Solving Linear Differential Equations with Polynomial Coefficients in Some Power Series Classes

Sergei A. Abramov[1]
Computer Center of the Russian Academy of Science
Vavilova 40, Moscow 117967, Russia
abramov@ccas.ru, sabramov@cs.msu.su

Let a linear homogeneous ordinary differential equation $Ly = 0$ with polynomial coefficients be given. First we demonstrate how to find all points a (ordinary or singular of the given equation) and all formal power series

$$y_a(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

which satisfy $Ly_a(x) = 0$ and whose coefficients a_n have a "nice" explicit representation in terms of n. We consider the cases where a_n is a polynomial or a rational function of n, and where a_n is a hypergeometric term (i.e., a_{n+1}/a_n is a rational function of n). Hypergeometric series are in the latter class of power series.

Then we introduce the notion of the m-sparse power series (e.g., expanding $\sin x$ and $\cos x$ at $x = 0$ gives 2-sparse power series: a coefficient a_n of the series can be nonzero only if $remainder(n, 2)$ is equal to a fixed number). We consider the problem of finding all m-points of a linear ordinary differential equation $Ly = 0$ with polynomial coefficients (i.e., the points at which the equation has a solution in the form of an m-sparse series). It is easy to find an upper bound for m. We prove that if m is fixed then either there exists a finite number of m-points and all of them can be found or all points are m-points and L can be factored $L = L_1 \circ C$ where C is an operator of a special kind with constant coefficients. Additionally we formulate simple necessary and sufficient conditions for the existence of m-points for an irreducible L.

Combining the algorithms described in the paper with Petkovšek-Salvy's algorithm we find all m-sparse m-hypergeometric solutions of the equation $Ly = 0$.