter can be bounded using bilinear theory of Carleson-Sjölin integrals developed by Hörmander [Hör73]. Hence the approach is closely related to investigations on the Fourier restriction and Bochner-Riesz problems.

My work in this area began in [BlSo15b], kicking off a sequence of joint works with Sogge on these problems. Here we extended bounds of the form (7) to higher dimensions $d \geq 3$, for $\frac{2(d+2)}{d} < p < \frac{2(d+1)}{d-1}$. Once again we appealed to a family of estimates emanating from the bilinear approach to the restriction conjecture, in particular, bounds of Lee [Lee06] for bilinear Carleson-Sjölin integrals in their most general form. Lee's work built on the bilinear bounds for the classical Fourier restriction problem due to Wolff [Wol01] and Tao [Tao03].

Given this higher dimensional extension, we complemented it in the same work by extending the results of Sogge-Zelditch [SZ14] for $d = 2$ to all dimensions, showing that when $(M, g)$ has nonpositive curvatures, then $\lim_{\lambda\to\infty} \|\varphi_\lambda\|_{KN(\lambda)} = 0$ (in contrast to the canonical sphere). The setting of nonpositive curvature is a natural one to work with for a couple of reasons. One is that classical results of Hopf and Anosov show that the geodesic flow on a negatively curved manifold is ergodic, and based on the prior heuristics that the mass of an eigenfunction should reflect the classical dynamics of the manifold, this means that it is reasonable to conjecture that the mass of an eigenfunction is in some sense evenly spread out throughout the manifold, meaning the $L^p$ norms and Kakeya-Nikodym norms would be relatively small. Even in the presence of nonpositive curvature, there is a notion of partial ergodicity which may lead to similar expectations. A second reason is that the Cartan-Hadamard theorem implies that such manifolds lack conjugate points (caustics), which complicate the analysis of the wave kernel. Hence this assumption removes the technicalities that may obscure the phenomena of interest.

It then became interesting to consider refinements to (7), ones which would provide a quantitative link between $L^p$ bounds and Kakeya-Nikodym norms. In [BlSo15c], Sogge and I proved

$$\|\varphi_\lambda\|_{L^p} \le C_p \lambda^{\delta(p)} \|\varphi_\lambda\|_{KN(\lambda)}^{\frac{2(d+1)}{p(d-1)}-1}, \qquad \tfrac{2(d+2)}{d} < p < \tfrac{2(d+1)}{d-1}, \tag{8}$$

where again $\delta(p)$ is as in (3). This built on our earlier work [BlSo14], which had an intermediate result when $d = 2$. Hence if one has bounds on the rate at which $\|\varphi_\lambda\|_{KN(\lambda)} \to 0$, then this yields a corresponding quantitative improvement on the $L^p$ bounds. In particular, our complementary work at the same time [BlSo15a] showed that for any Riemannian manifold with nonpositive curvatures there exists a power $\sigma_d > 0$ such that

$$\|\varphi_\lambda\|_{KN(\lambda)} = O\left((\log \lambda)^{-\sigma_d}\right), \qquad \text{as } \lambda \to \infty. \tag{9}$$

Hence (8) in turn implies that for some different power $\tilde{\sigma}_{d,p} > 0$,

$$\|\varphi_\lambda\|_{L^p} = O\left(\lambda^{\delta(p)} (\log \lambda)^{-\tilde{\sigma}_{d,p}}\right), \qquad 2 < p < \tfrac{2(d+1)}{d-1}. \tag{10}$$

As alluded to above, it is expected that in the presence of nonpositive or negative curvature, the eigenfunctions $\varphi_\lambda$ should satisfy much better $L^p$ and Kakeya-Nikodym bounds than these, in particular an improvement on the power $\delta(p)$ in (3) appearing in the universal bounds. However, aside from the torus and special arithmetic quotients of the hyperbolic plane, obtaining an improvement of this type continues to remain elusive. This is because in the setting of nonpositive curvature, there are natural limitations on the time scales over which the wave kernel at frequency $\lambda$ is currently understood. In particular, considerations relating to the *Ehrenfest time* in quantum mechanics suggest that working over time scales much larger than $\log \lambda$ should be subtle. So aside from a possible improvement on the exact powers $\sigma_d, \tilde{\sigma}_{d,p}$, any further improvement on the bounds (9), (10) expects to require the ingenious development of new techniques.

The bounds (9) (and their predecessors in [SZ14], [BlSo15b]) have significance in their own right in the analysis of the quantum scars alluded to in §1. As mentioned there, one problem of interest is to understand when and how eigenfunctions concentrate their mass on a lower dimensional manifolds. For example, the highest weight spherical harmonics above concentrate their mass in a $\lambda^{-\frac{1}{2}}$ neighborhood of a closed geodesic, and classical Gaussian beam constructions suggest there should be analogous families of eigenfunctions whenever there is an elliptic closed geodesic (one which is in some sense stable under perturbations). However, this phenomena for hyperbolic/unstable closed geodesics (and invariant manifolds with similar structure) is not fully understood. Hence (9) has significance in that it shows that in the presence of nonpositive curvature, mass cannot fully concentrate in $\lambda^{-1/2}$ neighborhoods of a closed geodesic (or even a geodesic segment!). So if scarring occurs, its profile must involve larger sets.

2.3. **Improvements at the critical exponent in the presence of nonpositive curvature: review of** [BlSo17]**.** The results discussed in (2.2) showed how to obtain improvements on $L^p$ norms when $2 < p < \frac{2(d+1)}{d-1}$ from mass nonconcentration bounds in $\mathcal{T}_{\lambda^{-1/2}}(\gamma)$, meaning that if one can disprove concentration akin to the highest weight harmonics on $(\mathbb{S}^d, \mathrm{can})$, improved $L^p$ bounds in this range follow as a result. However, when $p \geq \frac{2(d+1)}{d-1}$, the zonal harmonics saturate the $L^p$ bounds on the canonical sphere, exhibiting a different phase space concentration profile. Hence if one seeks to improve the $L^p$ bounds for this range of $p$, there is a premium on disproving mass concentration akin to the zonal harmonics when $p > \frac{2(d+1)}{d-1}$. Moreover, when $p = \frac{2(d+1)}{d-1}$, a full spectrum of concentration profiles must be ruled out, not just those akin to the zonal and highest weight spherical harmonics, leading to its designation as the *critical* exponent in the theory. Indeed, as shown in [Tac16], there exist other families of spherical harmonics with profiles intermediate to both the zonal and highest weight harmonics that saturate the $L^p$ bounds at this exponent.

All the while, in the setting of nonpositive curvature, results of Bérard [Bér77] and Hassell and Tacy [HT15] show that when $p > \frac{2(d+1)}{d-1}$ one has a bound of the form (10), and in fact the latter work shows this to be true with $\tilde{\sigma}_{d,p} = \frac{1}{2}$ (though the constant $C_p$ in that work tends to infinity as $p \searrow \frac{2(d+1)}{d-1}$). Given the success of Sogge and myself in obtaining improvements in the range $p < \frac{2(d+1)}{d-1}$, it then becomes interesting to consider if improvements are satisfied at the critical exponent $p = \frac{2(d+1)}{d-1}$, though here one must deal with the subtlety of managing and disproving a spectrum of potential concentration profiles.

A breakthrough on this problem came first from Sogge [Sog15], who showed how to obtain bounds for nonpositively curved $(M, g)$ with a gain of some power of $\log\log\lambda$ (i.e. a bound of the form (10) holds, but with $\log\log\lambda$ replacing $\log\lambda$). The novelty in that work was to focus on bounds in weak-$L^p$ spaces (a.k.a. $L^{p,\infty}$ spaces), at which point strong $L^p$ bounds follow by interpolation with bounds of Bak and Seeger [BS11] for Carleson-Sjölin integrals in the Lorentz space $L^{p,2}$. The strategy for bounds in $L^{p,\infty}$ spaces isolate certain ranges of superlevel sets where either the classical bounds of Bérard are effective or bootstrapping the bounds (10) are effective.

In [BlSo17], Sogge and I were able to improve the $\log\log\lambda$ gain at the critical exponent in [Sog15] to a $\log\lambda$ gain of the form (10). This is significant as it reflects a more efficient use of our knowledge of the frequency localized wave kernel up to the Ehrenfest time. As in [Sog15], bounds in weak-$L^p$ spaces play a key role, though here a proof by contradiction is employed. In short, we consider a sequence of eigenfunctions which does not satisfy the logarithmic gain, then perform a weak version of a profile decomposition, allowing us to identify the phase space profile of members of the sequence. It is then possible to either use the bounds of Bérard as before or appeal to the ideas behind the proof of (9) given in [BlSo15a] depending on the profile. The strategy is partially inspired by those in nonlinear dispersive PDE, where one looks to show well-posedness by characterizing the phase space profile of a singular solution, then disproving that such concentration can occur.

2.4. **Restrictions to geodesic segments: review of** [Bl16]**.** Another problem of interest related to those above is to establish $L^p$ bounds on restrictions of eigenfunctions to embedded submanifolds in $(M, g)$, replacing the full manifold $M$ in (2) by a submanifold endowed with the natural measure inherited by the one on $M$. Much like $L^p$ bounds over the full manifold, the case of submanifolds reflect the phase space concentration profiles of eigenfunctions. Works of Burq, Gérard, and Tzvetkov [BuGT07] and Hu [Hu09] established bounds which are universal in that they are valid on any Riemannian manifold and for any submanifold therein. In each dimension there is an exponent analogous to (3), and in several cases of interest there is a notion of a critical exponent, but this depends both on the dimension of the manifold and the submanifold. We omit the full details, but remark that for manifolds of dimension $d = 2, 3$, this notion of criticality holds for curve segments in $M$. As in §2.3, there is a spectrum of scenarios for phase space concentration which could potentially saturate the $L^p$ bounds and in fact do so on the canonical sphere. These works also show that for hypersurfaces whose second fundamental form is nonvanishing (such as a curve with nonvanishing curvature when $d = 2$), there is a universal improvement in the $L^p$ bounds for certain values of $p$. As shown in [Hu09], restricting the Carleson-Sjölin kernels (5) to $\Sigma$ yields an FIO associated to a canonical relation with (possibly generalized) folding singularities, at which point bounds on restrictions follow from $L^p$ bounds for such operators [PS90, GS94, Com99].

Geodesic segments have emerged as a submanifold of particular interest for a couple of reasons. One is that as in the discussions regarding scarring above, the mass of an eigenfunction expects to be asymptotically invariant under the geodesic flow. Hence $L^p$ norms on restrictions expect to be the largest when the submanifold is totally geodesic, and geodesic segments are ubiquitous in all Riemannian manifolds, unlike totally geodesic submanifolds of higher dimension. Consequently, if an $L^p$ bound on a restriction is smaller, this suggests the scarring effects are less pronounced. Another is that $L^p$ bounds on geodesic segments are closely related to the Kakeya-Nikodym norms, since in low dimension a bound on the $L^2$ norm over a tubular neighborhood as in (6) can be considered to be a thickening of an $L^2$ bound on a geodesic segment.

When $d = 2$, the critical exponent for restrictions to geodesic segments $\gamma$ is $p = 4$, and the universal bounds show that in this case $\|\varphi_\lambda\|_{L^4(\gamma)} = O(\lambda^{1/4})$. In [CS14], Chen and Sogge showed that on manifolds with nonpositive curvature, this can be improved to $o(\lambda^{1/4})$. That same work has analogous results in the $d = 3$ case, though the improvement was only seen for manifolds of constant negative curvature. Interestingly enough, these improvements at the critical exponent for restrictions to geodesic segments preceded the results in [Sog15], which treat $L^p$ bounds over all of $M$ (cf. §2.3), by a few years. The geodesic restriction problem seems to benefit from the fact that the tangent space over a geodesic is of lower dimension, which essentially allows for an effective splitting into cases where the segment is part of a periodic geodesic and otherwise. A work of Xi and Zhang [XZ17] showed that for surfaces of constant negative curvature ($d = 2$), the gain exhibited in [CS14] can be quantified as $\|\varphi_\lambda\|_{L^4(\gamma)} = O(\lambda^{\frac{1}{4}}(\log\lambda)^{-\frac{1}{4}})$. A crucial matter for them was to obtain estimates on derivatives of the Riemannian distance function restricted to geodesic segments in the universal cover. For $(M, g)$ of constant negative curvature, the universal cover for the surface is the hyperbolic plane, allowing these estimates to be performed in an explicit fashion.

In [Bl16], I extended the result of Xi and Zhang, $\|\varphi_\lambda\|_{L^4(\gamma)} = O(\lambda^{\frac{1}{4}}(\log\lambda)^{-\frac{1}{4}})$, to any manifold of nonpositive curvature. Once again the key matter was to treat the Riemannian distance function in the universal cover, so the novelty in my work was to express the derivatives on the distance function in terms of Jacobi fields, at which point classical comparison theorems can be used. This created a more robust infrastructure for treating derivatives of the Riemannian distance function, and allows for a geometric interpretation of these quantities. This also allowed for a logarithmic improvement in the critical bounds for manifolds of constant negative curvature when $d = 3$, analogous to the second result in [CS14].

## 3. Boundary value problems and low regularity metrics

3.1. **Introduction.** This section considers two type of problems, the Strichartz estimates on Riemannian manifolds $(M, g)$ with nonempty boundary $\partial M \neq \emptyset$ and $L^p$ bounds on eigenfunctions satisfying the homogenous Dirichlet and Neumann conditions in (1). Recall that this setting naturally includes $C^\infty$ domains in $\mathbb{R}^n$ with the flat metric as a special case.

Let $u, v : \mathbb{R} \times M \to \mathbb{C}$ denote solutions to the wave and free Schrödinger equations respectively

$$(\partial_t^2 - \Delta_g)u(t,x) = 0, \qquad (u(0,\cdot), \partial_t u(0,\cdot)) = (f,g), \qquad \text{(Wave equation)} \tag{11}$$

$$(i\partial_t + \Delta_g)v(t,x) = 0, \qquad v(0,\cdot) = h. \qquad \text{(Schrödinger equation)} \tag{12}$$

As in (1), we assume that one of the two boundary conditions are imposed in the spatial variable

$$u(t,x)\big|_{x\in\partial M} = 0 \quad \text{(Dirichlet)}, \qquad \text{or} \qquad \frac{\partial u}{\partial \vec{n}}(t,x)\big|_{x\in\partial M} = 0 \quad \text{(Neumann)}, \tag{13}$$

and similarly for the Schrödinger solutions $v$. Here $\partial/\partial\vec{n}$ denotes a normal vector field to $\partial M$.

Strichartz estimates are a family of space-time integrability estimates on solutions to (11), (12). To state them, we introduce the following norm for functions $u : [-T,T] \times M \to \mathbb{C}$ (the natural one on the $L^q(M)$-valued $L^p$ space $L^p([-T,T]; L^q(M))$)

$$\|u\|_{L^p_T L^q} := \left(\int_{-T}^{T}\left(\int_M |u(t,x)|^q\,dx\right)^{\frac{p}{q}} dt\right)^{\frac{1}{p}} = \left(\int_{-T}^{T} \|u(t,\cdot)\|^p_{L^q(M)}\,dt\right)^{\frac{1}{p}}.$$

Local Strichartz estimates for solutions to (11) and (12) take the form

$$\|u\|_{L^p_T L^q} \leq C_{T,p,q}\left(\|f\|_{H^s(M)} + \|g\|_{H^{s-1}(M)}\right) \qquad (u(0,\cdot), \partial_t u(0,\cdot)) = (f,g) \tag{14}$$

$$\|v\|_{L^p_T L^q} \leq C_{T,p,q}\|h\|_{H^s(M)} \qquad v(0,\cdot) = h \tag{15}$$

provided $p, q \geq 2$ satisfy admissibility conditions (with $d = \dim(M)$ as before)

$$\tfrac{1}{p} + \tfrac{d}{q} = \tfrac{d}{2} - s, \qquad \text{and} \qquad \tfrac{2}{p} + \tfrac{d-1}{q} \leq \tfrac{d-1}{2} \qquad \text{(Wave)} \tag{16}$$

$$\tfrac{2}{p} + \tfrac{d}{q} = \tfrac{d}{2} - s, \qquad \text{and} \qquad \tfrac{2}{p} + \tfrac{d}{q} \leq \tfrac{d}{2} \qquad \text{(Schrödinger)}\ . \tag{17}$$

In each case, the left identity yields the optimal Sobolev regularity in the inequality predicted by scaling and the inequality on the right is dictated by the so-called *Knapp example*, which gives a solution highly concentrated along a light ray, essentially with the worst possible dispersive properties, and in fact its concentration profile is analogous to the highest weight spherical harmonics. In certain cases where $M$ is noncompact, global Strichartz estimates can also be considered, taking $T = \infty$, but here the discussion is limited to local bounds.

The modern proof of (15) typically follows from a dispersive or decay inequality such as the following one, an easy consequence of the structure of the fundamental solution to the Schrödinger equation

$$\|v(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq C'|t|^{-\frac{d}{2}}\|v(0,\cdot)\|_{L^1(\mathbb{R}^n)}. \tag{18}$$

Similar considerations hold for (14) though the analogous estimate is weaker given finite propagation speed. From a phase space perspective, the key to dispersive estimates lies in the stability of light rays, namely that they are a Lipschitz function of initial position and momentum. This underscores the discussion from §1.1 that the evolution should disperse highly peaked initial data, and here this is reflected in the decay of the amplitude of the wave.

Strichartz estimates have played a crucial role in applications to nonlinear versions of these equations, such as semilinear equations of the form

$$(\partial_t^2 - \Delta_g)\tilde{u} = \pm|\tilde{u}|^{\kappa-1}\tilde{u} \qquad \text{and} \qquad (i\partial_t + \Delta_g)\tilde{v} = \pm|\tilde{v}|^{\kappa-1}\tilde{v},$$

particularly when the initial data is of low regularity, lying in some Sobolev space. The solutions to these model equations satisfy conservation laws which involve a limited number of derivatives, hence obtaining a low regularity theory for them is significant for exploiting such laws and for understanding the development of singularities. The bounds (14), (15) are significant in that they limit the extent to which the nonlinear interactions overwhelm dispersive effects. As a toy example, after applying energy or $L^2$ estimates, one might be led to measure the strength of the nonlinearity in a space such as $|u|^{\kappa-1}u \in L^1([-T,T];L^2(M))$, a function space which integrates in different Lebesgue exponents with respect to space and time. This in turn reduces to estimating $u \in L^\kappa([-T,T];L^{2\kappa}(M))$, and Strichartz estimates are well-suited for such purposes. To achieve this, estimates for nonhomogeneous equations with a source term are needed, but there are now duality tricks for obtaining such inequalities from the their homogeneous counterparts.

3.1.1. *Boundary value problems and Strichartz estimates.* There are now a reasonable family of *local* Strichartz estimates on manifolds $(M,g)$ for which $\partial M = \emptyset$, though it could be said the picture is still rather incomplete. Strichartz estimates for the wave equation follow from results on smoothing properties of FIOs [MSS93, Kap89]. However, even local Strichartz estimates for the Schrödinger equation are more intricate due to the fact that infinite speed of propagation will amplify the problems created by trapped rays and caustics. There are results [ST02, BuGT04] which show that one has the estimates (15) when $\partial M = \emptyset$, but with a loss of $1/p$ derivatives in that the right hand side places $h \in H^{s+1/p}$ with $s$ determined by scaling as in (17).

Much less is known about the validity of these estimates when one starts to consider boundary value problems. This is due to the fact that the conditions (1) affect the development of waves and the flow of energy. In particular, waves now propagate along *generalized* geodesics, ones which graze the boundary or reflect off of it according to Snell's law. Hence the techniques for proving these estimates when $\partial M = \emptyset$ don't always transfer well to the $\partial M \neq \emptyset$ case.

The characteristics of generalized geodesics are in turn impacted by the geometry of the boundary. To this end, two extremes are helpful to consider. One extreme is given by points of strict concavity in the boundary. Here the dynamics of the generalized geodesics are comparatively favorable to dispersion, but the analysis is still rather difficult. This is because the issue of determining stability estimates for rays as they interact with the boundary can be subtle. Indeed, a so called *grazing* ray which strikes the boundary tangentially, without reflection, tends to separate away from nearby reflecting rays in phase space significantly. At the other extreme, points of strict convexity in the boundary give rise to the phenomenon of multiply reflecting rays. These situations also complicate stability estimates and give rise to *whispering gallery* phenomena, where energy tends to concentrate near the boundary to a strong degree. There are a set of examples due to Ivanovici [Iva10a, Iva12], showing that for a proper subset of the admissible triples on the right in (16), (17) the Strichartz estimates fail to hold without a loss of derivatives. Despite the challenges already present in these two extremes, they are in some sense the best understood cases; cases intermediate to these two extremes include points of nonstrict concavity or convexity, points of inflection, and points of infinite tangency which complicate the geometry of wave propagation in more subtle ways.

One other difficulty comes from the imposition of Neumann boundary conditions, which can be more difficult to handle than their Dirichlet counterparts. This is due in part to the fact that Neumann conditions do not satisfy the so-called *uniform Lopatinski condition*, allowing for better control of boundary traces, and simplifying the analysis considerably.

In spite of these challenges, it is important to gain an understanding of how these inequalities manifest themselves when boundary conditions are present, given their applications. Many of the innovators in nonlinear wave and Schrödinger equations have considered problems over interior and exterior domains in $\mathbb{R}^n$. There has been a great deal of progress on large data global well-posedness and scattering of energy critical equations in the wave [SS95, BuLP08, BuP09] and Schrödinger [KVZ16] settings. The validity of Strichartz estimates in domains has played a key role in these results and this expects to be the case in future developments. Moreover, the discussions above show that the pursuit of Strichartz and eigenfunction estimates are of independent interest as they illuminate the role of boundary conditions in affecting the dispersive properties of waves.

3.1.2. *Boundary value problems and eigenfunction estimates.* Similar considerations also complicate the theory of $L^p$ bounds for eigenfunctions when the boundary conditions (1) are imposed. Here even understanding the optimal bounds on the operator $\mathbf{1}_{[\lambda,\lambda+1]}(\sqrt{-\Delta_g})$ as in (4) has proven to be subtle for much of the same considerations as in §3.1.1. Grieser [Gri92] observed that the whispering gallery effect on manifolds with a convex boundary furnishes examples of approximate eigenfunctions in the range of $\mathbf{1}_{[\lambda,\lambda+1]}(\sqrt{-\Delta_g})$ showing that for values of $p \in (2, \frac{6d+4}{3d-4})$, the bounds (2) cannot hold with the exponent in (3). In short, the multiply reflecting rays give rise to modes which concentrate in a $\lambda^{-2/3}$ collar of the boundary, much smaller than the profile of the tubular neighborhoods in geodesic segments considered above. These modes in turn satisfy much worse $L^p$ bounds than in the boundaryless case, since the effect of concentrating more mass into a smaller region increases their amplitude. Hence the family of problems of interest here is to investigate what can be said about the analog of the universal $L^p$ bounds on eigenfunctions, ones which result from analyzing $\mathbf{1}_{[\lambda,\lambda+1]}(\sqrt{-\Delta_g})$. That said, "universal" in this context may be relative to boundaries with a given geometry (e.g. strictly concave or convex), as improvements akin to the ones discussed in §2 appear to be subtle.

3.1.3. *Low regularity metrics.* The approaches to the boundary value problems above I have used were inspired by some of the successes in establishing Strichartz estimates for wave and Schrödinger equations when the metric coefficients are of *low regularity*, that is, they are not assumed to be infinitely differentiable. When the metric coefficients are no longer Lipschitz, the basic ODE theory of the geodesic equations do not guarantee existence of these curves. Moreover, if the coefficients are not $C^{1,1}$, geodesics may not be unique (bifurcation of solutions) and there may not be any sense of stability under variation of initial parameters, even locally in time. For these reasons, the dynamics of geodesic rays in this setting can suffer from pathologies similar to those for boundary value problems.

In addition, the homogeneous boundary value problems considered here can be treated by gluing two copies of the manifold together along its boundary, forming the *double* of the manifold.

This in some sense erases the boundary, but in order to extend solutions to the double (in an odd, even fashion corresponding to Dirichlet, Neumann conditions respectively) the metric coefficients also need to be extended via a reflection in the boundary. This reflection means that unless the boundary is flat, even $C^\infty$ coefficients will be extended to Lipschitz ones.

In the late 90's and early 00's there was significant progress in establishing Strichartz bounds for equations with rough coefficients due to Smith [Smi98, Smi06], Tataru [Tat01], and Staffilani and Tataru [ST02] and others, motivated in part by applications to low regularity solutions to quasilinear wave equations, ones where the coefficients depend on the solutions themselves. The main idea was to take dyadic decompositions in the frequency variable, then regularize the coefficients in a frequency dependent fashion (very similar to the paradifferential smoothing of pseudodifferential operators as in [Tay91] for example). From there, the frequency localized solutions can be represented as a superposition of *wave packets*. Wave packets are approximate solutions to the wave and Schrödinger equations which are highly concentrated along a geodesic in both space and frequency on scales which are consistent with the uncertainty principle. It can then be seen that this superposition of packets yields decay estimates akin to (18).

3.2. **Strichartz estimates for the wave and Schrödinger equations for general $C^\infty$ boundaries:** [BlSS09, BlSS12, BlSS08b]**.** These were joint works with H. Smith and C. Sogge that proved a family of Strichartz bounds (15) for the wave and Schrödinger equations (respectively) that operated under a fairly weak set of assumptions: namely $\partial M$ was only assumed to be $C^\infty$ and compact and that either type of boundary condition (1) was allowed. However, it was assumed that the exponents $p, q$ satisfy

$$\begin{cases} \frac{3}{p}+\frac{\sigma}{q}\leq\frac{\sigma}{2}, & \sigma\leq 3,\\ \frac{1}{p}+\frac{1}{q}\leq\frac{1}{2}, & \sigma\geq 3,\end{cases} \quad \text{with} \quad \sigma=\begin{cases} d-1, & \text{for the wave equation,}\\ d, & \text{for the Schrödinger equation,}\end{cases} \tag{19}$$

thus providing a restrictive condition on the admissibility of $p, q$ in comparison to the inequality in (17). When $M$ is compact, the estimates matched the loss of $\frac{1}{p}$ derivatives obtained for general boundaryless manifolds referenced above. However, when $M = \mathbb{R}^n \setminus \mathcal{K}$ is a domain exterior to a nontrapping obstacle $\mathcal{K}$, estimates with no loss of derivatives can be obtained. So while (19) does not give the sharp range of exponents for certain geometries, the hypothesis are broadly applicable and allow for either boundary condition in (13).

These results employed a parametrix construction originating in a work of Smith and Sogge [SS07] on eigenfunction estimates. The starting point is to eliminate the boundary by forming its double as described in §3.1.3 at the cost of introducing Lipschitz coefficients. The wave packet and coefficient smoothing techniques can then be used in an effective manner. The restriction (19) comes from the boundary interactions causing packets to lose their coherent structure on small time scales, ones which are comparable to the angle that the underlying momentum forms with the boundary. Hence the error term in the parametrix can only be bounded over these small time scales. However, this can be countered by certain gains in dispersive type estimates, using that these poorly behaved packets live in a comparatively small cone in the frequency variable.

3.3. **Estimates for Neumann problems in strictly concave $C^\infty$ boundaries: Review of** [Bl14, Bl17]**.** In spite of the virtues of the wave packet methods outlined above, the results fail to obtain the optimal range of estimates in certain geometries. Since my joint works with Smith and

Sogge in [BlSS09, BlSS12], use such a general set of assumptions, they neglect certain geometric features that might allow for stronger dispersive effects than what is used. Indeed, works of Smith and Sogge [SS95] and Ivanovici [Iva10b] show that the full range of Strichartz bounds (14), (15) are satisfied in domains with a strictly concave boundary when Dirichlet conditions are imposed (the Schrödinger case assuming $M$ is exterior to a strictly convex obstacle). The results [Gri92, SS94] show eigenfunction bounds (2) with exponent (3) in the same setting. Instead of wave packets, these works rely on a parametrix construction of Melrose and Taylor (see e.g. [MT, Zwo90]). However, it is not known if this approach can yield the Strichartz and eigenfunction estimates when the Dirichlet condition is replaced by a Neumann condition as one seems to be led to consider FIOs where the order of the operator is too high.

Therefore, another virtue of the wave packet approach outlined in §3.2 is that it is effective in treating Neumann conditions. It is thus interesting to see if the wave packet methods can be adapted to certain geometric settings, hoping to improve upon the stringent admissibility conditions in (19) when the dynamics of generalized geodesics are favorable. A natural starting point is to revisit strictly concave boundaries, as the full range of estimates are known for Dirichlet conditions, and it is expected that the same should hold for Neumann conditions.

In [Bl14, Bl17], I began a program which sought to accomplish this by first proving a black box theorem showing that whenever a family of integrated local energy estimates for the equations are satisfied, then Strichartz estimates follow provided the exponents $p, q$ in (14), (15) are subcritical with respect to the second condition in (16), (17) (and more precisely the Knapp example), that is, $\frac{2}{p} + \frac{d-1}{q} < \frac{d-1}{2}$, $\frac{2}{p} + \frac{d}{q} < \frac{d}{2}$ respectively. The key innovation was to use such local energy estimates to do a better job of bounding the error terms in the wave packet parametrix constructions used in the joint works with Smith and Sogge [BlSS09, BlSS12]. Moreover, the method yields *square function* estimates for the wave equation, which imply that the eigenfunction bounds (2) hold with exponent (3) when $p > \frac{2(d+1)}{d-1}$. Second, in [Bl17], it was shown that these local energy estimates are indeed satisfied in any manifold where $\partial M$ is strictly concave, thus verifying the hypothesis in the black box theorem.

To state these integrated local energy estimates, consider the case of the wave equation. Let $u_\lambda$ solve $(\partial_t^2 - \Delta_g)u_\lambda = 0$, with spectral localization to frequencies $\approx \lambda$ in that $\beta(\lambda^{-2}\Delta_g)u_\lambda(t,\cdot) = u_\lambda(t,\cdot)$ where $\beta \in C_c^\infty(1/2, 2)$ and $\beta(\lambda^{-2}\Delta_g)$ is defined by the functional calculus. Then define $S_{<j} := \{x : d_g(x, \partial M) \leq 2^{-j}\}$, the collar neighborhood of $\partial M$ of width $2^{-j}$. The integrated local energy estimates then assert that with $\nabla_{t,x}u_\lambda = (\partial_t u_\lambda, \nabla_g u_\lambda)$,

$$\|\nabla_{t,x}u_\lambda\|_{L^2([-T,T]\times S_{<j})} \leq C_T 2^{-\frac{j}{4}} \|\nabla_{t,x}u_\lambda(0,\cdot)\|_{L^2(M)}, \tag{20}$$

with $C_T$ independent of $\lambda$ and $j$ provided $2^{-j} \in [\lambda^{-2/3}, 1]$. One heuristic argument for such an estimate lies in wave packet analysis: since glancing rays behave parabolically with respect to the boundary, a wave packet concentrated along such a ray should spend a time $\approx 2^{-j/2}$ within $S_{<j}$, so that taking the square integral in time suggests the gain. These estimates were first proved in domains which are exterior to a ball in $\mathbb{R}^n$ by Ivanovici [Iva07], but this relied on explicit computations of the Green's kernel for the Helmholtz equation available in this setting. My method of proof for (20) in the general case relied on an intricate positive commutator argument that used Tataru's results on the regularity of boundary traces [Tat98] to estimate such contributions. The energy estimates for the Schrödinger equation are similar, the only

difference being that there is a further gain of $\lambda^{-1/2}$ since solutions localized at frequency $\lambda$ propagate at speed $\lambda$.

3.4. **Polygonal domains in $\mathbb{R}^2$: Review of** [BlFHM12, BlFM13, BlFM15]**.** In spite of the progress in establishing Strichartz and eigenfunction estimates for $C^\infty$ boundaries, very little is known about the validity of these estimates for equations posed on manifolds with corners (and other singular spaces). This is because diffractive effects arise when wave fronts interact with the corners, complicating the analysis on top of the issues created by the reflecting and grazing light rays discussed above. In fact, it was only recently that a reasonable propagation of singularities theory was developed for these manifolds by Melrose, Vasy, and Wunsch (see e.g. [MVW13]); in contrast, the theory for manifolds with boundary was established in the 70's. In their work, it is seen that a singularity carried along a generalized geodesic approaching a corner can be propagated along a family of rays emanating from the corner tip, instead of a single ray. Hence this new phenomena gives rise to a *diffractive wave front*, which creates further discontinuities in the fundamental solution of wave equation beyond the incident and reflected wave fronts.

A starting point for the development of Strichartz and eigenfunction estimates in this setting is to study polygonal domains in $\mathbb{R}^2$, those whose boundary is a union of line segments. Waves propagating in such domains incur nontrivial diffractive effects from the corners, but since the boundary is otherwise flat, it eliminates some of the other difficulties arising from curved boundaries discussed above. In my joint works with Ford and Marzuola, we proved that the full range of Strichartz estimates for the wave equation are satisfied [BlFM13] and also that Sogge's bounds for eigenfunctions (2) with exponent (3) are satisfied [BlFM15] for equations posed in polygonal domains. Strichartz estimates for the Schrödinger equation were treated by the same authors and Herr in [BlFHM12], obtaining estimates with a loss of $1/p$ derivatives.

The approach in these works is to first double the polygon in the same way as before; the resulting regularity is not a Lipschitz metric as before, rather a locally Euclidean surface which has *conical singularities*. The latter are flat manifolds which locally resemble $\mathbb{R}_+ \times (\mathbb{R}/2\pi\rho\mathbb{Z})$ with the metric $dr^2 + r^2 d\theta^2$ for some $\rho > 0$. Works of Cheeger and Taylor [CT82a, CT82b] and Ford [For10], then give explicit representations of the fundamental solutions to these equations near these singularities. However, understanding the diffractive contribution of such representations is highly nontrivial, particularly since they do not seem to fall into the usual classes of oscillatory integrals encountered in harmonic analysis. The joint work on eigenfunction estimates [BlFM15] presented unique challenges as inserting these formulae into (4) and integrating in $t$ means that the discontinuities in the fundamental solution created by the transitions in between incident, reflected, and diffracted wave fronts must be dealt with.

3.5. **Restrictions of eigenfunctions to curves and submanifolds: Review of** [Bl13]**.** As described in §2.4, one problem of interest is to understand $L^p$ bounds on restrictions of eigenfunctions to embedded submanifolds. However, by the same considerations as in §3.1.2,3.1.3 when one starts to consider either low regularity metrics or boundary conditions, even recovering the universal bounds known for $C^\infty$ metrics when $\partial M = \emptyset$ can be subtle.

In [Bl13], I proved a family of $L^p$ bounds on restrictions for Lipschitz and $C^{1,\alpha}$ metrics with $\alpha \in (0, 1]$. As observed in §3.1.3, problems on manifolds with boundary reduce to those on manifolds with Lipschitz metrics, hence the results here also apply to manifolds with boundary. These bounds were also significant in that they observed an improvement in the $L^p$ bounds for hypersurfaces where the second fundamental form is nonvanishing. Indeed, this operation is meaningful when the metric is in $C^{1,\alpha}$ for $\alpha \in (0,1]$, so the results in this work indicate that even though geodesics may not be unique in this setting, there are still improved bounds when the hypersurface does not contain geodesic segments. The key idea in this work was to see that the wave packet parametrices for rough metrics [Smi98, Tat01, Smi06] can be used to estimate traces of solutions to the wave equation over these submanifolds, which in turn implies $L^p$ bounds after appealing to operators such as (4). This possessed its own set of challenges given that formally, these parametrices yield FIOs with folding singularities.

3.6. **Multilinear eigenfunction bounds for manifolds with boundary: Review of** [BlSS08a]**.** Results of Burq, Gérard, and Tzvetkov [BuGT05] considered multilinear generalizations of the bound (2), proving a bound of the following type on the product of two $L^2$ normalized eigenfunctions $(\Delta_g + \lambda^2)\varphi_\lambda = 0$, $(\Delta_g + \mu^2)\varphi_\mu = 0$

$$\|\varphi_\lambda \, \varphi_\mu\|_{L^2} \le C \left(\min(\lambda,\mu)\right)^\alpha ,$$

for some power $\alpha > 0$. They in turn applied this to study the nonlinear Schrödinger equation on manifolds.

In [BlSS08a], Smith, Sogge, and I proved a generalization of this bilinear bound for boundary value problems. This meant dealing with the same challenges outlined in §3.1.1,3.1.2. Consequently the power $\alpha$ appearing was worse than in the boundaryless case $\partial M = \emptyset$, but given the whispering gallery modes, they were essentially sharp for functions in the range of the operators $\chi(\lambda - \sqrt{-\Delta_g}), \chi(\mu - \sqrt{-\Delta_g})$.

3.7. **Strichartz and eigenfunction bounds for metrics of Sobolev regularity: Review of** [Bl06, Bl09]**.** My dissertation work dealt with Strichartz estimates and $L^p$ bounds for eigenfunctions in the presence of low regularity metrics. As discussed in §3.1.3, a crucial consideration for the validity of such inequalities is the regularity of the geodesic flow and its stability under small perturbations in position and momentum (over local time scales). While $g \in C^{1,1}$ is sufficient to ensure these properties, there are weaker regularity hypotheses which will accomplish the same. In [Bl06, Bl09], I considered a family of metrics whose regularity was characterized by its inclusion in certain Sobolev space and proved Strichartz and eigenfunction bounds in this setting. In particular, the most significant cases considered were such that metric regularity was lower than $C^{1,1}$ but still allowed for the same regularity of the geodesic flow. The wave packet techniques outlined in §3.1.3 were the cornerstone of the approach and these investigations influenced my later work on boundary value problems.

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