How a code for verifying our conjecture opened new directions -Abstract

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A common tool used in enumerating combinatorial objects is the generating function, which is an algebraic way of presenting all the enumerative information in one glance. When the generating function is a polynomial which can be factorized, the factorization may provide important information about the objects themselves. Nowadays many mathematicians use computer code to test their conjectures before attempting to prove them in a rigorous form. While trying to find a closed formula for the length function of a certain group of symmetries, we used a Sage code to obtain a polynomial generating function. When we then used Mathematica to factorize this polynomial, the results provided us with a very significant insight: the formula we were looking for must consist of two parts, corresponding to a specific known decomposition of the group into cosets.

1 Complex reflection groups

Let S_n be the symmetric group on n letters $1, \ldots, n$. For $\sigma \in S_n$ with $\sigma(i) = r_i$, $1 \leq i \leq n$, we denote by $((a_1, \ldots, a_n), (r_1, \ldots, r_n)$ the $n \times n$ monomial matrix with non-zero entries a_i in the $(i, r_i)-$ positions. For $p|m$ in $\mathbb{N}$, we set:

$$G(r, p, n) = \{((a_1, \ldots, a_n), \sigma) \in GL_n(\mathbb{C}) \mid a_i^r = 1\}.$$

We denote an element of $G(r, p, n)$ in a more concise manner:

$$(\sigma, k) = a_1^{k_1} \cdots a_n^{k_n}$$

for $\sigma = a_1 \cdots a_n$ and $k = (k_1, \ldots, k_n)$.

Example 1.1.

$$\pi = (312, (1, 3, 3)) = 3^1 1^3 2^3$$

Various sets of generators have been defined for complex reflection groups but (as far as we know), no length function has been formulated.

In a separate paper [1] we provide such a function for the case of $G(r, r, n)$ with a specific choice of generating set proposed by Shi. (See [2]).

1.1 Shi's Generators for $G(r, r, n)$

For each $i \in \{1, \ldots, n-1\}$ let $s_i = (i, i+1)$ be the well-known adjacent transpositions generating S_n.

Define $t_0 = (1^{r-1}, n^1)$. In [2] the following theorem is proven.

Theorem 1.2. *The set* $\{t_0, s_1, \ldots, s_{n-1}\}$ *generates* $G(r, r, n)$.

After we found a length function for the elements of the group $G(r, r, n)$, we proceeded to seek a generating function. In order to be able to get a grasp on the form that generating function should take, we composed a simple Sage program which went over all the elements of $G(r, r, n)$ for some small values of r and n and calculated the length, using the length function we had discovered. When we used the Mathematica program to factor the resulting polynomial, we found out that in all the cases which had been checked, the factor $[n]_q! = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$ appeared. Here are two examples of the factorizations we have obtained:

Example 1.3.

$$G_{4,4,4}(q) = [4]_q!(1+2q^2+3q^3+4q^4+5q^5+7q^6+8q^7+10q^8+12q^9+7q^{10}+3q^{11}).$$

Example 1.4.

$$G_{6,6,3}(q) = [3]_q!(1+q+2q^2+2q^3+3q^4+3q^5+4q^6+4q^7+5q^8+5q^9+6q^{10}).$$

Since $[n]_q!$ is the generating function of the length function of S_n, these and other examples led us to the conclusion that the correct way of presenting the length function for the elements of $G(r, r, n)$ must be based on a decomposition of $G(r, r, n)$ into cosets of S_n.

In [1] we provide the following length function for $G(r, r, n)$.

Theorem 1.5. *Let* $\pi = a_1^{k_1} \cdots a_n^{k_n} \in G(r, r, n)$.

Write $\pi = u \cdot \sigma$ *where* $u \in S_n$ *and* σ *is the minimal length representative. Then:* $\ell(\pi) = \sum_{1 \le i < j \le n} |k_j - k_i| - noninv(k) + inv(u)$

References

[1] E. Bagno and M. Novick, *A length function for the complex reflection group* $G(r, r, n)$, in preparation.

[2] J. Y. Shi, *Certain imprimitive reflection groups and their generic versions*, Transcations of the A.M.S., Vol. 364, No. 5, pp. 2115-2129.