# Prove Inequalities by Solving Maximum/Minimum Problems Using a Computer Algebra System

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**Abstract**

This presentation offers an alternative to traditional approaches to proving non-trivial inequalities, such as applying AM-GM, Cauchy, Hölder and Minkowski inequalities. This alternative approach, as demonstrated by various examples, establishes the validity of an inequality through solving a maximization/minimization problem by commonly practiced procedures in Calculus. Since the procedures are algorithmic, a Computer Algebra System (CAS) can carry out the computation efficiently.

**Keywords**
Computer Aided Proofs in Analysis, Inequality, Calculus, Computer Algebra System, CAS

## 1 Introduction

It is common knowledge that among various quantities of certain kind, if one quantity $\alpha$ is larger than others, then $\alpha$ is the maximum of the said kind. If $\alpha$ is less than others, then it is the minimum.

Expressed in mathematical inequality,

$$y \leq \alpha \tag{1}$$

indicates that $\alpha$ is the *greatest* of all $y$'s, thus the *maximum* of $y$.

Similarly, inequality

$$y \geq \alpha \tag{2}$$

indicates that $\alpha$ is the *least* of all $y$'s, thus the *minimum* of $y$.

The above definitions of maximum and minimum imply that if an inequality is in the form of (1) or (2), with a quantity taking the place of $y$, then the quantity has $\alpha$ as its extreme. Hence, proving such inequality is equivalent to solving a maximization/minimization problem and showing the extreme of the quantity is indeed $\alpha$.

The algorithmic procedure of finding maxima/minima is well established. It is based on a number of theorems from Calculus.

For functions with a single variable, the following theorem is well-known.

**Theorem 1** *Let $f(x)$ be a continuous and twice differentiable function in a domain $D$, and suppose $f'(x_0) = 0$. Then at the point $x_0$, we have*

***Case 1.*** $f_{xx} > 0 \Rightarrow f$ *has a local minimum.*

***Case 2.*** $f_{xx} < 0 \Rightarrow f$ *has a local maximum.*

For two-variable functions, the appropriate generalization of this result is given in the following theorem, which is a standard result in multi-variable Calculus.

**Theorem 2** *Let $f(x, y)$ be a continuous and twice differentiable function in a domain $D$, and suppose $\nabla f(x_0, y_0) = 0$. Then at the point $(x_0, y_0)$, we have*

**Case 1.** $f_{xx}f_{yy} - f_{xy}^2 > 0,\ f_{xx} > 0 \Rightarrow f$ *has a local minimum.*

**Case 2.** $f_{xx}f_{yy} - f_{xy}^2 > 0,\ f_{xx} < 0 \Rightarrow f$ *has a local maximum.*

**Case 3.** $f_{xx}f_{yy} - f_{xy}^2 < 0 \Rightarrow f$ *has neither a local maximum nor minimum.*

## 2 Examples

We will now illustrate our approach through various examples.

**Example 1** *For $x \in (0, 1)$, prove:* $x(1 - x^2) \le \dfrac{2\sqrt{3}}{9}$.

**Example 2** *For $a, b \in \mathbb{R}^+$, and $a + b = 1$, prove:* $\sqrt{a + \dfrac{1}{2}} + \sqrt{b + \dfrac{1}{2}} \le 2$.

**Example 3** *Given $p^3 + q^3 = 2$, $p, q \in \mathbb{R}$, prove:*

1. $p + q \le 2$.
2. $pq \le 1$.

**Example 4** *For $x, y \in \mathbb{R}$, prove:* $x^2 + y^2 + 1 \ge xy + x + y$.

**Example 5** *For $a, b, c \in \mathbb{R}^+$, and $a + b + c = 1$, prove:*

1. $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} \ge 9$.
2. $\dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2} \ge 27$.
3. $a^2 + b^2 + c^2 \ge \dfrac{1}{3}$.

**Example 6** *For $a, b, c \in \mathbb{R}^+$, and $a + b + c = 1$, prove:* $\dfrac{1}{b+c} + \dfrac{1}{a+c} + \dfrac{1}{a+b} \ge \dfrac{9}{2}$.

**Example 7** *For $x, y, z \in \mathbb{R}^+$, and $x^2 + y^2 + z^2 = 1$, prove:* $\dfrac{yz}{x} + \dfrac{xz}{y} + \dfrac{xy}{z} \ge \sqrt{3}$.

**Example 8** *For $x, y, z \in \mathbb{R}^+$, prove:* $x(x - y)(x - z) + y(y - x)(y - z) + z(z - x)(z - y) \ge 0$.

Certain inequalities may not be in a form suitable for our approach at first glance, but can still be proved by solving a maximum/minimum problem first, followed by an extra step.

**Example 9** *For $x, y, z \in \mathbb{R}$, and $x + y + z = 1$, prove:* $xy + yz + xz < \dfrac{1}{2}$.

**Example 10** *For $a, b, c \in \mathbb{R}^+$, and $a + b + c = 1$, prove:* $\sqrt{4a + 1} + \sqrt{4b + 1} + \sqrt{4c + 1} < 5$.