Solutions of Knizhnik-Zamolodchikov equation by dévissage

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30th Applications of Computer Algebra - ACA 2025

In this work, solutions of universal differential equation (see (2) below, when the solutions exist) are provided using Volterra expansions for the Chen series. Ultimately, applied to the Knizhnik-Zamolodchikov (see (8) below) [1], this provides by *dévissage*[1] the unique grouplike solution satisfying asymptotic conditions. These solutions use a Picard-Vessiot theory of noncommutative differential equations and various factorizations of Chen series over the alphabet $\mathcal{T}_n := \{t_{i,j}\}_{1\le i<j\le n}$ and with coefficients in a commutative rings [2]. In particular, in the ring of holomorphic functions, $(\mathcal{H}(\mathcal{V}), 1_{\mathcal{H}(\mathcal{V})})$, over the simply connected differentiable manifold of $\mathbb{C}^n$, $\mathcal{V}$, the coefficients $\{\langle S \mid w\rangle\}_{w\in\mathcal{T}_n^*}$ of S are holomorphic and $\{\partial_i\langle S \mid w\rangle\}_{1\le i\le n}$ are well defined. So is the differential $d\langle S \mid w\rangle = \partial_1\langle S \mid w\rangle dz_1 + \cdots + \partial_n\langle S \mid w\rangle dz_n$. Thus, $\mathbf{d}$ can be defined over $\mathcal{H}(\mathcal{V})\langle\langle\mathcal{T}_n\rangle\rangle$ by

$$S = \sum_{w\in\mathcal{T}_n^*} \langle S \mid w\rangle w, \qquad \mathbf{d}S = \sum_{w\in\mathcal{T}_n^*} (d\langle S \mid w\rangle) w, \tag{1}$$

leading to the following noncommutative differential equation over $\mathcal{H}(\mathcal{V})\langle\langle\mathcal{T}_n\rangle\rangle$,

$$\mathbf{d}S = M_n S, \quad \text{where} \quad M_n := \sum_{1\le i<j\le n} \omega_{i,j} t_{i,j} \in \mathcal{L}ie_{\Omega(\mathcal{V})}\langle\mathcal{T}_n\rangle. \tag{2}$$

In particular, to the partition $\mathcal{T}_n$, onto $\mathcal{T}_{n-1}$ and $T_n := \{t_{k,n}\}_{1\le k\le n-1}$, corresponds the split of M_n:

$$M_n = \bar{M}_n + M_{n-1}, \quad \text{where } M_{n-1} \in \mathcal{L}ie_{\Omega(\mathcal{V})}\langle\mathcal{T}_{n-1}\rangle \quad \text{and } \bar{M}_n := \sum_{k=1}^{n-1} \omega_{k,n}\ t_{k,n} \in \mathcal{L}ie_{\Omega(\mathcal{V})}\langle T_n\rangle. \tag{3}$$

For $N = n(n-1)/2$, the forms $\{\omega_i\}_{1\le i\le N}$ and the alphabet $X := \{x_k\}_{1\le j\le N}$ in bijection with $\mathcal{T}_n$,

$$\mathbf{d}S = M_n S, \quad \text{where} \quad M_n := \sum_{i=1}^{N} \omega_i x_j \in \mathcal{L}ie_{\Omega(\mathcal{V})}\langle X\rangle, \tag{4}$$

$$M_n = \sum_{1\le k\le N} F_k\ x_k = \sum_{1\le l\le n} U_l\ dz_l, \quad \text{where} \quad F_k = \sum_{1\le l\le n} f_{l,k}\ dz_j \text{ and then } U_l = \sum_{1\le k\le N} f_{l,k}\ x_k. \tag{5}$$

For $S \neq 0$ in the integral ring $\mathcal{H}(\mathcal{V})\langle\langle\mathcal{T}_n\rangle\rangle$, if S satisfies (2) then, by (5), one might have

$$\mathbf{d}S = M_n S = \sum_{1\le l\le n} (\partial_l S)\ dz_l, \quad \text{with} \quad \partial_l S = U_l S. \tag{6}$$

[1] *I.e.* solutions of KZ_n, for $n \ge 3$, are obtained using those of KZ_{n-1} and the generating series of hyperlogarithms

Since $\partial_j\partial_i S = ((\partial_j U_i) + U_iU_j)S$ and $\partial_i\partial_j S = \partial_j\partial_i S$ then $((\partial_j U_i) - (\partial_i U_j) + [U_i, U_j])S = 0$ and then $\partial_i U_j - \partial_j U_i = [U_i, U_j]$, $1 \le i, j \le n$. Or equivalently, $\mathbf{d}M_n = M_n \wedge M_n$ inducing a Lie ideal of relators among $\{t_{i,j}\}_{1\le i<j\le n}$, $\mathcal{J}_n$, and (2) are solved over $\mathcal{H}(\mathcal{V})\langle\langle\mathcal{T}_n\rangle\rangle$ and then $\mathcal{H}(\mathcal{V})\langle\langle\mathcal{T}_n\rangle\rangle/\mathcal{J}_n$.

According to Drinfel'd, M_n is *flat* and (2) is *completely integrable* [1]. Solution of (2), when exists, can be computed by convergent Picard's iteration over the topological basis $\{w\}_{w\in\mathcal{T}_n^*}$, *i.e.*

$$F_0(\varsigma, z) = 1_{\mathcal{H}(\mathcal{V})}, \quad F_i(\varsigma, z) = F_{i-1}(\varsigma, z) + \int_\varsigma^z M_n(s)F_{i-1}(s), i \ge 1, \tag{7}$$

and the sequence $\{F_k\}_{k\ge0}$ admits the limit, called Chen series of the holomorphic forms $\{\omega_{i,j}\}_{1\le i<j\le n}$ and along a path $\varsigma \rightsquigarrow z$ over $\mathcal{V}$, modulo $\mathcal{J}_n$, is viewed as the fundamental solution of (2).

More generally, by a Ree's theorem, Chen series is grouplike belonging to $e^{\mathcal{L}ie_{\mathcal{H}(\mathcal{V})}\langle\langle\mathcal{T}_n\rangle\rangle}$ and can be put in the MRS factorization form [2] and [4]. Moreover, since the rank of the module of solutions of (2) is at most equals 1 then, under the action of the Haussdorf group, *i.e.* $e^{\mathcal{L}ie_{\mathbb{C}}\langle\langle\mathcal{T}_n\rangle\rangle}$ playing the rôle of the differential Galois group of (2) [2].

From these, in practice, infinite solutions of (2) can be computed using convergent iterations of pointwise convergence over $\mathcal{H}(\mathcal{V})\langle\langle\mathcal{T}_n\rangle\rangle$ and then $\mathcal{H}(\mathcal{V})\langle\langle\mathcal{T}_n\rangle\rangle/\mathcal{J}_n$. A challenge is to explicitly and exactly compute these limits of convergent sequences of (not necessarily grouplike) series on the dual topological ring and over various corresponding dual topological bases.

Applying (2)–(3), substituting $t_{i,j}$ by $t_{i,j}/2\mathrm{i}\pi$ and specializing $\omega_{i,j}$ to $d\log(z_i - z_j)$ and then $\mathcal{V}$ to the universal covering of the configuration space of n points on the complex plane $\mathbb{C}_*^n := \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n | z_i \ne z_j \text{ for } i \ne j\}$, denoted by $\widetilde{\mathbb{C}_*^n}$, various expansions of Chen series over $\mathcal{H}(\widetilde{\mathbb{C}_*^n})\langle\langle\mathcal{T}_n\rangle\rangle$ provide solutions of the differential equation $\mathbf{d}F = \Omega_n F$, so-called KZ_n equation and Ω_n is so-called universal KZ connection form, defined by

$$\Omega_n(z) := \sum_{1\le i<j\le n} \frac{t_{i,j}}{2\mathrm{i}\pi} d\log(z_i - z_j) = \bar{\Omega}_n + \Omega_{n-1}, \quad \text{where} \quad \bar{\Omega}_n(z) := \sum_{k=1}^{n-1} \frac{t_{k,n}}{2\mathrm{i}\pi} d\log(z_k - z_n). \tag{8}$$

In particular, let $\Sigma_{n-2} = \{z_1, \ldots, z_{n-2}\} \cup \{0\}$ (for $z_{n-1} = 0$) be the set of singularities and $s = z_n$. For $z_n \to z_{n-1}$, the connection $\bar{\Omega}_n$ behaves as $(2\mathrm{i}\pi)^{-1}N_{n-1}$, where N_{n-1} is nothing but the connection of the differential equation satisfied by the noncommutative generating series of hyperlogarithms

$$N_{n-1}(s) := t_{n-1,n}\frac{ds}{s} - \sum_{k=1}^{n-2} t_{k,n}\frac{ds}{z_k - s} \in \mathcal{L}ie_{\Omega(\widetilde{\mathbb{C}\setminus\Sigma_{n-2}})}\langle T_n\rangle. \tag{9}$$

Let α_ς^z be the function on $\mathcal{T}_n^*$, mapping words to iterated integrals over the holomorphic 1-forms $\{d\log(z_i - z_j)\}_{1\le i<j\le n}$ along the path $\varsigma \rightsquigarrow z$ over $\widetilde{\mathbb{C}_*^n}$. The Chen series of $\{d\log(z_i - z_j)\}_{1\le i<j\le n}$ can be used to determine solutions of (8) and depends on the differences $\{z_i - z_j\}_{1\le i<j\le n}$. Furthermore, the universal KZ connection form Ω_n satisfies $\mathbf{d}\Omega_n - \Omega_n \wedge \Omega_n = 0$, inducing the relators associated to following relations on $\{t_{i,j}\}_{1\le i<j\le n}$ and generating the Lie ideal $\mathcal{J}_{\mathcal{R}_n}$ of $\mathcal{L}ie_{\mathcal{H}(\mathcal{V})}\langle\mathcal{T}_n\rangle$,

$$\mathcal{R}_n = \begin{cases} [t_{i,k} + t_{j,k}, t_{i,j}] = 0 & \text{for distinct } i, j, k, \quad 1 \le i < j < k \le n, \\ [t_{i,j} + t_{i,k}, t_{j,k}] = 0 & \text{for distinct } i, j, k, \quad 1 \le i < j < k \le n, \\ [t_{i,j}, t_{k,l}] = 0 & \text{for distinct } i, j, k, l, \quad \begin{cases} 1 \le i < j \le n, \\ 1 \le k < l \le n. \end{cases} \end{cases} \tag{10}$$

Then solutions of (8) are expected in $\mathcal{H}(\widetilde{\mathbb{C}_*^n})\langle\langle\mathcal{T}_n\rangle\rangle/\mathcal{J}_{\mathcal{R}_n}$.

For $z_n \to z_{n-1}$, grouplike solutions of (8) are of the form $h(z_n)H(z_1,\ldots,z_{n-1})$, where h satisfies the differential equation $df = (2i\pi)^{-1}N_{n-1}f$ such that $h(z_n) \sim_{z_n\to z_{n-1}} (z_{n-1}-z_n)^{t_{n-1,n}/2i\pi}$ and H satisfies the following differential equation

$$\mathbf{d}S = \Omega^{\phi_n}_{n-1}S, \quad \text{where} \quad \Omega^{\phi_n}_{n-1}(z) = \sum_{1\le i<j\le n-1} d\log(z_i - z_j)\phi_n^{(z^0,z)}(t_{i,j})/2i\pi \tag{11}$$

and $\phi_n^{(z^0,z)}(t_{i,j}) \sim_{z_n\to z_{n-1}} e^{\mathrm{ad}_{-\log(z_{n-1}-z_n)t_{n-1,n}/2i\pi}}t_{i,j} \mod \mathcal{J}_n$.

Let $\mathcal{D}_{\mathcal{T}_n}$ (resp. $\mathcal{D}_{T_n}$) denote the diagonal series on the $\sqcup\!\sqcup$-bialgebra $(\mathbb{Q}\langle\mathcal{T}_n\rangle, \mathtt{conc}, 1_{\mathcal{T}_n^*}, \Delta_{\sqcup\!\sqcup})$ (resp. $(\mathbb{Q}\langle T_n\rangle, \mathtt{conc}, 1_{T_n^*}, \Delta_{\sqcup\!\sqcup})$) endowed the dual bases $\{P_l\}_{l\in\mathcal{L}yn\mathcal{T}_n}$ and $\{S_l\}_{l\in\mathcal{L}yn\mathcal{T}_n}$ (resp. $\{P_l\}_{l\in\mathcal{L}ynT_n}$ and $\{S_l\}_{l\in\mathcal{L}ynT_n}$) indexed by Lyndon words on $\mathcal{L}yn\mathcal{T}_n$ (resp. $\mathcal{L}ynT_n$) [4]. Then solutions of (8) can be computed by the following recursion

$$V_k(\varsigma,z) = V_0(\varsigma,z)\sum_{t_{i,j}\in\mathcal{T}_{n-1}}\int_\varsigma^z \omega_{i,j}(s)S_0^{-1}(\varsigma,s)t_{i,j}V_{k-1}(\varsigma,s) \tag{12}$$

with $V_0(\varsigma,z) = (\alpha_\varsigma^z\otimes\mathrm{Id})\mathcal{D}_{\mathcal{T}_n}$, or with $V_0(\varsigma,z) = (\alpha_\varsigma^z\otimes\mathrm{Id})\mathcal{D}_{T_n} \mod [\mathcal{L}ie_{\mathcal{H}(\mathcal{V})}\langle\langle T_n\rangle\rangle, \mathcal{L}ie_{\mathcal{H}(\mathcal{V})}\langle\langle T_n\rangle\rangle]$. Finally, there effectively exists $\{F_{S_l}\}_{l\in\mathcal{L}yn\mathcal{T}_n}$ such that the sequence $\{V_k\}_{k\ge0}$ in (12) converges, in the first case, to the unique solution of (8) satisfying asymptotic conditions and achieving the *dévissage*:

$$\begin{aligned}\mathbb{F}_{KZ_n} &= \prod_{l\in\mathcal{L}ynT_n}^{\searrow} e^{F_{S_l}P_l}\underbrace{\Big(1_{\mathcal{T}_n^*} + \sum_{v_1,\ldots,v_k\in T_n^*,t_1,\ldots,t_k\in\mathcal{T}_{n-1},k\ge1} F_{a(v_1t_1)\,\underline{\sqcup\!\sqcup}_2\ldots\underline{\sqcup\!\sqcup}_2\,a(v_kt_k)}r(v_1t_1)\ldots r(v_kt_k)\Big)}_{\text{functional expansion of solution of } KZ_{n-1}} &(13)\\ &= \prod_{l\in\mathcal{L}yn\mathcal{T}_{n-1}}^{\searrow} e^{F_{S_l}P_l}\Big(\prod_{l=l_1l_2,l_2\in\mathcal{L}yn\mathcal{T}_{n-1},l_1\in\mathcal{L}ynT_n}^{\searrow} e^{F_{S_l}P_l}\Big)\prod_{l\in\mathcal{L}ynT_n}^{\searrow} e^{F_{S_l}P_l} &(14)\end{aligned}$$

while in the second case, it leads to an approximation of (13):

$$\mathbb{F}_{KZ_n} \equiv e^{\sum_{t\in T_n}F_tt}\Big(1_{\mathcal{T}_n^*} + \sum_{v_1,\ldots,v_k\in T_n^*,t_1,\ldots,t_k\in\mathcal{T}_{n-1},k\ge1} F_{a(\hat{v}_1t_1)\,\underline{\sqcup\!\sqcup}_2(\ldots\underline{\sqcup\!\sqcup}_2(a(\hat{v}_kt_k))\ldots)}r(v_1t_1)\ldots r(v_kt_k)\Big), \tag{15}$$

where $\underline{\sqcup\!\sqcup}_2$ denotes the half-shuffle product [3] and, for any $w = t_1\ldots t_m \in \mathcal{T}_n^*$,

$$a(w) = (-1)^m t_m\ldots t_1, \quad r(w) = \mathrm{ad}_{t_1}\circ\cdots\circ\mathrm{ad}_{t_{m-1}}t_m, \quad \hat{w} = t_1 \sqcup\!\sqcup \ldots \sqcup\!\sqcup t_m. \tag{16}$$

References

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