Factoring analytic polynomials and non-standard Cauchy-Riemann conditions

Extended Abstract

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The well known Cauchy-Riemann (in short: C-R) equations provide necessary and sufficient conditions for a complex function $f(z)$ to be holomorphic (c.f. [4], [1]). One traditional framework to introduce the C-R conditions is through the consideration of *harmonic conjugates*, $\{u(x, y), v(x, y)\}$, as the real and imaginary parts of a holomorphic function $f(z)$, after performing the substitution $z \to x+iy$, yielding $f(x+iy) = u(x, y)+i\,v(x, y)$. The Cauchy-Riemann conditions are a cornerstone in Complex Analysis and an essential ingredient of its many applications to Physics, Engineering, etc. Computer Algebra seems, at first glance, to be quite alien to this context.

In our contribution we would like to show that Computer Algebra can be useful in the study of C-R equations and harmonic conjugates. We will consider two different, but related, issues: first, we will address the specific factorization properties of conjugate harmonic polynomials and, second, we will attempt generalizing C-R conditions by replacing the real/complex framework by some more general field extensions. Let us briefly describe both topics in what follows.

An *analytic polynomial* (a terminology taken from the popular textbook in Complex Analysis, see e.g. [1]), is a bivariate polynomial $P(x, y)$, with complex coefficients, which arises by substituting $z \to x + iy$ on a univariate polynomial $p(z) \in \mathbb{C}[z]$, i.e. $p(z) \to p(x + iy) = P(x, y)$. In our work we have studied the factorization properties of analytic polynomials, showing, among other remarkable facts, that *conjugate harmonic polynomials can not have a common factor*. This is a quite basic result, but we were not able to find it in the consulted bibliography, probably because it requires an algebraic approach which is usually missing in the Complex Variables framework. On the other hand we can generalize this result from polynomials to other functions (several variables, germs of holomorphic functions at a point, entire functions), all of them having in common being elements of rings with some good factorization properties.

A computational relevant context (and in fact our original motivation) of our work about the factorization of harmonic polynomials is the following situation. Consider a rational function $f(\mathbf{z}) \in \mathbb{C}(\mathbf{z})$ in several complex variables and with complex coefficients, then perform the substitution $\mathbf{z} = \mathbf{x}+i\,\mathbf{y}$ and compute the real and imaginary parts of the resulting *analytic rational function* $f(\mathbf{x}+i\,\mathbf{y}) = u(\mathbf{x}, \mathbf{y})+i\,v(\mathbf{x}, \mathbf{y})$. These two rational functions in $\mathbb{R}(\mathbf{x}, \mathbf{y})$ involve, usually, quite huge expressions, so it is reasonable to ask if there is a possibility of simplifying them by canceling out some common factors of the involved numerators and denominators. Moreover, such functions appear quite naturally when working

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with complex parametrizations of curves [2], [3], and the key to show that some time-consuming steps can be avoided is, precisely, the analysis of the potential common factors for the two numerators of u, v. Learning about factorization properties of harmonic polynomials is useful in this respect.

In fact, as a consequence of our study we can prove here that that the assertion $\gcd(\mathrm{numer}(u), \mathrm{numer}(v)) = 1$ holds under reasonable assumptions and also that, if a rational function $f(\mathbf{z})$ in *prime* (also called *irreducible*) form is given, then the standard way of obtaining u and v yields also rational functions in prime form, i.e. not simplifiable.

As stated above, a second goal of our contribution deals with generalizing C-R conditions when suitably replacing the pair real/complex numbers by some other field extension. For a simple example, take as base field $\mathbb{K} = \mathbb{Q}$ and then $\mathbb{K}(\alpha)$, with α such that $\alpha^3 + 2 = 0$. Then we will consider polynomials (or more complicated functions) $f(z) \in \mathbb{K}(\alpha)[z]$ and perform the substitution $z = x_0 + x_1\alpha + x_2\alpha^2$, yielding $f(x_0 + x_1\alpha + x_2\alpha^2) = u_0(x_0, x_1, x_2) + u_1(x_0, x_1, x_2)\alpha + u_2(x_0, x_1, x_2)\alpha^2$, where, $u_i \in \mathbb{K}[x_0, x_1, x_2]$. Finally, we will like to find the necessary and sufficient conditions on a collection of polynomials $\{u_i(x_0, x_1, x_2)\}_{i=0,1,2}$ to be, as above, the *components* of the expansion of a polynomial $f(z)$ in the given field extension.

More generally, suppose $\mathbb{K}$ is a field, $\overline{\mathbb{K}}$ is the algebraic closure of $\mathbb{K}$, and α is an algebraic element over $\mathbb{K}$ of degree $r + 1$. In this context we proceed, first, generalizing the concept of analytic polynomial, as follows:

Definition. A polynomial $p(x_0, \ldots, x_r) \in \mathbb{K}(\alpha)[x_0, \ldots, x_r]$ is called *analytic* if there exists a polynomial $f(z) \in \overline{\mathbb{K}}[z]$ such that

$$f(x_0 + x_1\alpha + \cdots + x_r\alpha^r) = p(x_0, \ldots, x_r).$$

We say that f is the *generating polynomial* of p. An analytic polynomial can be uniquely written as

$$p(x_0, \ldots, x_r) = u_0(x_0, \ldots, x_r) + u_1(x_0, \ldots, x_r)\alpha + \cdots + u_r(x_0, \ldots, x_r)\alpha^r,$$

where $u_i \in \mathbb{K}[x_0, \ldots, x_r]$. The polynomial u_i are called the $\mathbb{K}$–*components* of $p(x_0, \ldots, x_r)$.

We denote the ring of (α-)analytic polynomials by $\mathcal{A}_{\mathbb{K}(\alpha)}[x_0, \ldots, x_r]$. The main result in this setting is the following analogue of C-R conditions:

Theorem Let $u_0, \ldots, u_r$ be the $\mathbb{K}$–components of a polynomial $p(x_0, \ldots, x_r) \in \mathcal{A}_{\mathbb{K}(\alpha)}[x_0, \ldots, x_r]$, and let $\alpha^{r+i} = a_{i,0} + a_{i,1}\alpha + \cdots + a_{i,r}\alpha^r$, with $a_{i,j} \in \mathbb{K}$ and

$i = 1, \ldots, r$. It holds that

$$\nabla u_i = \begin{pmatrix} 0 & \overset{i)}{\cdots} & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & a_{1,i} \\ \vdots & \iddots & & & & & \iddots & \vdots \\ 1 & & \cdots\cdots & & 0 & a_{1,i} & \cdots & a_{i,i} \\ 0 & & \cdots\cdots & & a_{1,i} & a_{2,i} & \cdots & a_{i,i} \\ \vdots & & & \iddots & & \cdots\cdots & & \vdots \\ \vdots & & \iddots & & & & & \vdots \\ 0 & a_{1,i} & a_{2,i} & & & \cdots\cdots & & a_{r,i} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial u_0}{\partial x_0} \\ \vdots \\ \vdots \\ \frac{\partial u_r}{\partial x_0} \end{pmatrix}, \quad i = 0, \ldots, r$$

where ∇u_i denotes the gradient of $u_i(x_0, \ldots, x_r)$. And, conversely, if these equations hold among a collection of polynomials u_i, then they are the $\mathbb{K}$-components of an analytic polynomial.

As expected, the above statement gives, in the complex case, the well known Cauchy–Riemann conditions. In fact, let $\mathbb{K} = \mathbb{R}$, $\alpha = i$, and $P(x, y) \in \mathbb{C}[x_0, x_1]$ be an analytic polynomial. If u_0, u_1 are the real and imaginary parts of P, the Theorem states that

$$\nabla u_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial u_0}{\partial x_0} \\ \frac{\partial u_1}{\partial x_0} \end{pmatrix}, \quad \nabla u_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial u_0}{\partial x_0} \\ \frac{\partial u_1}{\partial x_0} \end{pmatrix}$$

which is a matrix form expression of the classic C-R equations:

$$\frac{\partial u_0}{\partial x_1} = -\frac{\partial u_1}{\partial x_0}, \quad \frac{\partial u_1}{\partial x_1} = \frac{\partial u_0}{\partial x_0}$$

It might be interesting to remark that the square matrix expressing the C-R conditions in the Theorem above, is a Hankel matrix ([5]), an ubiquitous companion of Computer Algebra practitioners.

References

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