CYCLE TYPES OF MINIMAL GENERATING SETS OF DIFFERENTIAL POWERS IN AFFINE SEMIGROUP RINGS

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Abstract. In this thesis we examine differential powers of interior principal ideals of affine semigroup rings. Using the lattice representation of these rings we explore consistencies in the minimal generating sets of differential powers of principal ideals and describe upper bounds on the cardinality of these sets.

Introduction

The ring of differential operators has long been an object of interest for its relationship to solutions to linear partial differential equations and singularities in algebraic geometry. A particular view of the ring of differential operators called the Weyl algebra entered the scene with the advent of quantum mechanics; this construction has since been the subject of many important conjectures and results across a range of disciplines. In recent years, the Weyl algebra and other generalizations of the ring of differential operators have been used to analyze the properties of rings of positive characteristic [BJNnB19], compute symbolic powers of ideals [DDSG+18], and introduce new closure operators on commutative algebras [KMP+26]. These results in particular make use of the *differential power* of an ideal: a relatively new tool which characterizes elements of an ideal based on the differential operators of a ring.

In [ST01], authors Saito and Traves use the Weyl algebra to explicitly construct differential operators on a class of rings called *semigroup rings*. Through their results we define and examine the differential powers of ideals in affine semigroup rings; these rings admit a particularly nice visualization that greatly simplifies the calculation of differential powers. This thesis is not the first text to explore differential powers of affine semigroup rings using Saito and Traves' description . In [CNC+24] the authors describe the long term behavior of ideals in a particular kind of affine semigroup ring called a rational normal curve: an algebraic variety that appears frequently in projective and combinatorial geometry.

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Through a different series of techniques we generalize the results on the long term behavior of principal ideals found in [CNC+24] to the larger class of affine semigroup rings, specifically those that are pointed, saturated, and two dimensional.

Section 1 begins by laying out the algebraic language and foundation necessary to begin studying affine semigroup rings. We will describe the restrictions put on the rings of study and acquaint the reader with the visual presentation of the monomials in these rings. Section 2 will first familiarize the reader with the Weyl algebra before using Saito and Traves' results to define differential operators on semigroup rings. The section concludes with a description of the geometric view differential powers that we use for proofs in the following section.

The main results of this paper are found in section 3. Here we conclude that differential powers of principal ideals interior to an affine semigroup ring are periodically principal, then give an explicit formula for the length of each period.

1. Preliminary

Definition 1.1. A **group** is a set G with a binary operation $\star$ where the following hold:

(1) Associative Property: For all g_1, g_2, and g_3 in G, $(g_1 \star g_2) \star g_3 = g_1 \star (g_2 \star g_3)$

(2) Identity Property: There exists a unique element $e \in G$ such that, for all $g \in G$, $g \star e = g = e \star g$

(3) Inverse Propery: for all $g \in G$, there exists a unique $g^{-1} \in G$ such that $g \star g^{-1} = e$

Unless the group operation is given by addition, the group operation will not be explicitly written. We instead adopt multiplicative notation, so $a \star b$ will be written ab.

Example 1.2. The set of rational numbers $\mathbb{Q}$ is a group under addition with identity 0 and inverses being the negation of an element. By removing zero from the rational numbers we get the group $\mathbb{Q}^*$ under multiplication. Here 1 is the identity element and inverses are given by reciprocals of fractions.

A subset H of a group G is a *subgroup* if H is a group under G's group operation. If H is such that $gHg^{-1} = H, \forall g \in G$, then H is a *normal subgroup*. The set of integers under addition, for example, form a subgroup of the additive group of rational numbers. If a group's operation is commutative then the group is called an *Abelian group*. All of the groups in the above example are abelian, though this is not a necessary property of all groups: the group of all invertible real $n \times n$ matrices $\mathrm{GL}_n(\mathbb{R})$ forms a group under matrix multiplication that is not abelian.

Remark 1.3. If G is an abelian group, then all subgroups of G are normal.

When a subgroup is normal, we are able to construct an equivalence relation that agrees with the group operation. For a normal subgroup H of a group G we define the relation $a \sim b \iff aH = bH$. aH is called a *left coset of* H and a (equivalently b) is a *coset representative* of aH.

Definition 1.4. Let H be a normal subgroup of G. The **index** of H in G, sometimes written $[G : H]$, is the number of distinct cosets of H in G. The set of all distinct cosets of H in G, denoted G/H, is called a **quotient group** and inherits the group operation of G, i.e., $(aH)(bH) = (ab)H$.

Since all subgroups of abelian groups are normal, we can take the *quoitient* of an abelian group G with any of its subgroups H. For example consider the set of integers $\mathbb{Z}$ as an abelian group under addition. The subgroup $2\mathbb{Z}$ consisting of all even integers partitions $\mathbb{Z}$ into two distinct cosets: the coset of even numbers $2\mathbb{Z}$ and the coset of odd numbers $1 + 2\mathbb{Z}$. Letting o represent the odd coset and e the even coset, the structure of $\mathbb{Z}/2\mathbb{Z}$ is given by the following group table.

$+$	e	o
e	e	o
o	o	e

The set $\mathbb{Z}^n$ consisting of all ordered pairs of n integers is also an abelian group under addition.

Definition 1.5. A **ring** is an abelian group R under addition equipped with a binary operation $\cdot : R \times R \to R$, called a *product*, satisfying the following properties:

(1) Associative Property: for x, y, and z in R, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
(2) Distributive Property: for x, y, and z in R, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ and $(x + y) \cdot z = (x \cdot z) + (y \cdot z)$

The *multiplicative identity*, or *unity*, is an element $1 \in R$ with the property that, for all $r \in R$, $1 \cdot r = r = r \cdot 1$. Rings with a multiplicative identity are called *rings with unity*, and rings where elements commute over the product are called *commutative rings*. When a proper subset S of a ring R is itself a ring, then we call S a *subring* of R and write $S \subset R$.

When it is clear that two elements a and b are in the same ring, we write ab in place of $a \cdot b$ and $a - b$ for the addition of a with the additive inverse of b. 0 will always denote the additive identity of R. When they exist, the multiplicative inverse of an element a, denoted a^{-1}, is the unique element where $aa^{-1} = 1$. Commutative rings with unity where every nonzero element has a multiplicative inverse are called *fields*.

Remark 1.6. While the additive structure of a ring is an abelian group, it is not necessarily true that a ring is a group under its product. For a field k, $k^* := k \setminus \{0\}$ is an abelian group under multiplication.

Definition 1.7. A **ring homomorphism** is a map of rings $f : R \to S$ where the following hold for all $a, b \in R$:

(1) $f(a + b) = f(a) + f(b)$
(2) $f(ab) = f(a)f(b)$

A homomorphism that maps a ring to iself is an *endomorphism* and a bijective homomorphism is an *isomorphism*. The set of homomorphisms from R to S is denoted $\text{Hom}(R, S)$ and the set of all endomorphisms over R is denoted $\text{End}(R)$.

Definition 1.8. A subring I of a ring R is called an **ideal** of R when, for any $r \in R$ and $x \in I$, $rx \in I$.

We say that an ideal I is *finitely generated* if there exists a finite set of *generators* $S \subseteq R$ so that each element in I is of the form $\sum_{i=1}^{n} r_i s_i$ for $r_i \in R$ and $s_i \in S$. When the generators can be written explicitly, we write $I = (s_1, \ldots, s_k)$.

The following are necessary and sufficient conditions for a subset I of a ring R to be an ideal.

$$\forall x, y \in I \, , x - y \in I$$

$$\forall x \in I, r \in R \implies xr \in I.$$

The first property establishes I as an additive abelian group and the second is the ideal property, often called the *absorption property*. I is a subset of R, so the distributive property of the product is inherited from R.

Definition 1.9. An ideal $I \subseteq R$ that can be generated by a single element is called **principal**.

Examples of rings whose ideals are all principally generated include the ring of integers $\mathbb{Z}$ and the polynomial ring over a field $k[x]$, the latter defined as the set of all polynomials with coefficients in k. Slightly modifying either of these rings gives us rings which have nonprincipal ideals. In $k[x, y]$ the ideal (x^2, y^2) is not principal, and in $\mathbb{Z}[\sqrt{-5}] = \{a + \sqrt{-5}b \mid a, b \in \mathbb{Z}\}$ the ideal $(2, 1 + \sqrt{-5})$ is also not principal.

Definition 1.10. Let R be a commutative ring with unity. An R**-algebra** is a ring A with unity and a ring homomorphism $\varphi : R \to A$ such that $\varphi(R)$ is a commutative subring of A. An **algebra homomorphism** is a ring homomorphism between R-algebras $\psi : A \to B$ such that $\forall r \in R$ and $\forall a \in A$, $\psi(r \cdot a) = r \cdot \psi(a)$.

The set of R-algebra homomorphisms between rings A and B is denoted $\mathrm{Hom}_R(A, B)$. The set of endomorphisms over an R-algebra is $\mathrm{End}_R(A)$. Since R-algebra homomorphisms commute with addition, the set $\mathrm{End}_R(A)$ forms a ring under addition and composition.

An example of a $\mathbb{C}$ algebra homomorphism will be the map $\psi : \mathbb{C}[x, y] \to \mathbb{C}[t]$ given by $x \mapsto t$ and $y \mapsto t^2$. That this is an algebra homomorphism will be taken without proof, though an example of how elements λ of $\mathbb{C}$ factor out of the homomorphism is given. Consider $2 + x^3y - y^2$

$$\begin{aligned}
\psi(2\lambda + \lambda x^3y - \lambda y^2) &= 2\lambda + \lambda t^5 - \lambda t^4 \\
&= \lambda(2 + t^5 - t^4) \\
&= \lambda\psi(2 + x^3y - y^2)
\end{aligned}$$

It is often convenient to work over the complex numbers as they are *algebraically closed*, which we take to mean that any polynomial in $\mathbb{C}[x]$ can be factored into linear terms. Therefore $\mathbb{C}$ will be the field of choice when working over polynomial rings, though many of the results in this paper concerning these rings apply to other fields as well.

Definition 1.11. A ring R is a **graded ring** if R has the decomposition

$$R = \bigoplus_{i=0}^{\infty} R_i = R_0 \oplus R_1 \oplus \cdots$$

where each R_i is an additive abelian group and $R_iR_j = \{r_ir_j \mid r_i \in R_i, r_j \in R_j\} \subseteq R_{i+j}$.

The ring of polynomials with coefficients in a field k, which we denoted $k[x]$, is a graded ring with respect to the degree of a polynomial. If f and g are monomials of degree p and

q respectively, then $\deg(fg) = p + q$, so $R_pR_q \subseteq R_{p+q}$. This grading, called the "standard grading", is valid in polynomial rings with one or multiple variables, although multivariate polynomial rings also admit a *multigrading.*

Definition 1.12. R is **multigraded** if R has the decomposition

$$R = \bigoplus_{(\alpha_1,\ldots,\alpha_n)\in\mathbb{Z}^n} R_{(\alpha_1,\ldots,\alpha_n)}$$

where each $R_{(a_1,\ldots,a_n)}$ is an additive abelain group and $R_{(\alpha_1,\ldots,\alpha_n)}R_{(\beta_1,\ldots,\beta_n)} \subseteq R_{(\alpha_1+\beta_1,\cdots,\alpha_n+\beta_n)}$

The usual multigrading on $k[x_1,\ldots,x_n]$ treats monomials $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ as having multidegree $(\alpha_1,\ldots,\alpha_n)$.

Definition 1.13. Let λ be an element of a ring R and let A, B, and C be R-algebras. A map $\phi : A \times B \to C$ is a **bilinear map** when the following properties hold for all $a_i \in A$ and $b_i \in B$:

(1) $\phi(a, \lambda \cdot b) = \phi(\lambda \cdot a, b) = \lambda \cdot \phi(a, b)$
(2) $\phi(a_1 + a_2, b) = \phi(a_1, b) + \phi(a_2, b)$
(3) $\phi(a, b_1 + b_2) = \phi(a, b_1) + \phi(a, b_2)$

Definition 1.14. Let M be an $n\times n$ matrix with entries in k. The **determinant** of the matrix is a map $\det\colon k^{n\times n} \to k$ with the following properties:

(1) The determinant of the identity matrix is 1
(2) Where w_i, v_i, and u_i are the ith column vectors of a matrix M and λ is a scalar in k,

$$\det\begin{pmatrix} u_1 & \cdots & (w_i + \lambda v_i)\cdots & u_n \end{pmatrix} = \det\begin{pmatrix} u_1 & \cdots & w_i\cdots & u_n \end{pmatrix} + \lambda\det\begin{pmatrix} u_1 & \cdots & v_i\cdots & u_n \end{pmatrix}$$

(3) If M is a matrix where at least two columns have identical entries, then $\det M = 0$

Using previous definitions, property (2) states that the determinant of a 2×2 matrix is bilinear with respect to the column vectors of the matrix.

Let e_i represent the basis vector of $\mathbb{R}^n$ with 1 in the ith component and zeros elsewhere. The determinant of a matrix $M \in \mathbb{R}n\times n$ also gives the volume of the parallelepiped formed by all transformed vectors Me_i.

Definition 1.15. Let x and y be elements of a (not necessarily commutative) ring R. The **commutator** of x and y is the map $[\cdot,\cdot] : R \times R \to R$ sending $[a, b]$ to $ab - ba$.

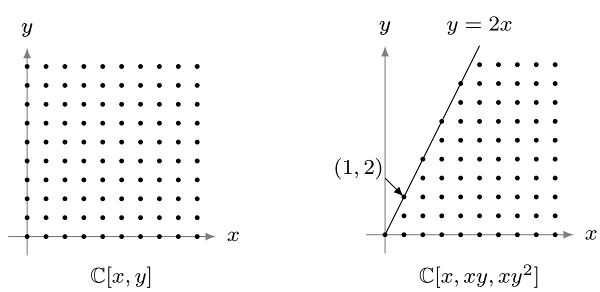
Remark 1.16. The commutator is a bilinear map. When R is a commutative ring, the commutator of any two elements in R is zero.

Definition 1.17. A **semigroup** S is a set with an associative binary operation $\circ : S \times S \to S$ and an identity element. A **semigroup ring** $R[S]$ is the set $\{\sum_{i=0}^{n} s_i r_i \mid s_i \in S, r_i \in R, n \in \mathbb{N}\}$, where multiplication is defined by the semigroup operation, i.e., $(s_i r_i)(s_j r_j) = (s_i \circ s_j) r_i r_j$.

Remark 1.18. Semigroups with an identity element are called *monoids*. We refer to the unity of the ring $R[S]$ when we say that a semigroup has an identity element, even when S lacks an identity of its own.

Semigroups whose operation is commutative and described by addition are *additive semigroups*. The results of this paper focus on subsemigroups S of the additive semigroup $(\mathbb{N}^d, +)$, where the operation is defined by adding components. In instances where the components of an element $(\alpha_1, \ldots, \alpha_d)$ do not need to be written explicitly, we denote by $\underline{\alpha}$ the same element in S.

Additive semigroup rings are presented as subrings of polynomial rings $\mathbb{C}[x_1, \ldots, x_n]$ where the base ring is the field of complex numbers and the semigroup is the set of monomials $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ under multiplication. Motivated by the additive rule of exponents, additive semigroup rings will represent such monomials by $x^{\underline{\alpha}}$. An example of a semigroup ring that is a proper subset of $\mathbb{C}[x, y]$ is the ring $\mathbb{C}[x, xy, xy^2]$, where the semigroup is defined by monomials whose factors are only x, xy, or xy^2. In both cases, the semigroup structure admits a nice visual representation, which will be important later, by identifying the multidegree of a monomial with a lattice point on a grid. Below are the representations of the rings $\mathbb{C}[x, y]$ and $\mathbb{C}[x, xy, xy^2]$ in lattice form.



The semigroup rings we investigate in this text are over subsemigroups of $\mathbb{N}^2$, hence our rings will have a corresponding two dimensional lattice. The presentation of monomials in the semigroup ring as points on the integer lattices is a *lattice embedding*, and we call semigroup rings whose semigroups are finitely generated with integer lattice embeddings "*affine semigroup rings*". Failure to be either finitely generated or to have an integer lattice embedding means a semigroup ring is not affine. For example, the ring $\mathbb{C}[x, xy, xy^{\sqrt{2}}]$ cannot be embedded into an integer lattice as some points have an irrational component (note that this ring is not even a subring of $\mathbb{C}[x, y]$, so it is already not under consideration). The subsemigroup generated by $\{(a, b) \mid a > 1,\ \gcd(a, b) = 1,\ \text{and}\ \frac{b}{a} < \sqrt{2}\}$ is a semigroup whose elements are integer pairs, thus has an integer lattice embedding, but is not finitely generated. We will not be considering non-affine semigroup rings for the remainder of this text as the tools we use in future sections were developed with affine semigroup rings in mind.

When we restrict our view to the semigroup, we denote their operation additively as one would with vector arithmetic. When talking about the semigroup S in a semigroup ring $R[S]$ we will often identify S with a matrix whose columns $\mathbb{N}$-span the lattice generated by S. Take, for example, the rings $\mathbb{C}[x, y]$ and $\mathbb{C}[x, xy, xy^2]$ with the following representation.

$$x^a y^b \sim \begin{pmatrix} a \\ b \end{pmatrix}$$

Then the matrices associated to these rings are respectively

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

We denote the set of lattice points associated to an affine semigroup ring $R[S]$ by $\mathbb{N}S$. This structure, representing all non-negative integer combinations of the column vectors, is referred to as an *integer cone*.

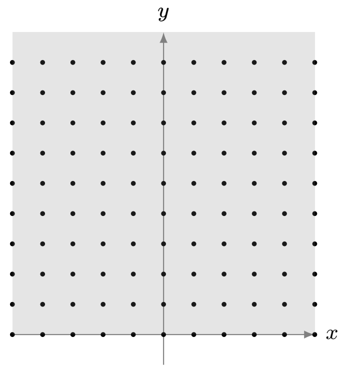
Definition 1.19. Let $\mathbb{R}_{\geq 0}S$ be the real cone formed by a semigroup S. The **facets** of S are the rays $\underline{v}_i$ defining the boundary of $\mathbb{R}_{\geq 0}S$. The column vector of S whose $\mathbb{R}_{\geq 0}$-span defines a facet is also denoted $\underline{v}_i$. When the components of $\underline{v}_i$ are relatively prime then we call $\underline{v}_i$ a **primitive vector**.

Remark 1.20. As we only concern ourselves with semigroups whose cones are two dimensional, there will always be exactly two facet rays. In texts whose semigroups are of dimension greater than two, the facets are defined as the $n-1$ dimensional (or codimension 1) faces that define the boundary of the cone.

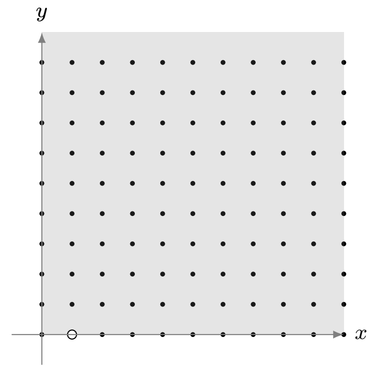
Definition 1.21. Let S be a semigroup and $k[S]$ a semigroup ring over a field k.

- S is **saturated** if $\mathbb{R}_{\geq 0}S \cap \mathbb{Z}^2 = \mathbb{N}S$, that is for any integer point in the real cone of S, the point is also in $\mathbb{N}S$.
- S is **pointed** if $\mathbb{R}_{\geq 0}S \cap \mathbb{R}_{\leq 0}S = \{0\}$.

As we have so far only seen semigroup rings with pointed and saturated lattices, examples of semigroups which fail to be pointed/saturated are given below.



$\mathbb{C}[S_1] = \mathbb{C}[x, x^{-1}, y]$



$\mathbb{C}[S_2] = \mathbb{C}[x^2, x^3, y, xy]$

The shaded gray area of the graphs are the $\mathbb{R}_{\geq 0}$ span of the semigroup generators. The semigroup S_1 is not pointed as the intersection $\mathbb{R}_{\geq 0}S_1 \cap \mathbb{R}_{\leq 0}S_1$ is the entire x-axis, and S_2 is not saturated since the point $(1, 0)$ is not in $\mathbb{R}_{\geq 0}S_2 \cap \mathbb{Z}^2$.

Remark 1.22. If the facet-defining monomial generators of a semigroup ring do not form primitive vectors, then the semigroup is not saturated.

Semigroups in this text will all be saturated and pointed unless otherwise specified. We therefore require that primitive vectors $\underline{v}_i$ are furthest reduced as integer ordered pairs. In other words if $\underline{v}_i = (a, b)$ then $\gcd(a, b) = 1$. Also, since the semigroups are all pointed, we may define the *chambers* $\mathcal{C}_\bullet$ of the semigroup in the following way.

$$\begin{cases} \mathcal{C}_{\mathrm{I}} & \operatorname{span}_{\mathbb{R}_{\geq 0}} \{\underline{v}_1, \underline{v}_2\} \\ \mathcal{C}_{\mathrm{II}} & \operatorname{span}_{\mathbb{R}_{\geq 0}} \{-\underline{v}_1, \underline{v}_2\} \\ \mathcal{C}_{\mathrm{III}} & \operatorname{span}_{\mathbb{R}_{\geq 0}} \{-\underline{v}_1, -\underline{v}_2\} \\ \mathcal{C}_{\mathrm{IV}} & \operatorname{span}_{\mathbb{R}_{\geq 0}} \{\underline{v}_1, -\underline{v}_2\} \end{cases}$$

Definition 1.23. Two semigroups A and B are **isomorphic** if there exists an invertible linear map T such that, as matrices, $TA = B$.

If A and B are isomorphic semigroups, then for k fixed, $k[A]$ and $k[B]$ are isomorphic through T as well. Hence when we say that two semigroup rings over the same field are isomorphic, we mean there exists a semigroup isomorphism between their generators.

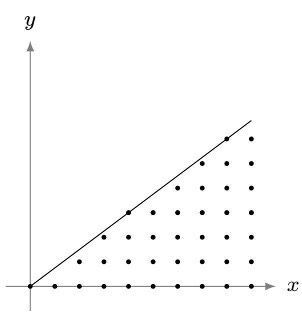
We observe that the entries of the semigroup isomosphism need not be integers. Take for example the *rational normal curve*, or RNS, of degree d $\mathbb{C}[x^d, x^{d-1}y, \dots, xy^{d-1}, y^d] = \mathbb{C}[A]$. This ring is isomorphic to the ring $\mathbb{C}[x, xy, \dots, xy^d] = \mathbb{C}[B]$ as the respective matrices of the semigroups are related by an invertible matrix T.

$$\underbrace{\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & d-1 & d \end{pmatrix}}_{B} = \underbrace{\begin{pmatrix} \frac{1}{d} & \frac{1}{d} \\ 0 & 1 \end{pmatrix}}_{T} \underbrace{\begin{pmatrix} d & d-1 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & d-1 & d \end{pmatrix}}_{A}$$

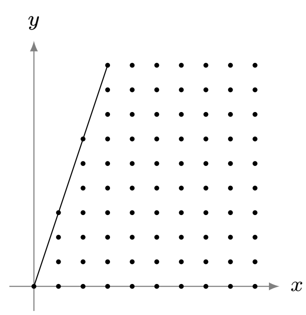
Elements of $\mathbb{C}[A]$ must all be of degree nd for some $n \in \mathbb{N}$, thus the ring is not saturated. Since rational normal curves of the form $\mathbb{C}[x, xy, \dots, xy^d]$ are saturated, we will present them as such in later sections.

Remark 1.24. The use of "rational normal curve" over the term "rational normal surface" comes from the projective setting $\mathbb{CP}^n := (\mathbb{C} \setminus \{0\})^{n+1}/\sim$, where, for points $P = (p_1, \dots, p_{n+1})$ and $Q = (q_1, \dots, q_{n+1})$ in $(\mathbb{C} \setminus \{0\})^{n+1}$, $P \sim Q \iff \exists \lambda \in \mathbb{C} \setminus \{0\}$ s.t. $P = \lambda Q$; equivalence classes within $\mathbb{CP}^n$ are presented in *homogeneous coordinates* $[p_1 : p_2 : \cdots : p_{n+1}]$. When $x \neq 0$ we get from the isomorphism T the point $[x : xy : \cdots : xy^n] = [1 : y : \cdots : y^n]$, which is a one-dimensional parameterization of the curve in projective space.

Two normal semigroup rings may be isomorphic as well. Below are rings $\mathbb{C}[x, x^2y, x^3y^2, x^4y^3]$ and $\mathbb{C}[x, xy, xy^2, xy^3]$ that are both isomorphic to the rational normal curve of degree 2.



$\mathbb{C}[x, x^2y, x^3y^2, x^4y^3]$



$\mathbb{C}[x, xy, xy^2, xy^3]$

The above semigroup lattices are isomorphic through the linear transformation

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

As the next proposition will show, there exist infinitely many normal affine semigroup rings that are isomorphic to a rational normal curve of any given degree.

Proposition 1.25. If $\underline{v}_1 = (1,0)$ and the slope of $\underline{v}_2$ is of the form $\frac{n}{mn+1}$ with $m, n \in \mathbb{N}$, then the semigroup is isomorphic to the rational normal curve of degree n.

Proof. Let n and m be positive integers and R the semigroup ring with facets $\underline{v}_1 = (1,0)$ and $\underline{v}_2 = (mn+1, n)$. $\underline{v}_2$ is a primitive vector as $\gcd(mn+1, n) = 1$, so we can now find generators so that the ring is saturated. As we already have x as a generator, it is enough to find generators (a, b) such that a is minimal and b ranges between 0 and n. For a given b, the smallest value of a such that $x^a y^b$ is a monomial in $\mathcal{C}_I$ is $\lceil \frac{mn+1}{n} b \rceil \implies a = mb+1$. The ring is therefore $\mathbb{C}[x, x^{m+1}y, x^{2m+1}y^2, \ldots, x^{mn+1}y^n]$. The isomorphism of this semigroup's matrix is

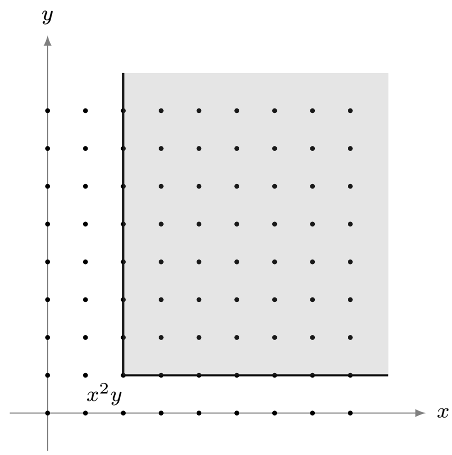
$$\begin{pmatrix} 1 & -m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & m+1 & \cdots & mn+1 \\ 0 & 1 & \cdots & n \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & n \end{pmatrix}$$

As m is arbitrary, we have a family of infinitely many saturated semigroup rings isomorphic to a rational normal curve. □

The fact that the generators lie on the line $y = \frac{1}{m}(x-1)$ is essential to the existence of an isomorphism as RNS the semigroup isomorphism is a linear transformation, and all the generators of the RNS lie on a vertical line. This means we can look at the "alignment" of the semigroup ring's maximal ideals in order to determine behavior, or at least their similarity.

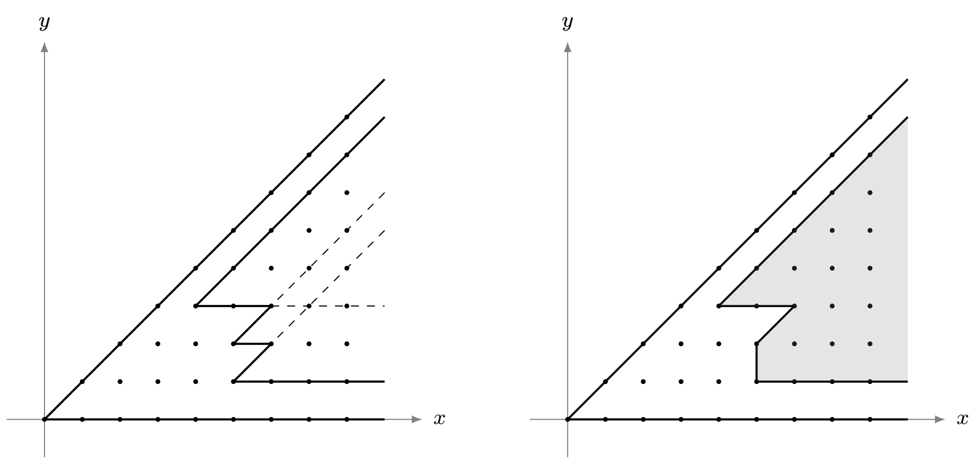
Remark 1.26. Every two dimensional saturated affine semigroup ring is isomorphic to one with facet $\underline{v}_1 = (1,0)$.

Ideals in semigroup rings are presented in a similar manner to the ring itself. Each monomial $x^{\underline{\alpha}_i}$ generating an ideal I defines the point of its own translated integer cone, which we denote cone(I). For example, the ideal $I = (x^2y)$ consists of all polynomials in $\mathbb{C}[x,y]$ where x^2y is a factor. Thus cone(I) (shaded gray in the following figure) is the set of all tuples $(a,b) = (2,1) + (a_0, b_0)$ for some $a_0, b_0 \in \mathbb{N}$.



$\mathbb{C}[x, y]$

If an ideal is not principal, then the translated cones of the generators will overlap. Graphically, this is presented as the union of overlapping cones. When the cone of a generator is "close enough" to that of another, we draw a line connecting the generators. This connecting of cones is done so that we may better discern a single ideal whose generators are in close proximity from two or more ideals whose generators are in close proximity. Below is the ideal (x^5y, x^5y^2, x^4y^3) in $\mathbb{C}[x, xy]$, included to show how non-principal ideals will be drawn.



Definition 1.27. Let $R[S]$ be an affine semigroup ring with facets corresponding to monomials x^{α_1} and x^{α_2}. An ideal $I \subset R[S]$ is an **interior ideal** if $x^{n\alpha_i} \notin I$ for any i and all $n > 0$. In other words I is interior if $\text{cone}(I)$ does not intersect the boundary $\mathbb{R}_{\geq 0}S$.

2. The Weyl Algebra

In this section we define the *Weyl algebra* and use it as the foundation on which the differential operators from [ST01] are defined on affine semigroup rings.

Definition 2.1. The **Weyl algebra**, denoted $\mathbb{A}_n$, is the ring of differential operators $\mathbb{C}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]$ where the variables are subject to the following relations:

(1) $[x_i, x_j] = 0$ for all $i, j \in \{1, \ldots, n\}$.
(2) $[\partial_i, \partial_j] = 0$ for all $i, j \in \{1, \ldots, n\}$.
(3) when $i \neq j$, $[\partial_i, x_j] = 0$.
(4) when $i = j$, $[\partial_j, x_j] = 1$.

The variables of the Weyl algebra are linear operators on the ring of polynomials, where ∂_i acts on polynomials $f \in \mathbb{C}[x_1, \ldots, x_n]$ by $\partial_i(f) = \frac{\partial f}{\partial x_i}$ and x_i by $x_i(f) = x_i f$. Since the composition of linear operators is associative and distributes over multiplication, the ring structure is well defined. With the operator description of this ring, we are able to make further sense of the fourth relation:

$$\begin{aligned}
[\partial_i, x_i](f) &= (\partial_i x_i)(f) - (x_i \partial_i)(f) \\
&= \partial_i(x_i f) - x_i(\frac{\partial f}{\partial x_i}) \\
&= f + x_i \frac{\partial f}{\partial x_i} - x_i \frac{\partial f}{\partial x_i} \\
&= f.
\end{aligned}$$

With 1 representing the identity operator in $\mathbb{A}_n$, we get $[\partial_i, x_i] = 1$. This commutation relation extends to general polynomials in $\mathbb{A}_n$ as the following proposition will show.

Proposition 2.2. Let $f \in \mathbb{A}_n$ be a polynomial generated by $\{x_1, \ldots, x_n\}$ over $\mathbb{C}$. The commutator $[\partial_i, f]$ is $\frac{\partial f}{\partial x_i}$: the derivative of f with respect to x_i.

Proof. Recall that the commutator is a bilinear map, so we need only examine the behavior on monomial summands ${x_1}^{\alpha_1} \cdots {x_n}^{\alpha_n}$ of f. Furthermore, we may focus solely on the monomial ${x_i}^{\alpha_i}$ as ∂_i is free to commute with all variables ∂_j and x_j when $j \neq i$. We thus omit indices on ∂ and x for the proof and show by induction that $[\partial, x^\alpha] = \alpha {x_i}^{\alpha-1}$ for all $\alpha \in \mathbb{N}$. For $\alpha = 1$, we get $[\partial, x] = x^0 = 1$ by relation 4. Now suppose that $[\partial, x^\alpha] = \alpha x^{\alpha-1}$.

Then

$$\begin{aligned}[\partial, x^{\alpha+1}] &= \partial x^{\alpha+1} - x^{\alpha+1}\partial \\ &= \partial x^{\alpha} x - x^{\alpha}\partial x + x^{\alpha}\partial x - x^{\alpha+1}\partial \\ &= [\partial, x^{\alpha}]x + x^{\alpha}[\partial, x] \\ &= \alpha x^{\alpha} + x^{\alpha} \\ &= (\alpha+1)x^{\alpha}\end{aligned}$$

□

Just as monomials in a polynomial have a multi-index representation, so too do differential operators over the Weyl algebra. Consider the polynomial ring $\mathbb{C}[x, y]$ and differential operators ∂_x, ∂_y. We denote the product $\partial_x^{\beta_1}\partial_y^{\beta_2}$ with $\partial^{\underline{\beta}}$. If $x^{\underline{\alpha}}$ is a monomial not annihilated by $\partial^{\underline{\beta}}$, then $\partial^{\underline{\beta}}(x^{\underline{\alpha}}) = cx^{\underline{\alpha}-\underline{\beta}}$ for some constant c. We will use this property in the future to classify differential operators based on a multidegree lying outside of $\mathcal{C}_I$.

When δ is a differential operator of order n in a ring R, then we say that $\mathrm{ord}(\delta) = n$. The definition of order for a differential operator δ in the Weyl algebra is defined in a recursive fashion which agrees with the usual definition used in the theory of differential operators.

$$\begin{cases} \mathrm{ord}(\delta) = 0 & [\delta, f] = 0, \forall f \in R \\ \mathrm{ord}(\delta) = n & \mathrm{ord}(\delta) > n-1 \text{ and } \mathrm{ord}([\delta, f]) < n, \forall f \in R \end{cases}$$

The set of differential operators of order less than or equal to n is $\mathrm{D}^n(R)$ and the set of all differential operators is $\mathrm{D}(R)$. An element δ that commutes with all $f \in R$ is an element in R, so $\mathrm{D}^0(R) = R$.

The set of differential operators is a filtered object. In other words we have a chain of inclusions $R = \mathrm{D}^0(R) \subset \mathrm{D}^1(R) \subset \cdots$ where each $\mathrm{D}^i(R)$ is a subset of $\mathrm{D}(R)$.

Note that this construction is not graded by its order as any differential operator has at least one other presentation as a sum of terms of different order. An example of this is the operator $x\partial_x$ as its other form $\partial_x x - 1$ is composed of an order 1 and order 0 term instead of a single order 1 term. Moreover the difference of first order terms $x\partial_x - \partial_x x$ has order zero, so we can't even count the largest order of the summands to define an order grading on $\mathrm{D}^i(R)$. The best we can do when working only with order is the filtration, where elements in $\mathrm{D}^i(R)$ have order less than or equal to i. The ramifications of defining $\mathrm{D}^i(R)$ through strict order will be examined further in Example 3.3 once differential powers are defined.

We will later see how to get a graded structure through the use of the multidegree of the differential operators instead of just order. Even though elements of $\mathbb{A}_n$ are not unique in their presentation, we still have a basis of $\mathbb{A}_n$ as an infinite dimensional vector space.

Proposition 2.3. [Cou95, Proposition 1.2.1] The set $B = \{x^{\underline{\alpha}}\partial^{\underline{\beta}} \mid \alpha, \beta \in \mathbb{N}^n\}$ is a basis of $\mathbb{A}_n$ over $\mathbb{C}$. In other words, elements of $\mathbb{A}_n$ can be represented as a unique linear combination of terms $x^{\underline{\alpha}}\partial^{\underline{\beta}}$.

Writing operators with the basis from proposition 2.3 will help avoid confusion when defining certain types of differential operators, particularly those over semigroup rings. These operators must take a different form as we run into a problem of closure when trying to use the differential operators from the standard polynomial ring. To see this in action, consider the action of ∂_x on the generator x^2y^3 in $\mathbb{C}[x, xy, x^2y^3]$. The resulting monomial xy^3 is not an element of our ring as it lives in chamber $\mathcal{C}_{II}$ and hence ∂_x may not even be considered an operator in $\mathbb{C}[x, xy, x^2y^3]$.

This takes us back to the polynomial case, where differential operators that would otherwise take a monomial outside of chamber $\mathcal{C}_I$ instead map the monomial to zero. We therefore expect that a particular differential operator will annihilate monomials that are in some way "too close" to a facet. Finding elements of $\mathbb{A}_n$ for which this holds in an affine semigroup ring motivates Saito and Traves' θ operators and *integral support* in [ST01], which we will now define.

Lemma 2.4. Define $\theta_i \in \mathbb{A}_n$ by $\theta_i := x_i\partial_i$. If $f(\theta_1, \ldots, \theta_n)$ is a polynomial in $\mathbb{C}[\theta_1, \ldots, \theta_n] \subseteq \mathbb{A}_n$ and $x^{\underline{\alpha}} \in \mathbb{C}[x_1, \ldots, x_n]$ a monomial, then $f(\theta_1, \ldots, \theta_n)(x^{\underline{\alpha}}) = f(\underline{\alpha})x^{\underline{\alpha}}$.

Proof. We will first show this property in the single variable monomial case by induction on the index of θ_x. Consider the monomial $f(\theta) = \theta$ acting on x^α. $\theta(x^\alpha) = x\partial(x^\alpha) = \alpha x^\alpha$, equaling $f(\alpha)x^\alpha$ as desired. Suppose this is holds for $\theta^{\beta-1}$ where $\beta \in \mathbb{N}$

$$\begin{aligned}\theta^\beta(x^\alpha) &= \theta^{\beta-1}x\partial(x^\alpha) \\ &= \alpha\theta^{\beta-1}(x \cdot x^{\alpha-1}) \\ &= \alpha\theta^{\beta-1}(x^\alpha)\end{aligned}$$

implying by induction

$$\theta^\beta(x^\alpha) = \alpha^\beta x^\alpha$$

Expanding back to the polynomial case $\mathbb{C}[x_1, \ldots, x_n]$, let $f(\theta_1, \ldots, \theta_n) = \theta^{\underline{\beta}} \in \mathbb{A}_n$. We reorder this monomial of differential operators before acting on a monomial of indeterminants $x^{\underline{\alpha}}$ using the commutation relations.

$$\begin{aligned}
(\theta_1^{\beta_1}\theta_2^{\beta_2}\cdots\theta_n^{\beta_n})(x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}) &= \theta_1^{\beta_1}(x_1^{\alpha_1})\theta_2^{\beta_2}(x_2^{\alpha_2})\cdots\theta_n^{\beta_n}(x_n^{\alpha_n}) \\
&= \alpha_1^{\beta_1}x_1^{\alpha_1}\alpha_2^{\beta_2}x_2^{\alpha_2}\cdots\alpha_n^{\beta_n}x_n^{\alpha_n} \\
&= \underline{\alpha}^{\underline{\beta}}x^{\underline{\alpha}} \\
&= f(\underline{\alpha})x^{\underline{\alpha}}
\end{aligned}$$

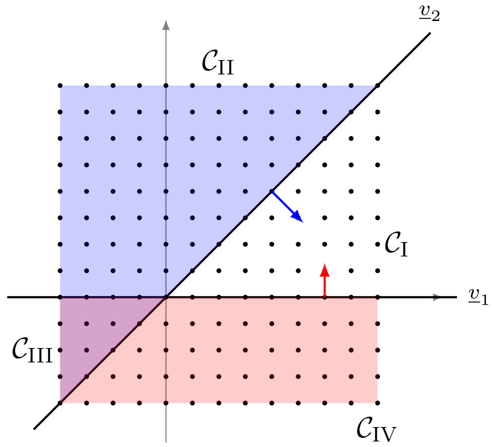
Now let f be a polynomial in $\mathbb{C}[\theta_1, \ldots, \theta_n]$ of degree k

$$\begin{aligned}
f(\theta_1, \ldots, \theta_n)(x^{\underline{\alpha}}) &= (\sum_{|\underline{\beta}_i|<k} c_i\theta^{\underline{\beta}_i})(x^{\underline{\alpha}}) \\
&= \sum_{|\underline{\beta}_i|<k} \left(c_i\theta^{\underline{\beta}_i}(x^{\underline{\alpha}})\right) \\
&= \sum_{|\underline{\beta}_i|<k} \left(c_i\underline{\alpha}^{\underline{\beta}_i}x^{\underline{\alpha}}\right) \\
&= f(\underline{\alpha})x^{\underline{\alpha}}
\end{aligned}$$

□

Definition 2.5. Given facets $\underline{v}_1 = (a, b)$ and $\underline{v}_2 = (c, d)$ of a semigroup ring, the corresponding **integral support** functions are $h_1(x, y) = ay - bx$ and $h_2(x, y) = dx - cy$.

The integral support functions are defined in such a way that points outside of the cone evaluate to a negative integer for at least one of the integeral support functions. In other words, the functions are such that the normal vectors $(-b, a)$ and (d, c) point inward toward the cone. Thus which of the functions evaluate to a negative integer depends on the chamber the point belongs to.



Chamber	$h_1(x,y)$	$h_2(x,y)$
$\mathcal{C}_{\mathrm{I}}$	positive	positive
$\mathcal{C}_{\mathrm{II}}$	positive	negative
$\mathcal{C}_{\mathrm{III}}$	negative	negative
$\mathcal{C}_{\mathrm{IV}}$	negative	positive

We use the fact that polynomials in $\mathbb{C}[\theta_1,\ldots,\theta_n]$ act on monomials by evaluating their exponent vectors to guarantee closure. Recall for the following construction that $h_i(\theta_1,\theta_2)(x^{\underline{\alpha}}) = h_i(\underline{\alpha})x^{\underline{\alpha}}$. The order of a differential operator with multidegree $\underline{\beta}$ is described as follows

Chamber	Order
$\underline{\beta} \in \mathcal{C}_{\mathrm{I}}$	0
$\underline{\beta} \in \mathcal{C}_{\mathrm{II}}$	$-h_2(\underline{\beta})$
$\underline{\beta} \in \mathcal{C}_{\mathrm{III}}$	$-h_1(\underline{\beta}) - h_2(\underline{\beta})$
$\underline{\beta} \in \mathcal{C}_{\mathrm{IV}}$	$-h_1(\underline{\beta})$

Thus we have the definition of the order of a differential operator $\underline{\beta}$. So differential operators are defined by their multidegrees, but we still have a way of describing them through the Weyl algebra, particularly with polynomials in $\mathbb{C}[\theta_1,\ldots,\theta_n]$.

Theorem 2.6. [ST01, Theorem 3.2.2] The ring of differential operators over an affine semigroup ring R has a $\mathbb{Z}^2$-graded description.

$$\mathrm{D}(R) = \bigoplus_{\underline{\alpha}\in\mathbb{Z}^2} x^{\underline{\alpha}} \cdot \left(\prod_{\substack{i\in\{1,2\} \\ h_i(\underline{\alpha})<0}} \prod_{j=0}^{h_i(-\underline{\alpha})-1} (h_i(\theta_1,\theta_2) - j) \right)$$

Corollary 2.7. [ST01, Corollary 3.2.4] If two operators with degrees $\underline{\beta}_1$ and $\underline{\beta}_2$ lie in the same chamber, then the order of $\underline{\beta}_1 + \underline{\beta}_2$ is the sum of the orders of $\underline{\beta}_i$.

Note that this description relies on the fact that affine semigroup rings have integer lattice embeddings, so defining differential operators over these rings through this definition is only well defined for operators corresponding to points with integer coordinates; we will not consider fractional derivatives or exponents.

Remark 2.8. The multidegree description of order in the polynomial case agrees with the recursive description.

Proofs of this fact can be found in [CNC+24], so we forego the proof here. Rather, we will compute some examples using the Weyl commutation relations with the goal of providing insight as to why these definitions agree. Our natural example of an order 3 differential operator in $\mathbb{C}[x]$ is the operator ∂^3. This operator corresponds to the point (-3) in the one-dimensional lattice as it annihilates monomials x^a when $a < 3$ and, ignoring coefficients, sends x^a to x^{a-3} when $a \geq 3$. The differential operator of order 3 in the semigroup description is $x^{-3}(\theta)(\theta-1)(\theta-2)$, which, if the order of operators is consistent, will be the differential operator ∂^3.

$$\begin{aligned} x^{-3}(x\partial)(x\partial-1)(x\partial-2) &= x^{-3}(x\partial x\partial - x\partial)(x\partial-2) \\ &= x^{-3}(x\partial + x^2\partial^2 - x\partial)(x\partial - 2) \\ &= x^{-3}(x^2\partial^2)(x\partial-2) \end{aligned}$$

Already the operator $x^2\partial^2$ appears, which when multiplied by x^{-2} (as one would do to find a second order operator) becomes ∂^2. In this case, we would say that ∂^2 corresponds to the point (-2). We have yet to reach a furthest simplified form for the operator corresponding to (-3), so we continue.

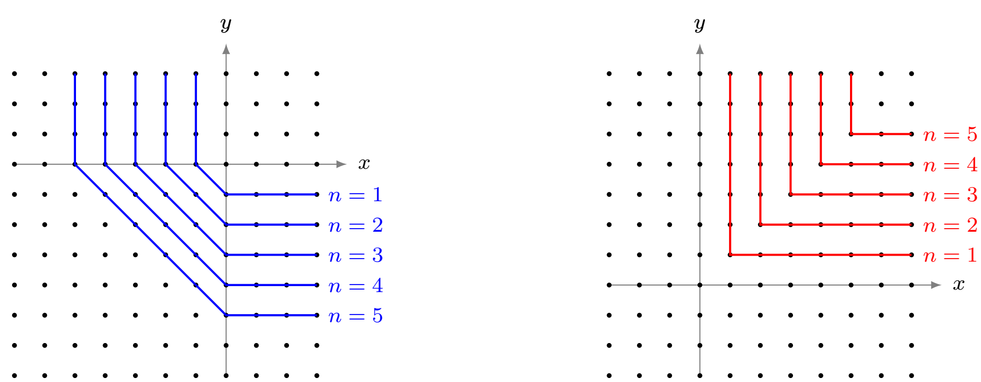
$$\begin{aligned} x^{-3}(x^2\partial^2)(x\partial-2) &= x^{-3}(x^2\partial^2 x\partial - 2x^2\partial^2) \\ &= x^{-3}(x^2\partial(\partial x)\partial - 2x^2\partial^2) \\ &= x^{-3}(x^2\partial(1+x\partial)\partial - 2x^2\partial^2) \\ &= x^{-3}(x^2\partial^2 + x^2\partial x\partial^2 - 2x^2\partial^2) \\ &= x^{-3}(x^2\partial^2 + x^2(1+x\partial)\partial^2 - 2x^2\partial^2) \\ &= x^{-3}(2x^2\partial^2 + x^3\partial^3 - 2x^2\partial^2) \\ &= x^{-3}(x^3\partial^3) \\ &= \partial^3 \end{aligned}$$

Remark 2.9. In [Boo59] the author demonstrates that *descending factorials* $(x\partial)(x\partial - 1)\cdots(x\partial - n + 1)$ are equal to the operator $x^n\partial^n$. This is given a more combinatorial treatment in [MS16] where the result is generalized to other operators defined by commutation relations. This means the differential operator corresponding to the point $(-a,-b)$ for $a, b \in \mathbb{N}$ in chambers II through IV is exactly $\partial_x^a\partial_y^b$ in the Weyl algebra.

With the descending factorial description, we are able to safely define differential operators based only on how they alter the degrees of monomials in an affine semigroup ring.

Though we disregard the order of a monomial in $\mathcal{C}_I$ when defining the order of a differential operator, the value still finds use in defining the monomials that are annihilated by differential operators of a given order. Suppose $\underline{\beta}$ is the multidegree of a differential operator and $\underline{\alpha}$ the multidegree of a monomial. Since the individual integral support functions are linear maps, $h_i(\underline{\alpha} + \underline{\beta}) = 0$ (so $\underline{\alpha} + \underline{\beta}$ lies on one or both facets) if and only if $h_i(\underline{\alpha}) = -h_i(\underline{\beta})$. Furthermore, if δ is a differential operator of order $n \geq 0$ then the only elements of the ring that it annihilates are either those who evaluate to $n-1$ through at least one of the integral support functions or those lying on the facet of the lattice.

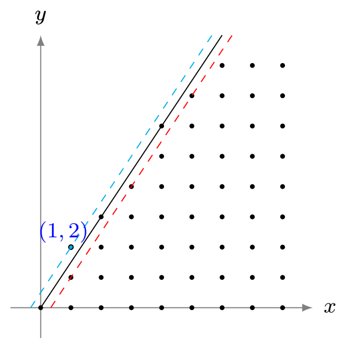
We observe this complimentary behavior for the monomials in $\mathcal{C}_I$ through the polynomial case first.



differential operators of order n smallest order n annihilating each monomial

Suppose $\underline{m}$ is a point corresponding to a monomial $x^{\underline{m}} \in R$ and α a point corresponding to a differential operator in $\mathrm{D}(R)$. The order of $x^{\underline{m}}$ as an operator on R is the signed "distance" (with respect to the primitive integral support functions) of m from either one or both facets. So if the primitive vector $\underline{v}_2$ for example has components (a, b), then the differential powers strictly of order n lie on the line $y = \frac{b}{a}x - \frac{n}{a}$.

Example 2.10. The ring $\mathbb{C}[x, xy, x^2y^3]$ has integral support functions $h_1 = y$ and $h_2 = 3x - 2y$. A differential operator with multidegree $\underline{\alpha} = (1, 2)$ lies in the second chamber, so we use the second integral support function to find its order:



As this point lies in the second chamber, we use the second integral support function on the multidegree. $h_2(\underline{\alpha}) = 3(1) - 2(2) = -1$ which means the order of the monomial x^1y^2 as a differential operator over this ring is 1.

All points on the dashed line in $\mathcal{C}_I$ are zeroed when acted on by the differential operators with multidegree $(1, 2)$ as they all have a factor of $(3\theta_x - 2\theta_y)$ by Theorem 2.6.

3. Differential Powers

Definition 3.1. [DDSG+18, Definition 2.2] Let I be an ideal of R. The n**th differential power** of I is defined

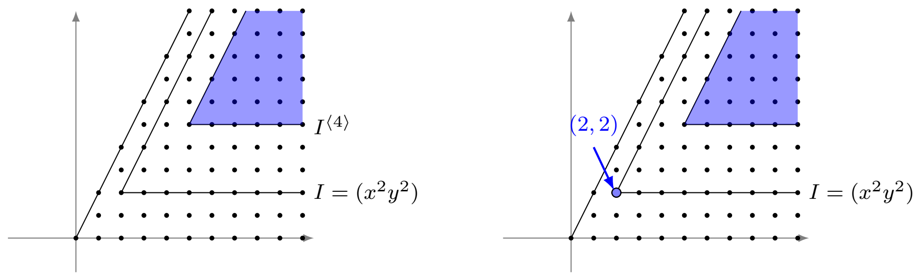
$$I^{\langle n \rangle} = \left\{ f \in R \;\middle|\; \delta(f) \in I \text{ for all } \delta \in \mathrm{D}^{n-1}(R) \right\}$$

Proposition 3.2. Let I be an ideal of a normal semigroup ring R. Then $I^{\langle n \rangle}$ is an ideal for positive integer n.

Proof. Let f and g be elements of $I^{\langle n \rangle}$ and $\delta \in \mathrm{D}^{n-1}(R)$. $\delta(f - g) = \delta(f) - \delta(g)$ is a difference of elements in the ideal I, so $I^{\langle n \rangle}$ is closed under addition and inverses. We now prove by induction on n that $I^{\langle n \rangle}$ is an ideal. As multiplying by an arbitrary $g \in R$ is the same as applying an arbitrary order 0 differential operator, we have already shown our base case. Now let δ be a differential operator of order $n - 1$, g an arbitrary element of R, and suppose $I^{\langle n-1 \rangle}$ is an ideal. Since $[\delta, g](f) = \delta(gf) - g\delta(f)$, we have that $\delta(gf) = [\delta, g](f) + g\delta(f)$. $\delta(f)$ is already an element of I, so then $g\delta(f)$ is as well. Since $[\delta, g]$ is of strictly lower order than $n - 1$, we have that $[\delta, g](f)$ is an element of I and so $\delta(gf)$ is also an element of I. Thus if $f \in I^{\langle n \rangle}$, then $gf \in I^{\langle n \rangle}$. □

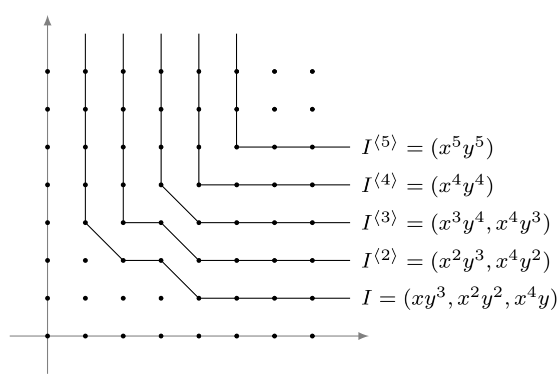
As the following example demonstrates, treating the membership of $\delta(f)$ in I as being solely dependent on the operators strictly of order n instead of all operators in $\mathrm{D}^n(R)$ keeps differential powers from being ideals.

Example 3.3. Take the rational normal curve of degree two $R = \mathbb{C}[x, xy, xy^2]$ and the ideal $I = (x^2y^2)$. The graph on the left represents the fourth differential power $I^{\langle 4\rangle}$ and the graph on the right represents the monomials in R sent to I by differential operators of order strictly equal to 3.



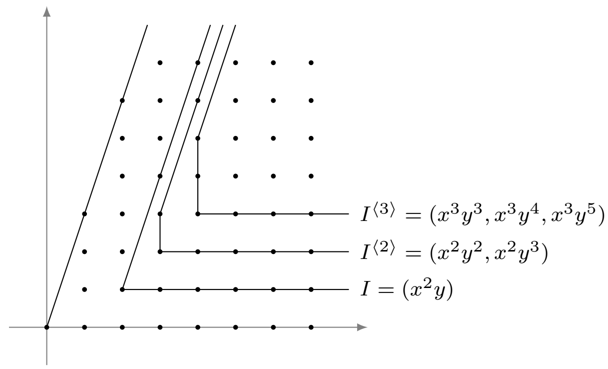
Without considering differential operators of lower order, the set of monomials on the right do not form an ideal as, for example, x^2y^2 is in the set while $x^3y^3 = xy \cdot x^2y^2$ is not.

Example 3.4. In the ring $\mathbb{C}[x, y]$, the differential operators in $\mathrm{D}^n(R)$ correspond to the operators $\partial_x^{-i}\partial_y^{-j}$ corresponding to points (i, j) in chambers II, III, and IV where $-i-j \leq n$. If adding (i, j) to the multidegree of a monomial keeps us within an ideal for all such i, j, then the point is in the differential power $I^{\langle n+1\rangle}$. Let $I = (xy^3, x^2y^2, x^4y)$.



Remark 3.5. In [KMP+26], the authors examine the asymptotic behavior of differential powers in the polynomial case and conclude that interior ideals are eventually principal.

Differential powers in the semigroup case are not so well-behaved in that the next differential power of a principal ideal is not always principal. The ideal (x^2y) in the rational normal curve of degree 3 for example does not stay principal.



It seems that one of the properties preserved between different rings is that sequential differential powers are "smaller" in the sense that subsequent powers are contained in previous ones. We make this thought rigorous through the following proposition.

Proposition 3.6. The following hold for ideals I, J, and K in an affine semigroup ring R:

(1) $I^{\langle n+1\rangle} \subseteq I^{\langle n\rangle}$
(2) $I \subseteq J \implies I^{\langle n\rangle} \subseteq J^{\langle n\rangle}$
(3) $I^{\langle n\rangle} \subsetneq J^{\langle n\rangle} \implies I \subsetneq J$
(4) $\left(J^{\langle n\rangle} \cap K^{\langle n\rangle}\right) = (J \cap K)^{\langle n\rangle}$

Proof.

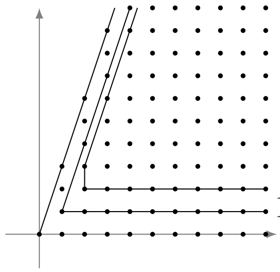
1) and 2) follow from the definition of differential power.
3) The non-strict containment follows inductively from property 1); what remains is to show the containment is proper. Suppose that f is an element of $J^{\langle n\rangle}$ and not $I^{\langle n\rangle}$. For all differential operators δ of order $n-1$ or less, $\delta(f) \notin I$ and $\delta(f) \in J$, thus there exists an element of J not in I. Therefore $I \subsetneq I$.
4) Let f be a polynomial in $(J \cap K)^{\langle n\rangle}$, so $\forall \delta \in \mathrm{D}^{n-1}(R)$, $\delta(f) \in J \cap K$, so $\delta(f)$ is an element of both J and $K \implies \delta(f) \in J^{\langle n\rangle}$ and $\delta(f) \in K^{\langle n\rangle}$

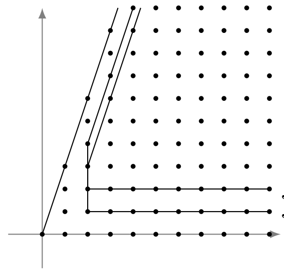
Now suppose that f is in $J^{\langle n\rangle}$ and $K^{\langle n\rangle}$. $\delta(f)$ is then in both J and K so $\delta(f) \in (J \cap K) \implies f \in (J \cap K)^{\langle n\rangle}$. Therefore $\left(J^{\langle n\rangle} \cap K^{\langle n\rangle}\right) = (J \cap K)^{\langle n\rangle}$

□

Corollary 3.7. Let I and J be ideals and let m and n be positive integers. If $I^{\langle n\rangle} \subseteq J^{\langle n+m\rangle}$, then $I \subsetneq J$.

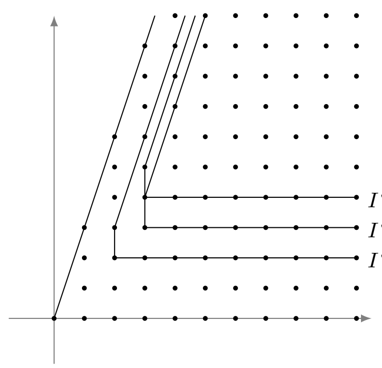
Note that properties 2 and 3 from proposition 3.6 are not biconditional statements. For these properties, we see from the following example that their converses fail when the containment $I^{\langle n\rangle} \subseteq J^{\langle n\rangle}$ is not proper.





The converse of the second property fails as $I^{\langle 2 \rangle} \subseteq I^{\langle 2 \rangle}$ and $J \subseteq I$; the converse of the third property fails as $J \subsetneq I$ and $J^{\langle 2 \rangle} = I^{\langle 2 \rangle}$. From the second differential power onward the ideals $J^{\langle n \rangle}$ and $I^{\langle n \rangle}$ will continue to be equal.

The examples of differential powers of principal ideals in the affine semigroup ring setting have so far seen the number of generators grow with each differential power. The question of whether this growth will continue forever is quickly answered by taking the third differential power of either one of the ideals from the previous example.

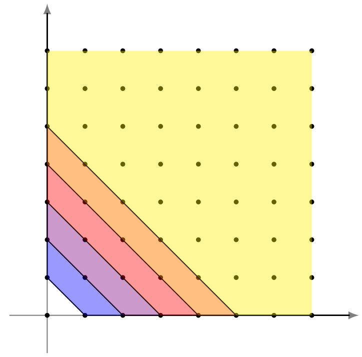


Although the number of generators increases by one going from $I^{\langle 2 \rangle}$ to $I^{\langle 3 \rangle}$, the ideal is once again principal for the fourth differential power.

Definition 3.8. Given an ideal I of an affine semigroup ring, the function $\pi_I(n)$ returns the number of generators of $I^{\langle n \rangle}$.

The rest of this section is an investigation of the way the number of generators of an ideal changes with each differential power. While the focus will be mainly on ideals which are principal and interior, we'll quickly look at the properties of differential powers of non-principal ideals for the sake of completeness.

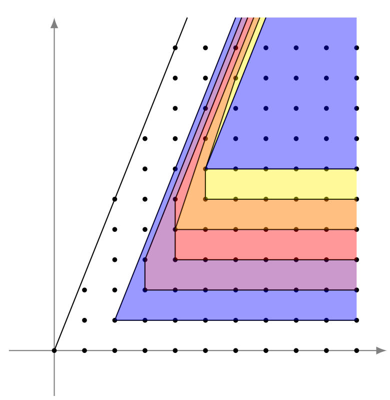
For $I = (x, y)$ in $\mathbb{C}[x, y]$, $\pi_I(n)$ grows continually. As usual we include a figure exhibiting this behavior, though the labeling of such the differential powers must be different as the facets of $\text{cone}(I^{\langle n \rangle})$ will overlap. The region $\text{cone}(I^{\langle n \rangle}) \setminus \text{cone}(I^{\langle n+1 \rangle})$ will be colored according to the value of $\pi_I(n)$.



The number of generators matches the function $\pi_I(n) = n+1$, which holds for all $n \geq 1$. As seen in section 2, $\pi_I(n)$ becomes the number of points (x_0, y_0) such that $x_0 + y_0 = n$. This is exactly $n + 1$. We once again direct the reader to [KMP+26] for more information on the long-term behavior of differential powers in polynomial rings.

Generally if an ideal I has a generator on the facet of an affine semigroup ring, then so will $I^{\langle n \rangle}$ for any $n \in \mathbb{N}$. If $x^{m\underline{v}_2}$ lies on $\underline{v}_2$ for example, then any differential operator in chambers $\mathcal{C}_{II}$ or $\mathcal{C}_{III}$ will send $x^{m\underline{v}_2}$ to zero, which is always an element of the ideal. This means that the only differential operators which can annihilate $x^{m\underline{v}_2}$ are those in $\mathcal{C}_{IV}$, though for any given order there are only finitely many elements on $\underline{v}_2$ that are taken out of the ideal generated by $x^{\underline{\alpha}}$. Therefore some $p > m$ exists such that $x^{p\underline{v}_2}$ is a part of $I^{\langle n \rangle}$.

We now restrict our view of differential powers to principal interior ideals of affine semigroup rings. As we saw with the rational normal curve of degree 3, $\pi_I(n)$ would grow linearly before dropping to 1 again. We should check that whether it is a unique property of rational normal curves that the ideals become principal again, so we look at differential powers of the ring $\mathbb{C}[x, xy, xy^2, x^2y^5]$ and the ideal (x^2y). As this ideal will have more than one differential power with 2 generators, we will instead color the principal ideal in blue and other differential powers according to n.



Observe that the growth in this ring is not linear as it was in the rational normal setting and that we see two differential powers with 2 generators. What the rings agree on is that the principal ideal is eventually principal again.

Definition 3.9. Let I be an interior principal ideal in an affine semigroup ring. For p the smallest natural number greater than 1 with $\pi_I(p) = 1$, the sequence $\{\pi_I(1), \dots, \pi_I(p-1)\}$ is the **cycle type** of I and $p-1$ is the **period** of π_I. If $\pi_I(n) = 1$ for all n, then the cycle type of I is $\{1\}$.

Using the previous example, the cycle type of (x^2y) in $\mathbb{C}[x, xy, xy^2, x^2y^5]$ is $\{1, 2, 3, 2, 3\}$. The remaining part of this section is dedicated to proving that π_I is actually periodic. For this we will use Pick's theorem, a result from geometry, in order to find a maximum value of π_I for an interior principal ideal I.

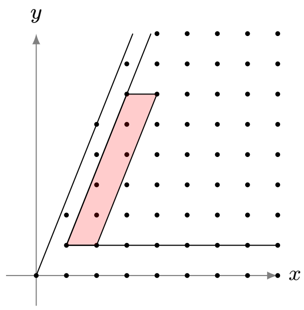
Theorem 3.10 (Pick's Theorem [Gru07])**.** The area of a simple closed polygon whose vertices have integer coordinates is given by $i + \frac{\beta}{2} - 1$, where i is the number of interior points and β is the number of boundary points.

Lemma 3.11. Let L be the lattice generated by the primitive vectors of a semigroup ring. The index of L in $\mathbb{Z}^2$ as abelian groups is the determinant of the matrix whose columns are the primitive vectors generating L.

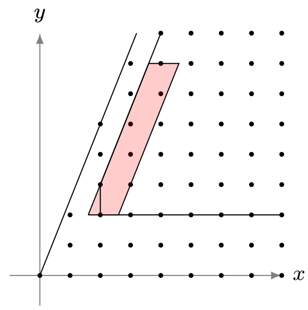
Proof. Let L be the subgroup of $\mathbb{Z}^2$ generated by vectors $v_1 = (a, b)$ and $v_2 = (c, d)$ and consider the parallelogram formed by v_1 and v_2. No point lying strictly within the parallelogram is in the $\mathbb{Z}$-span of $\{\underline{v}_1, \underline{v}_2\}$ and any two distinct points in the parallelogram are too close to differ by an integer combination of $\underline{v}_2$ and $\underline{v}_2$. Thus the number of points in the interior of the parallelogram is one less than the number of cosets in the quotient $\mathbb{Z}^2$. By Pick's Theorem, the number of interior points is equal to $A - \frac{\beta}{2} + 1$, where A is the area of the parallelogram. There are four points on the boundary of the parallelogram, so we have $A - 1$ interior points. Since all points on the boundary of the parallelogram are in the same coset, they count towards one more point in $\mathbb{Z}^2/L$ and thus $A = [\mathbb{Z}^2 : L]$. $\square$

Definition 3.12. The **index** of a semigroup ring is the index of its primitive lattice L in the semigroup $\mathbb{Z}^2$ and is denoted σ_L.

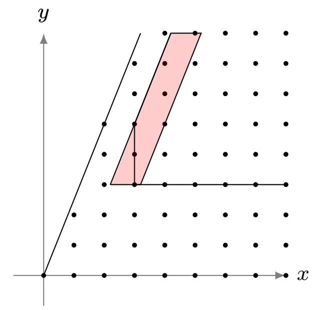
When $(x^{\underline{\alpha}})^{\langle n \rangle}$ is not principal, monomials in the parallelogram are candidates to be minimal generator of the ideal. Because all points outside of the parallelogram differ by some sum of $\underline{v}_1$ and $\underline{v}_2$, these points make up all the possible minimal generators of $I^{\langle n \rangle}$.



(xy) - principal



$(xy)^{\langle 2\rangle}$ - two-generated



$(xy)^{\langle 3\rangle}$ - three-generated

So not only does the index of the ring set an upper bound on $\pi_I(n)$ but, as the next theorem will show, the length of the cycle is also bounded above by this number.

Theorem 3.13. Let S be an affine semigroup with primitive lattice L and I a principal ideal in the interior of an affine semigroup ring $k[S]$. The cycle length of $\pi_I(n)$ is bounded above by the index of L in $\mathbb{Z}^2$.

Proof. Let L, generated by $\underline{v}_1$ and $\underline{v}_2$, have index σ_L. We have that, for a principal ideal $I = (x^{\underline{u}})$, a monomial $x^{\underline{\alpha}}$ is in $I^{\langle n\rangle}$ if and only if the following holds for $i = 1, 2$.

$$h_i(\underline{\alpha}) - h_i(\underline{u}) \geq n - 1$$

Given a coset represntative $\omega_r \in \mathbb{Z}^2/L$, the exponent vector $\underline{\alpha}$ of a monomial has a presentation $\omega_r + k_1\underline{v}_1 + k_2\underline{v}_2$ for unique integers k_1 and k_2. As the vectors $\underline{v}_1$ and $\underline{v}_2$ are linearly independent, we can solve for the k_i when ω_r and n are given.

$$\begin{aligned} h_i(\underline{\alpha}) - h_i(\underline{u}) &= h_i(\omega_r + k_1\underline{v}_1 + k_2\underline{v}_2) - h_i(\underline{u}) \\ &= h_i(\omega_r) + k_1 h_i(\underline{v}_1) + k_2 h_i(\underline{v}_2) - h_i(\underline{u}) \geq n - 1 \end{aligned}$$

The primitive vectors each vanish on their respective integral support functions, so $k_j h_i(v_j)$ is zero when $j = i$. When $i \neq j$, the value of $h_i(v_j)$ is exactly equal to σ_L. Without loss of generality, let $i \neq j$. Making appropriate substitutions, the inequality is now

$$h_i(\omega_r) + k_j\sigma_L - h_i(\underline{u}) \geq n - 1$$

solving for the integer k_j, we get the inequality

$$k_j \geq \left\lceil \frac{h_i(\underline{u}) - h_i(\omega_r) + n - 1}{\sigma_L} \right\rceil$$

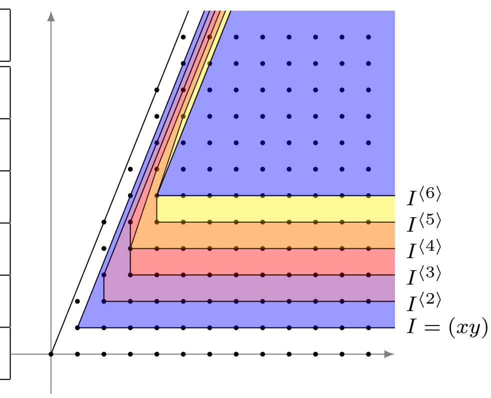
Since the fractional part of k_j before the ceiling are periodic with respect to n, differential powers of I that differ by a multiple of σ_L are shifts of the other by an integer combination of $\underline{v}_1 + \underline{v}_2$. The generators of the ideals are then in the same cosets of the quotient lattice

$\mathbb{Z}^2/L$, thus the number of generators of the ideal are the same for each nth differential power. $\square$

Remark 3.14. For a fixed n, the map $(r,p,q) \mapsto (\omega_r + (k_1+p)\underline{v}_1 + (k_1+q)\underline{v}_2)$ with $(r,p,q) \in \{1,\ldots,\sigma_L\} \times \mathbb{N}^2$ provides us with an algorithm to generate the lattice of an affine semigroup ring using only the primitive vectors and appropriate coset representatives ω_r. For affine semigroups with $\underline{v}_1 = (1,0)$ distinct representatives ω_r may be given by $(0,r)$, so with an appropriate isomorphism T to such a ring there are representatives $T^{-1}(\omega_r)$ for any affine semigroup ring.

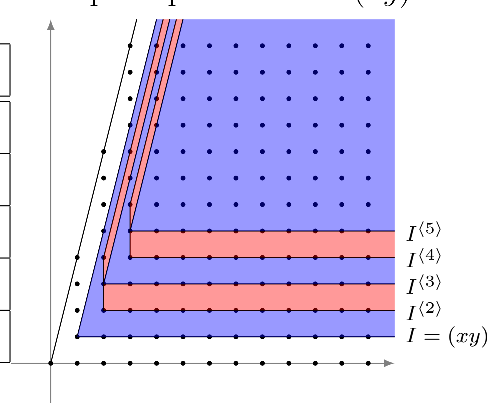
To check the upper bound against some examples, consider the semigroup ring $R = \mathbb{C}[x, xy, xy^2, x^2y^5]$ with primitive vectors $(1,0)$ and $(2,5)$. The period of $\pi_I(n)$ has length at most 5, so we expect $\pi_I(6) = 1$.

n	$\pi_I(n)$	$I^{\langle n\rangle}$
1	1	(xy)
2	2	(x^2y^2, x^2y^3)
3	3	(x^3y^3, x^3y^4, x^3y^5)
4	2	(x^3y^4, x^4y^7)
5	3	(x^4y^5, x^4y^6, x^5y^9)
6	1	(x^4y^6)



When the primitive index σ_L of an affine semigroup ring is prime, the cycle length of $\pi_I(n)$ is equal to either σ_L or 1 as in the example above. If the primitive index of a ring R is not prime, then it is possible that the length of $\pi_I(n)$ will be some factor of this index. Take the rational normal curve of degree 4 and the principal ideal $I = (xy)$.

n	$\pi_I(n)$	$I^{\langle n\rangle}$
1	1	(xy)
2	3	(x^2y^2, x^2y^3, x^2y^4)
3	1	(x^2y^3)
4	3	(x^3y^4, x^3y^5, x^3y^6)
5	1	(x^3y^5)



In this case, the cycle length is 2 and we observe that the period of $\pi_I(n)$ is not exactly equal to σ_L. This behavior is perhaps better explained by the vertex of cone$(I^{\langle n\rangle})$. In particular, the vertex only lands on an integer point when n is odd. We verify by solving for the intersection of the facets given a differential power n with a system of linear equations.

$$\begin{pmatrix} 4 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} n-1 \\ n-1 \end{pmatrix}$$

The solution to this system is given by

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \frac{n-1}{2} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

The primitive rays of the differential power intersect at an integer point when n is even, thus every other differential power is generated by a single element.

Theorem 3.15. Let $\underline{v}_1 = (a, b)$ and $\underline{v}_2 = (c, d)$ be primitive vectors of an affine semigroup ring. For a principal ideal I, the cycle length of $\pi_I(n)$ is $\dfrac{\sigma_L}{\gcd(a + c, b + d, \sigma_L)}$.

Proof. Suppose the ideal is generated by the monomial $x^{\alpha_1} y^{\alpha_2}$. Since the ideal is principally generated when point of the cone$(I^{\langle n \rangle})$ lies on an integer point, the cycle will end with a length of n. The smallest such n is found by solving a system of linear equations.

$$\begin{aligned} d(x - \alpha_1) - c(y - \alpha_2) &= n - 1 \\ -b(x - \alpha_1) + a(y - \alpha_2) &= n - 1 \end{aligned}$$

With the cyclic nature of the ideals being independent of where inside the lattice the ideal is placed, we assume $\underline{\alpha} = (0, 0)$. The solution for a given n is found by applying Cramer's rule.

$$\frac{n-1}{ad - bc} \begin{pmatrix} a + c \\ b + d \end{pmatrix}$$

This vector is in $\mathbb{Z}^2$ if and only if the product of $\frac{n-1}{\sigma_L}$ with $(a + c)$ and $(b + d)$ are integers. This is trivial when $n = 1$, so suppose $n \neq 1$ and let p be the greatest common divisor of $a+c$, $b+d$, and σ_L so that the values are the integer products pA, pB, and $p\sigma_0$ respectively.

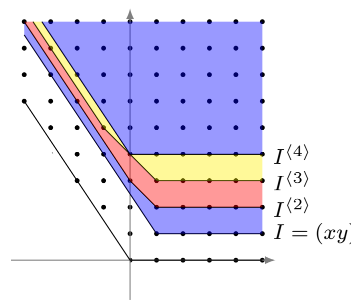
$$\frac{n-1}{\sigma_0} \begin{pmatrix} A \\ B \end{pmatrix}$$

A, B, and σ_0 are setwise coprime, so the only time the point is an integer is when $n-1 \equiv 0 \mod \sigma_0$. Thus the ideal $I^{\langle n \rangle}$ is principal only when $n \equiv 1 \mod \sigma_0$, therefore the cycle length is $\sigma_0 = \dfrac{\sigma_L}{\gcd(a + c, b + d, \sigma_L)}$ as desired. □

As we have yet to do so, we should see that this also holds for semigroups whose $\underline{v}_2$ facet is not in the first quadrant. The following semigroup is a subset of the *Laurent polynomial ring* $\mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ but the construction of differential operators, and hence the geometric interpretation of them, is the same.

Example 3.16. The ring $\mathbb{C}[x^{-2}y^3, x^{-1}y^2, y, x]$ has primitive vectors $\underline{v}_1 = (1, 0)$ and $\underline{v}_2 = (-2, 3)$. Using the formula from theorem 3.15 we find that the cycle length should be $\frac{3}{\gcd(-1,3,3)} = 3$.

n	$\pi_I(n)$	$I^{\langle n \rangle}$
1	1	(xy)
2	2	(y^3, xy^2)
3	3	$(x^{-1}y^5, y^4, xy^3)$
4	1	(y^4)



4. Future Work

We've seen that the facets of a two dimensional affine semigroup ring completely characterize the behavior of the differential powers of principal ideals, but we have yet to consider the reverse direction: what do differential powers say about the ring? The following corollary to Theorem 3.15 answers this in a restricted case.

Corollary 4.1. If two affine semigroup rings over rational normal curves of degree greater than 1 have the same cycle type, then the rings are isomorphic.

Proof. The formula from Theorem 3.15 implies that, for any $n \in \mathbb{N}$, there are exactly two rational normal curves with cycle length n. This will not cause issues as it is sufficient to show that the cycle types of any two non isomorphic curves differ in at least one place. Let $I = (x^{\underline{\alpha}})$ be a principal ideal in a rational normal curve of degree d. The facets of $I^{\langle 2 \rangle}$ intersect their first integer coordinates at $\underline{\alpha} + (1, 1)$ and $\underline{\alpha} + (1, b - 1)$. Besides these points, we have candidate generators $\{(1, 2), \ldots, (1, b - 2)\}$ added to $\underline{\alpha}$. As each of these points are vertically aligned and our ring does not have the primitive vector $(0, 1)$, all candidates must be in separate cosets. The only point left to consider is $\underline{\alpha} + (2, b - 1)$, though this is point is a combination of $\alpha + (1, b - 1)$ and $(1, 0)$. Therefore $I^{\langle 2 \rangle}$ has $b - 1$ minimal generators and we conclude that cycle types differ between rational normal curves of separate degree. $\square$

We ask now whether the cycle type classifies *all* affine semigroup rings up to isomorphism. The rings $\mathbb{C}[x, xy, xy^2]$ and $\mathbb{C}[x, y]$ make quick work of this proposal as they are not isomorphic and have cycle type $\{1\}$. Even worse for our case, the issue of cycle types not being unique extends beyond relatively small rings; the sublattices $\begin{pmatrix} 1 & 3 \\ 0 & 11 \end{pmatrix}$ and $\begin{pmatrix} 1 & 4 \\ 0 & 11 \end{pmatrix}$, once given appropriate generators, form affine semigroup rings which are not isomorphic and both have cycle type $\{1, 4, 2, 5, 3, 3, 4, 4, 2, 5, 3\}$. Tables for these differential powers on the ideal $I = (1, 1)$ are given below, with the first and second rings' powers being given below.

n	$\pi_I(n)$	$I^{\langle n \rangle}$ for $v_2 = (3, 11)$	$I^{\langle n \rangle}$ for $v_2 = (4, 11)$
1	1	(xy)	(xy)
2	4	$(x^2y^2, x^2y^3, x^2y^4, x^3y^8)$	$(x^2y^2, x^2y^3, x^3y^6, x^4y^9)$
3	2	(x^2y^3, x^2y^4)	(x^2y^3, x^3y^6)
4	5	$(x^3y^4, x^3y^5, x^3y^6, x^3y^7, x^4y^{11})$	$(x^3y^4, x^3y^5, x^4y^8, x^5y^{11}, x^6y^{14})$
5	3	(x^3y^5, x^3y^6, x^3y^7)	$(x^3y^5, x^4y^8, x^5y^{11})$
6	3	$(x^3y^6, x^4y^{10}, x^5y^{14})$	(x^4y^6, x^4y^7, x^5y^8)
7	4	$(x^4y^7, x^4y^8, x^4y^9, x^4y^{10})$	$(x^4y^7, x^5y^{10}, x^6y^{13}, x^7y^{16})$
8	4	$(x^4y^8, x^4y^9, x^5y^{13}, x^6y^{17})$	$(x^5y^8, x^5y^9, x^5y^{10}, x^6y^{13})$
9	2	(x^4y^9, x^5y^{13})	(x^5y^8, x^5y^9)
10	5	$(x^5y^{10}, x^5y^{11}, x^5y^{12}, x^6y^{16}, x^7y^{20})$	$(x^6y^{10}, x^6y^{11}, x^6y^{12}, x^7y^{15}, x^8y^{18})$
11	3	$(x^5y^{11}, x^5y^{12}, x^6y^{16})$	$(x^6y^{11}, x^6y^{12}, x^7y^{15})$
12	1	(x^5y^{12})	(x^6y^{12})

We are left then with a different question: what property in conjunction with cycle type determines the ring up to isomorphism?

One route we can take to answer this question is to consider the affine semigroup rings as the quotient of a polynomial ring and a binomial ideal. Such quotient representations exist for all affine semigroup rings in the purview of this text. These quotient rings encode a linear relation between the generators of the affine semigroup ring, so knowing that the differential powers are also governed by a linear relation makes exploring this relationship worthwhile for the author.

Another avenue for research on this subject is the behavior of differential powers in general n-dimensional affine semigroup rings. The subject of differential operators and the geometric presentation of them in three dimensions has already been the subject of exploration (see [BCK$^+$21]), though applying the results of this paper to higher dimensions will require a different suite of tools. The argument for the number of candidates for minimal generators of an ideal relied on Pick's theorem, which unfortunately does not apply to dimensions 3 and over. Approaches based on the Smith normal form of the quotient lattice may be general enough to relate our results to this expanded context.

Lastly, we consider the growth of $\pi_I(n)$ and ask what linear numerical function, if any, describes the function for the duration of its cycle. We have seen in the polynomial case that non-interior ideals intersecting both facets grow linearly, but we so far do not know what equation describes cycle types. We propose that cycle types of affine semigroup rings in n-dimensions described by polynomials of degree at most $n-1$, and we hope to verify this in future work on this topic.

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