# Automated methods applied for the exploration of singularities of some curves

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*In memoriam Josef Böhm, a good friend dedicated to our community*

**Abstract**

We explore singularities of curves defined by geometric constructions using automated methods. A central feature of the work consists in networking between different kinds of software in order to use their respective strengths (dynamic geometry, strong algebraic computations, etc.). The activities have been proposed to in-service teachers learning towards an advanced degree.

## 1 Automated methods

This talk shows reports of an ongoing work about various purely geometric constructions which yield plane algebraic curves, studied in a technology-rich environment.

Automated methods are developed and implemented in software [3], requesting both Dynamic Geometry (for interactive exploration and conjectures) and a Computer Algebra System (for algebraic manipulations). We use GeoGebra[1], and its companion package GeoGebra-Discovery[2]. The 2nd

[1]freely downloadable from http://wwww.geogebra.org)
[2]freely downloadable from https://github.com/kovzol/geogebra-discovery.

one has symbolic implementations of commands, which work numerically in the standard GeoGebra (such as **Locus**, **Envelope**). GeoGebra has an embedded CAS, but when more computational power is needed, we use Maple 2024.

Moreover, it is well-known that software has hard time to plot a curve in a neighborhood of a singular point. Here, GeoGebra-Discovery command **Plot2D** is an efficient tool to overcome this pitfall.

Both parametric presentations and implicit equations are used, switching from parametric to polynomial representation being based on packages for Gröbner bases and Elimination. We wish to emphasize the importance of the networking between different kinds of software, and wish to see more possibilities of communication between them. Since [6], a few advances have been performed, but we have still to copy-paste by hand.

What we present switches mathematics education from the traditional sequence definition-theorem-proof-examples towards a experimental approach: exploration-conjecture-proof-theorem. Of course, this is quite clear for geometry, but can be adapted to other domains. We experienced this with a group of in-service mathematics teachers learning towards a 2nd degree in Mathematics Education. At the end of the course, they expressed their happiness to have discovered a new approach to mathematics.

# 2 Envelopes and offsets

Envelopes of parameterized families of plane curves are important, both for the mathematics involved and for applications [2]. Offsets of a plane curve $\mathcal{C}$ are sometimes considered as a special case of envelopes, but this is not accurate. Due to Euler, offsets are also called parallel curves, a misleading name. Envelopes and offsets are generally different objects, similar only in particular cases, e.g., when the curve $\mathcal{C}$ is smooth and the offset distance small enough. Studying their topology may be fascinating, as the topology of an offset can be more complicated than that of its progenitor [1].

Let be given a plane curve $\mathcal{C}$ and a positive real number. At every non singular point $A$ of $\mathcal{C}$, we define a unit normal vector $\overrightarrow{N_A}$. The offset of $\mathcal{C}$ at distance $d$ is the geometric locus of points $M$ such that $\overrightarrow{AM} = d \cdot \overrightarrow{N_A}$. Two components can be determined, which may have different topologies; see Figure 1 for an ellipse whose equation is $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and $d = 1, 1.7, 2$.

Some vectors, obtained with *Trace On*, have been kept in the plot in order to emphasize the dynamic aspect of the work.

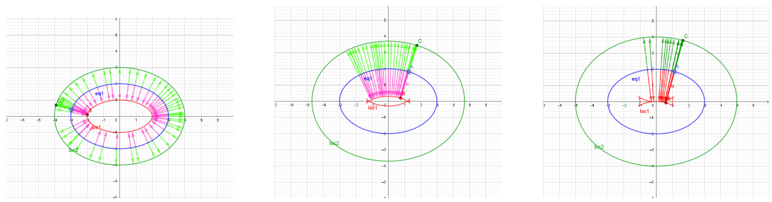

Figure 1: The two components of an offset of an ellipse

We present also a study of envelopes of families of circles centered on the so-called kiss curve and offsets of this curve, observing the differences between constructs. Some are displayed in the following figures. Figure 2(a) and (b) enhances the cusps, and Figure 2(c) the self-intersection (crunodes).

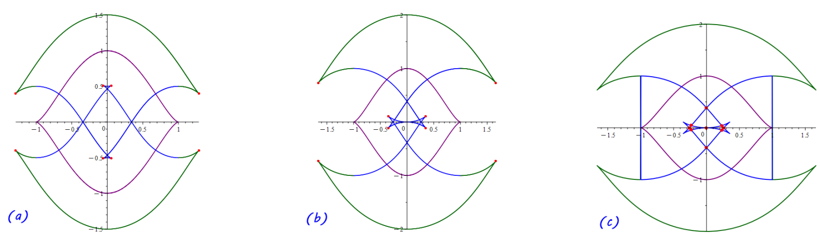


Figure 2: Singular points of an offset

# 3 Pedal curves

Another example that we present is the construction of pedal curves. Let be given a plane curve $\mathcal{C}$ and a point $A$. The *pedal* of $\mathcal{C}$ with respect to $A$ (called the *pole*) is the geometric locus of the feet of the lines passing by $A$ and perpendicular to the tangents to $\mathcal{C}$. Here too,GeoGebra's **Locus** commands are useful. Their output is a plane curve, which is sometimes not totally accurate, as in Figure 3(a). Figure 3(b) shows the efficiency of GeoGebra-Discovery command **Plot2D** to fulfill the gaps and provide an accurate plot, emphasizing the singular points.

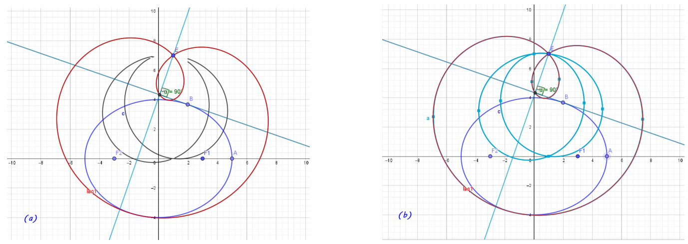

Figure 3: Pedal curve of a canonical ellipse with respect to an external point

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