Combining Sparsity and Symmetry Exploitation for SOS-Certificates

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SOS-Certificates

Let A be a graded real *-algebra. Given $f \in A$, a sums-of-squares (SOS) certificate is a representation of f in the form

$$f = \sum_t q_t\, q_t^*$$

with finitely many $q_t \in A$.

As a historical motivation, we take Hilbert's proof from 1888 that every nonnegative homogeneous polynomial $f \in A = \mathbb{R}[X] = \mathbb{R}[X_1, \ldots, X_n]$ (with * the identity) of degree $2r$ can be written as a sum of squares if and only if $(n, 2r) \in \{(2, 2r), (n, 2), (3, 4)\}$. The first example however of a nonnegative polynomial which is not a sum of squares was given later in 1967 by Motzkin, indicating that it is far from trivial to find an explicit SOS-certificate (or to disprove its existence).

Theorems that state the existence of an SOS-certificate are called Positivstellensätze, see for example [7] and [9], and enable solving computational problems with techniques from real algebra geometry [3] and [10]. Some applications are polynomial optimization

$$\begin{array}{rllcll} f^* = & \min & f(X) & \geq & \max & \lambda \\ & \text{s.t.} & X \in \mathbb{R}^n & & \text{s.t.} & \lambda \in \mathbb{R}, \\ & & & & & f - \lambda \text{ is SOS in } \mathbb{R}[X] \end{array} \tag{POP}$$

with $f \in \mathbb{R}[X]$, see [2], computing a maximal positive invariant set of a dynamical system $\dot{X}(t) = \mathbf{f}(X(t))$, see [1], or verifying Kazhdan's property (T) for a finitely generated group $\mathfrak{G}$, which holds if and only if

$$\Delta^2 - \lambda\, \Delta \text{ is SOS in } \mathbb{R}[\mathfrak{G}] \tag{T}$$

for some $\lambda > 0$ with Laplacian Δ, see [6].

Computing an explicit SOS representation can give not only the solution to the problem but also an optimizer in which the solution is attained. In practice, this often boils down to solving a semidefinite program (SDP), which is obtained by restricting the degrees of the sums of squares and constructing a hierarchy of numerical bounds up to a satisfying precision. Naturally, these problems become very difficult to handle computationally and tools to gain efficiency whilst preserving numerical accuracy are required.

Symmetry Reduction

Let G be a finite group acting on the algebra A and its subspaces A_r of degree at most r, which are assumed to be finite dimensional. As a vector space, each A_r (or more precisely its complexification) has an isotypic decomposition

$$A_r = \bigoplus_{i=1}^{h} \bigoplus_{j=1}^{m_r^{(i)}} V_j^{(i)},$$

where h is the number of irreducible characters of G and $m_r^{(i)}$ are their multiplicities [11]. A vector space basis admitting this decomposition is called symmetry-adapted. By Schur's Lemma, we may choose a total of $m_r^{(i)}$ distinguished basis elements from the i–th component and denote them by $w_j^{(i)} \in V_j^{(i)}$.

Let $f \in A$ be a G-invariant objective function of degree $2\,r_{\min}$ for which we seek an SOS-certificate in A and denote by $\mathcal{R}^G$ the Reynolds operator. For $r \geq r_{\min}$, we approximate f as

$$f = \sum_{i=1}^{h} \mathcal{R}^G(q_r^{(i)}) \quad \text{with sums of squares} \quad q_r^{(i)} = (\mathbf{w}_r^{(i)})^t \cdot \mathbf{Q}_r^{(i)} \cdot (\mathbf{w}_r^{(i)})^*.$$

Here, $\mathbf{w}_r^{(i)}$ is the vector of basis elements $w_j^{(i)}$, $1 \leq j \leq m_r^{(i)}$, and $\mathbf{Q}_r^{(i)}$ is a Hermitian positive semidefinite matrix, that is, $q_r^{(i)}$ is a sum of squares in the vector space generated by the $w_j^{(i)}$, see [4] and [8].

Adding Term Sparsity

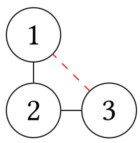
The term sparsity pattern (tsp) is encoded by a graph with nodes given by a basis for A_r. Without going into the technical details of the construction of the edges, one can follow [12] to construct a sequence of binary matrices

$$\mathbf{B}_{r,s}^{(i)} \subseteq \mathbf{B}_{r,s+1}^{(i)} \subseteq \mathbf{B}_{r,s+2}^{(i)} \subseteq \ldots \in \{0,1\}^{m_r^{(i)} \times m_r^{(i)}},$$

such that one only considers sums of squares with term sparsity (TSSOS) of the form

$$q_{r,s}^{(i)} = (\mathbf{w}_r^{(i)})^t \cdot (\mathbf{B}_{r,s}^{(i)} \circ \mathbf{Q}_r^{(i)}) \cdot (\mathbf{w}_r^{(i)})^*.$$

Here, $\circ$ denotes the Hadamard product and s is the sparse order. For example, the graph



is represented by the binary matrix $\mathbf{B} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

and encodes that basis elements $\mathfrak{b}_1$, $\mathfrak{b}_2$, $\mathfrak{b}_3$ appear in the problem data as $\mathfrak{b}_1\,\mathfrak{b}_2$ or $\mathfrak{b}_2\,\mathfrak{b}_3$, but not $\mathfrak{b}_1\,\mathfrak{b}_3$.

Symmetric TSSOS Hierarchy

For POP, we obtain a semidefinite lower bound

$$\begin{aligned} f^* \geq f_{\text{sos}}^{r,s} := \quad \max \quad & \lambda \\ \text{s.t.} \quad & \lambda \in \mathbb{R}, \\ & f - \lambda \in \text{SOS}^G(\mathbf{B}_{r,s}^{(1)}) \oplus \ldots \oplus \text{SOS}^G(\mathbf{B}_{r,s}^{(h)}), \end{aligned}$$

where $\mathrm{SOS}^G(\mathbf{B}_{r,s}^{(i)})$ is the convex cone of sparse G-invariant sums of squares $\mathcal{R}^G(q_{r,s}^{(i)})$.

Theorem. *For fixed $r \geq r_{\min}$, the sequence $(f_{\mathrm{sos}}^{r,s})_{s\geq 1}$ is monotonously nondecreasing and converges in finitely many steps to some $f_{\mathrm{sos}}^{r,*} \leq f^*$. For fixed $s \geq 1$, the sequence $(f_{\mathrm{sos}}^{r,s})_{r\geq r_{\min}}$ is monotonously nondecreasing. Under additional algebraic assumptions and constraints, one has asymptotic convergence $f_{\mathrm{sos}}^{\infty,*} = f^*$.*

Conclusion, Work in Progress, Outlook

By symmetry reduction, a matrix representation of a sum of squares is not of size $\dim(A_r)^2$, but splits into potentially much smaller blocks $\mathbf{Q}_r^{(i)}$ of combined size $(m_r^{(1)})^2 + \ldots + (m_r^{(h)})^2$. By sparsity exploitation, one removes further entries of these matrices according to tsp graphs.

The preprocess of achieving such a reduction involves the computation of a symmetry adapted basis. However, this basis does not depend on the specific form of the objective function, but only on the group G and the degree r. Hence, one such preprocess can be reapplied for multiple problems. Afterwards, computing the reduced SDP is more efficient than the original one.

In the talk, I will quantify these computational gains via benchmarks on a selection of polynomial optimization problems, for which we used the JULIA package TSSOS:

https://github.com/wangjie212/TSSOS

We are currently working on the combination of symmetry with further sparsity types, see [5], and on the generalization of the above convergence result to noncommutative algebras.

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