Two Methods for Proving "Japanese Theorem II" Using Maxima and KeTCindy: An Application of the MNR Method

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Abstract

Many of the plane geometry problems that appear in Wasan(Japanese Mathematics) are beautiful to look at but difficult to solve using computer algebra. In particular, triangular problems involve simultaneous equations with irrational expressions, which are extremely difficult to solve using normal methods. The author therefore devised a method to express the quantities of a triangle using $m = \tan\frac{B}{2}, n = \tan\frac{C}{2}$ and the radius of the inscribed circle r, which he named the MNR method. The author developed the Maxima MNR library and confirmed that it can solve various problems. It is possible to solve even more problems with techniques using quarter angles $M = \tan\frac{B}{4}, N = \tan\frac{C}{4}$.

1 Introduction

Wasan[1] contains various plane geometry problems that are beautiful but difficult to solve. Even if one attempts to solve them using computer algebra, many of them become simultaneous equations that contain irrational expressions, making them difficult to solve using normal methods. In particular, problems involving triangles are almost always unsolvable. Therefore, the author devised a method to express various quantities of a triangle using the tangent of the half angles m, n, and the radius of the inscribed circle r, which he calls the MNR method, and developed a Maxima library for it([7]). KeTCindy([1]), which the authors have been developing since 2014, is a collection of function libraries for the dynamic geometry system Cinderella, and can call Maxima functions. In the following sections, we explain how to solve "Japanese Theorem II"[2] using the MNR method.

[1]Wasan is a Japanese mathematics that developed during the Edo period, from the 17th to the 19th centuries.
[2]Uegaki named it([3],[4]). He called the theorem introduced by [2] "Japanese Theorem I". See [6].

2 Japanese Theorem II

Fig.1 is a mathematical tablet that was displayed at Hakusan Shrine in Niigata Prefecture but was later lost[3]. The problem on the right is called Japanese Theorem II.

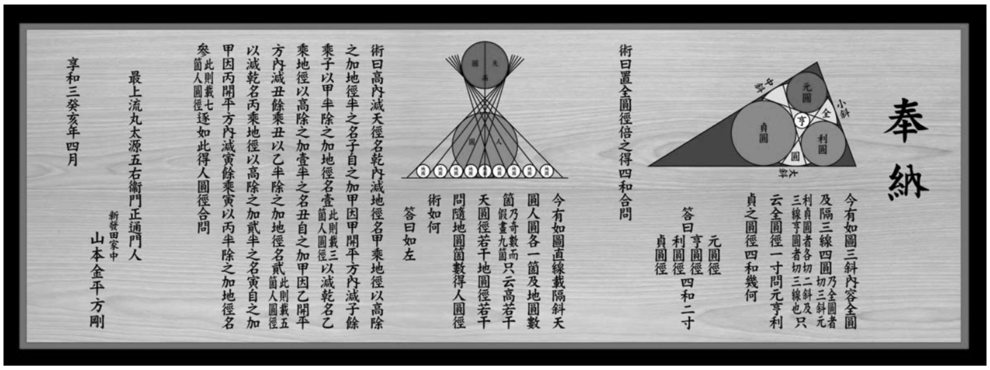


Fig.1 Sangaku Lost from the Hakusan Shrine (Reconstruction)

The letters are written vertically from right to left, and the question, answer, and solution are written in that order. Here, we will explain them. However, the question sentences are paraphrased.

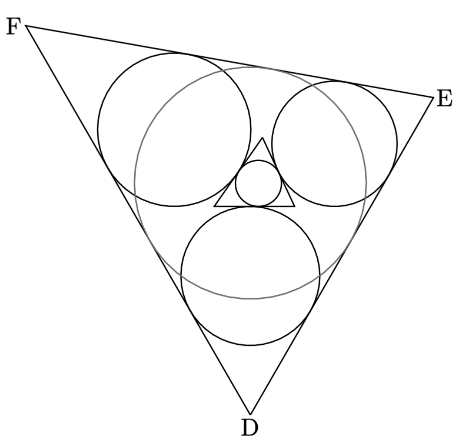
Question: Let the radius of the inscribed circle of the large triangle be R, the radius of the inscribed circle, the radii of the three circumscribed circles of the small triangle be r, r_1, r_2, r_3. Express the sum $r + r_1 + r_2 + r_3$ by R.

Answer: The sum of r, r_1, r_2, r_3 is twice R.

Solution: Double the radius R to get the sum of the radii of the four circles.

2.1 Solution1

Apply the MNR method to the triangle inside. Then the three outer circles become excircles of the triangle, and the theorem can be proven by finding the radius of the incircle of the triangle formed by their common tangents[4].



$$r0 : r$$

$$sum : \frac{(m^2n^2 - n^2 - 2mn - m^2 - 1)r}{mn(mn-1)}$$

$$CR : [-(\frac{(n-m)r}{mn}), -(\frac{(3m^2n^2 + n^2 + m^2 - 1)r}{2mn(mn-1)})]$$

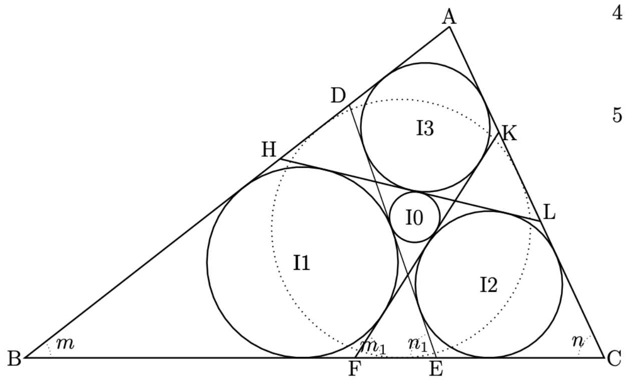
$$R : \frac{(m^2n^2 - n^2 - 2mn - m^2 - 1)r}{2mn(mn-1)}$$

Fig.2 Solution starting from the inner triangle

[3]Wakuta and Togawa created a reconstruction using Adobe Illustrator([5]).
[4]The MNR method supports the inscribed circle and the common tangent.

2.2 Solution2

Put the outer triangle ABC in Fig.3 first as per the question. In that case, the intermediate circles will be inscribed in the pentagon, and the equations for this contain irrational expressions of m, n, therefore, they are unsolvable as is. So, let $M = \tan \frac{B}{4}, N = \tan \frac{N}{4}$. Then, the equations are expressed as rational expressions in M, N, and the solutions can be obtained.



$$4 \quad ansm1 = \sqrt{n^2+1} - n$$

$$ansn1 = \sqrt{m^2+1} - m$$

$$5 \quad r0 = \frac{(M-1)(N-1)(N+M)R}{M^2N^2 + MN^2 + M^2N - 2MN + N + M + 1}$$

$$r1 = \frac{(M-1)(N+1)(MN-1)R}{M^2N^2 + MN^2 + M^2N - 2MN + N + M + 1}$$

$$r2 = \frac{(M+1)(N-1)(MN-1)R}{M^2N^2 + MN^2 + M^2N - 2MN + N + M + 1}$$

$$r3 = \frac{(M+1)(N+1)(N+M)R}{M^2N^2 + MN^2 + M^2N - 2MN + N + M + 1}$$

$$S = r0 + r1 + r2 + r3 = 2R$$

Fig.3 Solution starting from the outer triangle

3 Conclusion

The MNR method is effective when solving plane geometry problems using computer algebra. Further ingenuity may be required to make a system of rational equations. One such ingenuity is the use of the tangent of quarter angles.

References

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[7] S. Takato, H. Makishita, A Method to Prove Japanese Theorems and Others Appeared in Wasan Using Maxima, SCSS 2024, LNAI 14991, 57–78, 2024