Closed forms of power series with hypergeometric-type terms

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This talk focuses on power series representations of univariate D-finite and D-algebraic functions whose general coefficients linearly involve hypergeometric terms. For D-finite functions, we present an algorithmic improvement over [8] designed to further simplify outputs not in normal forms. For D-algebraic (and non-D-finite) functions, we detail an ongoing investigation into detecting closed forms represented as linear polynomials in $\mathcal{H}[S(n)]$, where $\mathcal{H}$ is the ring of hypergeometric-type terms, and $S(n)$ is the nth term of Bernoulli or Euler numbers. The presentation is structured into these two distinct parts.

Simplifying FPS outputs

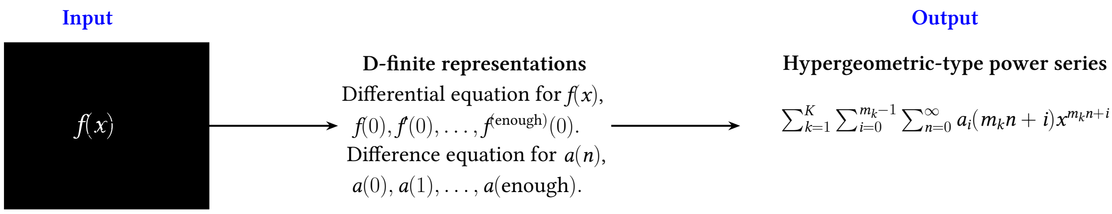

Figure 1: The FPS algorithm for hypergeometric-type power series.

Drawing from [5] and [6], this part of the talk will introduce the computable ring of hypergeometric-type sequences ($\mathcal{H}$) and showcase its properties using our Maple package. Figure 1 provides a high-level overview of the FPS algorithm, with arrows indicating its key steps. This algorithm is designed to take a black-box mathematical expression and first construct a D-finite representation. If successful, it solves the corresponding D-finite recurrence for m-fold hypergeometric term solutions, ultimately constructing the power series as an appropriate linear combination of these terms.

Sometimes, the order of the obtained recurrence equation is too small to enable the algorithm to escape unnecessary splitting fields. The generating function $f(x) := \frac{x\left(-x^4+7x^3+6x^2+7x+5\right)}{(1-x)^4(x^2+x+1)}$ of the OEIS sequence A208946 is a typical example. The FPS algorithm internally solves a 7th-order recurrence equation and returns

$$\mathrm{FPS}(f(x),x,n) = \sum_{n=0}^{\infty}\left(-\frac{2\cos\left(\frac{2n\pi}{3}\right)}{3} - \frac{2\sqrt{3}\,\sin\left(\frac{2n\pi}{3}\right)}{9} + \frac{4n^3}{3} + 2n^2 + n + \frac{2}{3}\right)x^n. \tag{1}$$

An equivalent result is obtained with the Maple command `convert/FormalPowerSeries`, which implements a variant of FPS. Using our software from [5], we compute the normal form

$$\texttt{HyperTypeSeq}: -\texttt{HTS}\left(-\frac{2\cos(\frac{2n\pi}{3})}{3}-\frac{2\sqrt{3}\,\sin(\frac{2n\pi}{3})}{9}, n\right) = -\frac{2\chi_{\{modp(n,3)=0\}}}{3}+\frac{2\chi_{\{modp(n,3)=2\}}}{3}. \tag{2}$$

By the correspondence between hypergeometric-type power series and hypergeometric-type terms, we deduce the simplified closed form below.

$$f(x) = \sum_{n=0}^{\infty}\left(\frac{4n^3}{3}+2n^2+n+\frac{2}{3}\right)x^n - \frac{2}{3}\sum_{n=0}^{\infty}x^{3n}+\frac{2}{3}\sum_{n=0}^{\infty}x^{3n+2}. \tag{3}$$

D-algebraic series solutions of quadratic ODEs

We aim to consider D-algebraic power series whose general coefficients have the closed form:

$$\alpha(n)+\beta(n)\,S(n), \tag{4}$$

where $\alpha(n), \beta(n) \in \mathcal{H}$, and $(S(n))$ is a non-D-finite sequence which has the zero sequence as a subsequence. The target algorithm assumes that $S(n)$ is known. For example, $S(n)$ could be the nth Bernoulli number B_n, which has the following properties.

$$B_0 = 1,\ B_{2n} = \frac{(-1)^{n+1}2\,(2n)!}{(2\pi)^{2n}}\zeta(2n),\ n \geq 1, \tag{5}$$

$$B_{2n+1} = 0,\ n \geq 1. \tag{6}$$

In (5), ζ is the Riemann Zeta function. Such a formula is not supposed to be known; what the algorithm requires is the ability to (efficiently) compute terms of $S(n)$ (B_n in this case). We aim to recover closed forms of power series involving numbers such as Bernoulli and Euler numbers, and potentially discover hidden formulae in the form of (4). At present, we are investigating proofs for the correctness of the results. Indeed, the algorithm combines D-finite and D-algebraic guessing (see [2] to [4]) together with the hypergeometric-type representation algorithm from [5] and [6].

A simple situation corresponds to when $\alpha(n) = 0$ in (4). In [7], we proposed an approach to extend the FPS algorithm for non-D-finite functions that satisfy quadratic differential equations. For $f(x) := \tan(x)$, the algorithm uses the following differential equation to return a recursive formula for the series.

$$y''(x) - 2\,y(x)\,y'(x) = 0. \tag{7}$$

Using quadratic guessing [4], one can obtain the same equation from the first few coefficients of the power series of $f(x)$.

Assuming $f(x) = \sum_{n=0}^{\infty} a_n x^n$, we use D-finite guessing from [3] to construct a holonomic recurrence equation for $\frac{a_{2n-1}}{B_{2n}}$, $n \geq 1$. Using the guessed recurrence and the initial values we detect the identity

$$a_n = \frac{(-1)^{\frac{n}{2}-\frac{1}{2}}\,2^{n+1}\,(2^{n+1}-1)}{(n+1)\,!}\,\chi_{\{n\equiv 1 \mod 2\}}B_{2n},\ \ n \geq 1, \tag{8}$$

where $\chi_{\{n\equiv 1 \mod 2\}}$ is our mathematical notation of interlacement, implemented in Maple with the notation given in (2). The final step is to use quadratic guessing to construct (7) from the first terms of the right-hand side in (8) and verify that $f(x)$ satisfies it. This is indeed successful, and we deduce the classical formula for the tangent power series.

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 4^{n+1}(4^{n+1}-1)B_{2(n+1)}}{(2(n+1))!} x^{2n+1}. \tag{9}$$

When $\alpha(n) \neq 0$, we investigate shape lemmas related to Ritt factorizations of differential polynomials. Our goal is to construct a specific type of algebraic differential equations satisfied by sums of D-finite and D-algebraic (and not D-finite) functions. This enables us to determine $\alpha(n)$ from the nonlinear recursion of $\alpha(n) + \beta(n)\, S(n)$ and the indices where $S(n) = 0$. A particular challenge arises when $\alpha(n)$ vanishes at the same indices as $S(n)$, making it difficult to pinpoint the initial terms of $\alpha(n)$. We mention the work of Gao and Zang [1] on the decomposition of differential polynomials, which could be relevant for our context and warrants further exploration.

Keywords: Hypergeometric-type terms, Bernoulli numbers, Euler numbers, quadratic differential equations, guessing.

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References

[1] Gao, X.S., Zhang, M.: Decomposition of differential polynomials with constant coefficients. In: Proceedings of the 2004 international symposium on Symbolic and algebraic computation. pp. 175–182 (2004)

[2] Kauers, M., Koutschan, C.: Guessing with little data. In: Proceedings of the 2022 International Symposium on Symbolic and Algebraic Computation. pp. 83–90 (2022)

[3] Salvy, B., Zimmermann, P.: GFUN: a Maple package for the manipulation of generating and holonomic functions in one variable. ACM Transactions on Mathematical Software (TOMS) **20**(2), 163–177 (1994)

[4] Teguia Tabuguia, B.: Guessing with quadratic differential equations. Software Demo at ISSAC'22. arXiv preprint arXiv:2207.01037 (2022)

[5] Teguia Tabuguia, B.: Computing with hypergeometric-type terms. ACM Communication in Computer Algebra **58**(2), 23 – 26 (2024)

[6] Teguia Tabuguia, B.: Hypergeometric-type sequences. Journal of Symbolic Computation **125**, 102328 (2024).

[7] Teguia Tabuguia, B., Koepf, W.: On the representation of non-holonomic univariate power series. Maple Transactions **2**(1) (2021)

[8] Teguia Tabuguia, B., Koepf, W.: Symbolic conversion of holonomic functions to hypergeometric type power series. Programming and Computer Software **48**(2), 125–146 (2022)