Nonlinear Differential Equations with Inexactly Known Boundary Conditions and Gröbner bases

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An algorithm is presented for constructing approximate solutions of boundary problems for second-order differential equations of the form

$$\Phi(y'', y', x) = 0, \quad y(a) = y_a, \quad y_b = y_b\,, \tag{1}$$

where $y(x)$ is a real function, Φ is a polynomial in its arguments over rationals, and the numbers a, b, y_a, y_b are also rational. Here y_a or y_b or both of them are assumed to be known with some reasonably small errors. The algorithm is based on the use of the quadratic penalty functions for the approximately given boundary conditions and solving the corresponding unconditional extremum problem. In particular, if equation $\Phi(y'', y', x) = 0$ admits the variational formulation $(\phi_{y'})' - \phi_y = 0$, and both y_a, y_b are inexactly known, then an approximate solution of (1) can be constructed from the condition

$$F(\{c_i\}, \alpha, \beta) = \int_a^b \phi(\tilde{y}'', \tilde{y}', x)dx + \alpha[\tilde{y}(a) - y_a]^2 + \beta[\tilde{y}(b) - y_b]^2 \implies \min_{\{c_i\}},$$

$$y \approx \tilde{y} = \sum_{i=1}^{n} c_i v_i(x)\,,$$

The arising system of nonlinear algebraic equations in the coefficients c_i for some appropriate set of basic functions $v_i(x)$ is solved by construction of a lexicographical Gröbner basis. It is shown that the construction of such a basis allows one to develop a perturbation scheme in the inverse degrees of the penalty parameters α and β. The related computational aspects for parametric Gröbner bases are considered. The proposed algorithm is illustrated by an example of the boundary problem with the use of computer algebra system Reduce. The obtained accuracy is analysed in comparison with some other methods used to solve that particular boundary problem.