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# Familiarizing students with definition of Lebesgue measure using Mathematica - some examples of calculation directly from its definition

Włodzimierz Wojas[1], Jan Krupa[1] Jarosław Bojarski[1]

*"Young man, in mathematics you don't understand things. You just get used to them"*

John von Neumann

The following definitions and lemmas we will use in our talk (see [9, 3]):

**Rectangles.** A closed rectangle $R$ in $\mathbb{R}^d$ is given by the product of d one-dimensional closed and bounded intervals: $R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$, where $a_j \leq b_j$ are real numbers, $j = 1, 2, \ldots, d$. In other words, we have $R = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : a_j \leq x_j \leq b_j$ for all $j = 1, 2, \ldots, d\}$. We remark that in our definition, a rectangle is closed and has sides parallel to the coordinate axis. In $\mathbb{R}$, the rectangles are precisely the closed and bounded intervals, while in $\mathbb{R}^2$ they are the usual four-sided rectangles. In $\mathbb{R}^3$ they are the closed parallelepipeds.

An open rectangle is the product of open intervals, and the interior of the rectangle $R$ is then

$$(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_d, b_d).$$

We say that the lengths of the sides of the rectangle $R$ (open or closed) are $b_1 - a_1, \ldots, b_d - a_d$. The volume of the rectangle $R$ (open or closed) is denoted by $\mathrm{vol}\,(R)$, and is defined to be $\mathrm{vol}\,(R) = (b_1 - a_1) \cdots (b_d - a_d)$.

A union of rectangles is said to be almost disjoint if the interiors of the rectangles are disjoint. In this presentation, coverings by rectangles play a major role, so we isolate here three important lemmas.

**Lemma 1.** *If a closed rectangle is the almost disjoint union of finitely many other closed rectangles, say $R = \bigcup_{k=1}^{N} R_k$, then $\mathrm{vol}\,(R) = \sum_{k=1}^{N} \mathrm{vol}(R_k)$.*

**Lemma 2.** *If $R, R_1, \ldots, R_N$ are closed rectangles, and $R \subset \bigcup_{k=1}^{N} R_k$, then*

$$\mathrm{vol}(R) \leq \sum_{k=1}^{N} \mathrm{vol}(R_k).$$

**Lemma 3.** *If $R_1, \ldots, R_N$ are almost disjoint closed rectangles, $Q_1, \ldots, Q_M$ are some closed rectangles and $\bigcup_{k=1}^{N} R_k \subset \bigcup_{k=1}^{M} Q_k$, then $\sum_{k=1}^{N} \mathrm{vol}(R_k) \leq \sum_{k=1}^{M} \mathrm{vol}(Q_k)$.*

Applying the previous Lemma 3 we can prove:

**Lemma 4.** *If $R_1, \ldots, R_N$ are almost disjoint closed rectangles, $Q_k$, $k = 1, 2, \ldots$ are some closed rectangles and $\bigcup_{k=1}^{N} R_k \subset \bigcup_{k=1}^{\infty} Q_k$, then $\sum_{k=1}^{N} \mathrm{vol}(R_k) \leq \sum_{k=1}^{\infty} \mathrm{vol}(Q_k)$.*

*Proof.* Let $A = \bigcup_{k=1}^{N} R_k$. For a fixed $\varepsilon > 0$ we choose for each $k$ an open rectangle $P_k$ which contains $R_k$, and such that $\mathrm{vol}(P_k) \leq (1+\varepsilon)\,\mathrm{vol}(Q_k)$. From the open covering $\bigcup_{k=1}^{\infty} P_k$ of the compact set $A$, we may select a finite subcovering which, after possibly renumbering the rectangles, we may write as $A \subset \bigcup_{k=1}^{M} P_k$. Taking the closure of the rectangles $P_k$, we may apply Lemma 3 to conclude that for any $\varepsilon$ we have: $\sum_{k=1}^{N} \mathrm{vol}(R_k) \le \sum_{k=1}^{M} \mathrm{vol}(P_k) \le (1+\varepsilon)\sum_{k=1}^{M} \mathrm{vol}(Q_k) \le (1+\varepsilon)\sum_{k=1}^{\infty} \mathrm{vol}(Q_k)$. Since $\varepsilon$ is arbitrary, we find that $\sum_{k=1}^{N} \mathrm{vol}(R_k) \le \sum_{k=1}^{\infty} \mathrm{vol}(Q_k)$. □

**Definition 1.** (see [3, 7, 8, 9]) Let $(\mathbb{R}^2, \mathfrak{M}, m)$ be measure space, where $\mathfrak{M}$ is $\sigma-$ algebra of Lebesgue measurable subsets in $\mathbb{R}^2$, and $m$- Lebesgue measure on $\mathbb{R}^2$. The measure $m$ for any $A \in \mathfrak{M}$ is defined by the following formula:

$$m(A) = \inf\Big\{ \sum_{j=1}^{\infty} \mathrm{vol}\,(R_j) : A \subset \bigcup_{j=1}^{\infty} R_j,\;\; R_j \text{ is closed rectangle in } \mathbb{R}^2, j \in \mathbb{N}\Big\}. \tag{1}$$

In this talk we present some examples of calculation the Lebesgue measure of some subsets of $\mathbb{R}^2$ directly from definition 1. We will consider the following subsets of $\mathbb{R}^2$: $\{(x,y) \in \mathbb{R}^2 : 0 \le y \le x^2, 0 \le x \le 1\}$, $\{(x,y) \in \mathbb{R}^2 : 0 \le y \le \sin x, 0 \le x \le \pi/2\}$, $\{(x,y) \in \mathbb{R}^2 : 0 \le y \le \exp(x), 0 \le x \le 1\}$, $\{(x,y) \in \mathbb{R}^2 : 0 \le y \le \ln(1-2r\cos x + r^2), 0 \le x \le \pi\}$, $r > 1$. We calculate sums, limits and plot graphs and dynamic plots of needed sets and unions of rectangles sums of which volumes approximate Lebesgue measure of the sets, using Mathematica. The title of this talk is very similar to the title of author's article [1] which deals with definition of Lebesgue integral but our talk deals with definition of Lebesgue measure instead.

Let $A$ be the set in $R^2$ bounded by curves: $y = x^2$, $y = 0$, $x = 1$, which means that $A = \big\{(x,y) \in \mathbb{R}^2 : 0 \le y \le x^2, 0 \le x \le 1\big\}$. Let us calculate Lebesgue measure of $A$ using only formula (1) from definition 1 and lemma 4

For $n \in \mathbb{N}$ define: $\bar{R}_j^n = [\frac{j-1}{n}, \frac{j}{n}] \times [0, \frac{j^2}{n^2}]$, $j = 1, 2, \ldots n$ and $\underline{R}_j^n = [\frac{j-1}{n}, \frac{j}{n}] \times [0, \frac{(j-1)^2}{n^2}]$, $j = 1, 2, \ldots n$.

**Step 1.** $\sum_{j=1}^{n} \mathrm{vol}(\bar{R}_j^n) = \sum_{j=1}^{n} \frac{1}{n}\frac{j^2}{n^2} = \frac{1}{n^3}\sum_{j=1}^{n} j^2 = \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} =$
$= \frac{1}{n^2}\frac{(n+1)(2n+1)}{6} \to \frac{1}{3}$. Hence because $A \subset \bigcup_{j=1}^{n} \bar{R}_j^n$, we have:

$$\begin{aligned} m(A) =& \inf\Big\{ \sum_{j=1}^{\infty} \mathrm{vol}(R_j) : A \subset \bigcup_{j=1}^{\infty} R_j,\;\; R_j \text{ is closed rectangle in } \mathbb{R}^2, j \in \mathbb{N}\Big\} \\ &\le \inf\Big\{ \sum_{j=1}^{n} \mathrm{vol}(\bar{R}_j^n) : j \in \mathbb{N}\Big\} \le \frac{1}{3}. \end{aligned} \tag{2}$$

**Step 2.** Note that:

$$\begin{aligned} \sum_{j=1}^{n} \mathrm{vol}(\underline{R}_j^n) = \sum_{j=1}^{n} \frac{1}{n}\frac{(j-1)^2}{n^2} = \frac{1}{n^3}\sum_{j=1}^{n-1} j^2 = \\ = \frac{1}{n^3}\frac{(n-1)n(2n-1)}{6} = \frac{1}{n^2}\frac{(n-1)(2n-1)}{6} \to \frac{1}{3}. \end{aligned} \tag{3}$$

We consider an arbitrary covering $A \subset \bigcup_{j=1}^{\infty} R_j$ by closed rectangles.
Hence because $\bigcup_{j=1}^{n} \underline{R}_j^n \subset A \subset \bigcup_{j=1}^{\infty} R_j$ for any $n$ from lemma 4 we have:

$\sum_{j=1}^{n} \mathrm{vol}(\underline{R}_j^n) \leq \sum_{j=1}^{\infty} \mathrm{vol}(R_j)$.
Applying the formula 3 we have: $\sum_{j=1}^{\infty} \mathrm{vol}(R_j) \geq \lim_{n\to\infty} \sum_{j=1}^{n} \mathrm{vol}(\underline{R}_j^n) = \frac{1}{3}$. Consequently,

$$\begin{aligned} m(A) = &\inf\left\{ \sum_{j=1}^{\infty} \mathrm{vol}(R_j) : A \subset \bigcup_{j=1}^{\infty} R_j,\ \ R_j \ \text{ is closed rectangle in } \mathbb{R}^2, j \in \mathbb{N} \right\} \\ &\geq \frac{1}{3}. \end{aligned} \tag{4}$$

From inequalities (2) and (4) we have $m(A) = \frac{1}{3}$ directly from formula 1 from definition 1.

**Keywords:** Higher education, Lebesgue measure, Application of CAS, Mathematica, Mathematical didactics

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[1]Department of Applied Mathematics
Warsaw University of Life Sciences (SGGW)
Warsaw Poland
wlodzimierz_wojas@sggw.pl
jan_krupa@sggw.pl
jaroslaw_bojarski@sggw.pl