On Bond, Invasion, and First Passage Percolation

by

Aldo Morelli

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Abstract

For the past 75 years, percolation models have been studied both for their mathematical content and their applicability in modeling phenomena in many scientific fields. In this thesis we provide brief introductions to the models of bond, invasion, and first passage percolation, and describe their connections, including a coupling between invasion and first passage percolation with log-uniform passage times.

Chapter 1

Introduction

The study of percolation models began in 1957 with Broadbent and Hammersley [BH57], who introduced, for directed crystal lattices, both Bernoulli bond percolation—a process of forming a random subgraph where each edge is independently included according to a Bernoulli random variable—and its associated critical threshold. This model has been used to study a variety of phenomena, including infectious disease, cyber security, and quantum magnetism [Mey07, CJLP24, ZYH+24].

Many other percolative models have been introduced and studied both in connection with physical processes that they model and to one another. One such model is invasion percolation, introduced in 1983 by Wilkinson and Willemsen [WW83], inspired by the work of Chandler, Koplik, Lerman, and Willemsen [CKLW82] from the previous year on the flow of fluids in porous media. Invasion percolation is a random process where edges in a graph are independently given a weight uniformly from 0 to 1, and non-invaded edges and vertices are sequentially invaded according to the minimal weight of a non-invaded edge incident to an invaded vertex. In 1985, Chayes, Chayes, and Newman [CCN85] utilized Bernoulli bond percolation to study invasion percolation. In so doing, they provided an equivalence between the density of the invaded set and the existence of an infinite connected component at the critical threshold of bond percolation.

Another widely studied model is first passage percolation, introduced in 1965 by Hammersley and Welsh [HW65], which has been used to model cell biology and study path-finding algorithms [Bre22, KSG$^+$24], among other topics. First passage percolation concerns the random assignment of passage times to edges in a graph, and the associated induced metric. An intuitive description of this process is given by water flowing from a source at a distinguished vertex, where the time taken to traverse an edge is randomly determined. We may then ask how the set of vertices that the water has reached evolves over time. Building upon previous work, Cox and Durrett proved in 1981 that for each passage time distribution in a general class, the set of wetted vertices tends toward a deterministic shape [CD81].

In this thesis, we provide a brief introduction to these three percolation models and some of their connections. These connections are formed in relation to bond percolation at the critical threshold on the lattice $\mathbb{L}^d$, and serve to translate the question of the existence of an infinite connected component at this threshold into questions concerning invasion and first passage percolation. The main contribution of this thesis concerns a coupling of invasion and first passage percolation, and associated simulations, discussed in **Chapters** 6 and 7. The bulk of the material within these chapters has already appeared in the manuscript [Mor25], which, to the author's knowledge, is novel.

Table 1.1: List of Symbols

Symbol	**Description**
Ω	Sample space
ω	A configuration in Ω
Σ	Event Space
$\mathcal{E}$	Event
$\mathbb{P}$	Probability Measure
$\mathscr{P}$	Power set
$[\![a, b]\!]$	The set $\{a, a+1, \ldots, b\}$
$G = (V, E)$	A Graph G with vertices V and edges E
$d(x, y)$	Graph metric
$e \sim e'$	Edges e and e' are adjacent
B_R	Ball of radius R about 0 in the graph metric d
E_R	Set of edges between vertices of B_R
∂X	Vertex inner boundary of X
$\partial_e X$	Edge boundary of X
$\mathbb{L}^d$	d-dimensional nearest neighbor lattice
$C_p(v)$	Cluster of a vertex v in p-bond percolation
$\theta(p)$	Percolation probability function
$p_c(d)$	Critical threshold for d-dimensional bond percolation
$x \leftrightarrow y$	There exists an open path between x and y in p-bond percolation
I	Invaded subgraph in invasion percolation
$<_{\texttt{IP}}$	Total ordering on invaded vertices and edges of $\mathbb{L}^d$
Λ_m	Box about 0 of side length $2m$ in $\mathbb{L}^d$
$\mathcal{I}_x$	Event that x is eventually invaded
τ_e	Passage time of an edge e

Symbol	**Description**
$T(\gamma)$	Total passage time of path γ
$T(x, y)$	Passage time between vertices x, y
$B(t)$	Ball of radius t about 0 in first passage percolation
$\tau_{K,e}$	Passage time of an edge e in log-uniform first passage percolation
T_K	Passage time in log-uniform first passage percolation
$<_{\texttt{FPP}}$	Total ordering on $\mathbb{L}^d$ in first passage percolation
$v_R^{\texttt{IP}}$	The first vertex invaded in ∂B_R

Chapter 2

Background

2.1 Probability theory

The material of this section is based on *Measure Theory Vol. 1* by Bogachev [Bog07] and *Probability Theory and Examples by Durrett* [Dur19]. For a more thorough treatment see [Bog07, Chapter 1 & Section 3.5] and [Dur19, Chapters 1 & 2].

Given a set Ω, a σ-algebra $\Sigma \subset \mathscr{P}(\Omega)$ is a collection of sets satisfying the following properties

i) $\Omega \in \Sigma$

ii) If $\mathcal{E} \in \Sigma$, then $\mathcal{E}^c \in \Sigma$

iii) If $\{\mathcal{E}_i\}_{i \in I} \subset \Sigma$ is a countable collection of sets, then $\cup_i \mathcal{E}_i \in \Sigma$.

By De Morgan's laws, we see that a σ-algebra Σ is also closed under countable intersections,

$$\bigcap_i \mathcal{E}_i = \left(\bigcup_i \mathcal{E}_i^c \right)^c \in \Sigma$$

whenever $\mathcal{E}_i \in \Sigma$.

Given a collection of sets $\{S_\alpha\}_{\alpha \in A} \subset \mathscr{P}(\Omega)$, we say Σ is a σ-algebra *generated* by $\{S_\alpha\}_{\alpha \in A}$ if Σ is the intersection of all σ-algebras containing $\{S_\alpha\}_{\alpha \in A}$.

A set Ω and a σ-algebra Σ together form a *measure space* (Ω, Σ), where we say a set $\mathcal{E} \subset \Omega$ is *measurable* whenever $\mathcal{E} \in \Sigma$.

A *probability measure* $\mathbb{P}$ on a pair (Ω, Σ) is a function $\mathbb{P} : \Sigma \to [0, 1]$ which satisfies

i) $\mathbb{P}(\mathcal{E}) \geq 0$ for all $\mathcal{E} \in \Sigma$.

ii) If $\{\mathcal{E}_i\}$ is a countable subset of Σ with $\{\mathcal{E}_i\}$ pairwise disjoint, then

$$\mathbb{P}\left(\bigcup_i \mathcal{E}_i\right) = \sum_i \mathbb{P}(\mathcal{E}_i).$$

iii) $\mathbb{P}(\Omega) = 1$.

A useful consequence of these properties is the *union bound* given by

$$\mathbb{P}\left(\bigcup_i \mathcal{E}_i\right) \leq \sum_i \mathbb{P}(\mathcal{E}_i)$$

for any countable $\{\mathcal{E}_i\} \subset \Sigma$.

A triple $(\Omega, \Sigma, \mathbb{P})$ defines a *probability space*, where we refer to Ω as the *sample space*, and Σ as the *event space*, with elements $\mathcal{E} \in \Sigma$ as *events*. Two events $\mathcal{E}_1, \mathcal{E}_2 \in \Sigma$ are said to be *independent* whenever $\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2) = \mathbb{P}(\mathcal{E}_1)\mathbb{P}(\mathcal{E}_2)$. Given events $\mathcal{E}_1, \mathcal{E}_2 \in \Sigma$ with $\mathbb{P}(\mathcal{E}_2) > 0$, we define the *conditional probability* of $\mathcal{E}_1$ given $\mathcal{E}_2$ by

$$\mathbb{P}(\mathcal{E}_1 | \mathcal{E}_2) = \frac{\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2)}{\mathbb{P}(\mathcal{E}_2)}.$$

We see that whenever $\mathcal{E}_1, \mathcal{E}_2$ are independent, $\mathbb{P}(\mathcal{E}_1 | \mathcal{E}_2) = \mathbb{P}(\mathcal{E}_1)$.

It is often convenient to describe a measure $\mathbb{P}$ only on a subclass of sets in $\mathscr{P}(\Omega)$. The following proposition allows us to extend such a description uniquely to the σ-algebra generated by this subclass.

Proposition 2.1.1 ([Bog07], Theorem 1.5.6.)**.** Let $\{S_\alpha\} \subset \mathscr{P}(\Omega)$ be a collection of sets which is closed under finite unions and compliments, and contains $\emptyset$. Then any function $\mathbb{P} : \{S_\alpha\} \to \mathbb{R}$ which satisfies the properties of a probability measure on $\{S_\alpha\}$ extends uniquely to a probability measure on the σ-algebra generated by $\{S_\alpha\}$.

One important probability space is that of $([0,1], \mathcal{B}_I, \lambda)$, where $\mathcal{B}_I$ is the *Borel* σ-algebra generated by open subsets of $[0,1]$ with respect to the subspace topology, and λ is the *Lebesgue* measure, extending the natural notion of length of an interval given by

$$\lambda([a,b]) = b - a$$

to all subsets of $[0,1]$ in $\mathcal{B}_I$.

2.1.1 Product Measures

Finite Products

Given two probability spaces $(\Omega_1, \Sigma_1, \mathbb{P}_1), (\Omega_2, \Sigma_2, \mathbb{P}_2)$, we might hope to construct a product probability space $(\Omega_1 \times \Omega_2, \Sigma, \mathbb{P}_1 \otimes \mathbb{P}_2)$ such that for any events $\mathcal{E}_1 \in \Sigma_1, \mathcal{E}_2 \in \Sigma_2$,

$$(\mathbb{P}_1 \otimes \mathbb{P}_2)(\mathcal{E}_1 \times \mathcal{E}_2) = \mathbb{P}_1(\mathcal{E}_1)\mathbb{P}_2(\mathcal{E}_2). \tag{2.1}$$

Fortunately this property suffices to uniquely generate a product probability measure $\mathbb{P}_1 \otimes \mathbb{P}_2$. Letting $\Sigma_1 \otimes \Sigma_2$ denote the σ-algebra generated by sets of the form $\mathcal{E}_1 \times \mathcal{E}_2$ where $\mathcal{E}_i \in \Sigma_i$, we have the following proposition,

Proposition 2.1.2 ([Bog07], Theorem 3.3.1)**.** A measure $\mathbb{P}_1 \otimes \mathbb{P}_2$ satisfying (2.1) may be constructed in a unique way such that $(\Omega_1 \times \Omega_2, \Sigma_1 \otimes \Sigma_2, \mathbb{P}_1 \otimes \mathbb{P}_2)$ is a probability space.

The phrase "unique way" above hints at some detail being swept under the rug. The precise statement guarantees uniqueness of a probability measure on the Lebesgue completion of $\Sigma_1 \otimes \Sigma_2$, which, once restricted to the sub-σ-algebra $\Sigma_1 \otimes \Sigma_2$, will be the probability measure we consider.

By induction this defines any finite product of probability spaces, with the product of their respective measures being associative. We now wish to extend this construction to countably infinite products of probability spaces.

Infinite Products

Given a countable collection of probability spaces $(\Omega_i, \Sigma_i, \mathbb{P}_i)$, it is possible to uniquely define a product probability space $(\prod_i \Omega_i, \Sigma, \mathbb{P})$ which extends the $\mathbb{P}_i$'s in the following way. For an index set $I \subset \mathbb{N}$, define $\bigotimes_{i \in I} \Sigma_i$ to be the σ-algebra generated by events of the form $\prod_{i \in I} \mathcal{E}_i$ where $\mathcal{E}_i \in \Sigma_i$ and $\mathcal{E}_i = \Omega_i$ for all but finitely many indices i.

For each $n \in \mathbb{N}$ define

$$\mathcal{F}_n = \left\{ C \times \prod_{i=n+1}^{\infty} \Omega_i \;\middle|\; C \in \bigotimes_{i=1}^{n} \Sigma_i \right\}$$

to be the set of events which depend only on the first n coordinates of the product space. Note that C need not be of the form $\prod_{i=1}^n \mathcal{E}_i$ for $\mathcal{E}_i \in \Sigma_i$. For $\mathcal{F} := \bigcup_n \mathcal{F}_n$, the set of events which depend only on finitely many coordinates, we may define a function

$$\mathbb{P} : \mathcal{F} \to [0, 1]$$

given by

$$\mathbb{P}\left(C \times \prod_{i=n+1}^{\infty} \Omega_i \right) = \mathbb{P}_1 \otimes \cdots \otimes \mathbb{P}_n(C),$$

where $\mathbb{P}_1 \otimes \cdots \otimes \mathbb{P}_n$ is the finite product measure on $\prod_{i=1}^n \Omega_i$. This $\mathbb{P}$ is well defined, as each $\mathbb{P}_i(\Omega_i) = 1$.

Let $\Sigma = \bigotimes_{n=1}^{\infty} \Sigma_n$. The following proposition allows us to extend $\mathbb{P}$ to a probability measure on Σ.

Proposition 2.1.3 ([Bog07], Theorem 3.5.1)**.** $\mathbb{P} : \mathcal{F} \to [0,1]$ extends uniquely to a probability measure $\mathbb{P} : \Sigma \to [0,1]$.

Utilizing this proposition we may construct a product probability space $(\prod_i \Omega_i, \Sigma, \mathbb{P})$ from any countable collection of probability spaces $(\Omega_i, \Sigma_i, \mathbb{P}_i)$.

2.1.2 Random Variables and Distributions

Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{R}$ generated by the open subsets of $\mathbb{R}$ in the standard topology. Given a probability space $(\Omega, \Sigma, \mathbb{P})$, we say a function $X : \Omega \to \mathbb{R}$ is *measurable* if for any set $B \in \mathcal{B}$, $X^{-1}(B) \in \Sigma$, and we call such functions *random variables*. For a random variable X and measurable set $B \in \mathcal{B}$, we utilize the notation

$$\mathbb{P}(X \in B) := \mathbb{P}\{\omega \in \Omega : X(\omega) \in B\},$$

and for $t \in \mathbb{R}$,

$$\mathbb{P}(X \le t) = \mathbb{P}(X \in (-\infty, t]).$$

A finite collection of random variables $\{X\}_{i=1}^{n}$ are said to be *independent* whenever for any measurable sets $B_i \in \mathcal{B}$,

$$\mathbb{P}(X_i \in B_i, \forall i \in \{1, \dots n\}) = \prod_{i=1}^{n} \mathbb{P}(X_i \in B_i),$$

and similarly a countable set of random variables $\{X_i\}_{i \in \mathbb{N}}$ are independent whenever any finite sub collection is independent.

We often hope to deal with independent copies of the 'same' random variable. This is made precise with the notion of distributions.

We define a *cumulative distribution function* $F_X : \mathbb{R} \to [0,1]$ of a random variable X by,

$$F_X(t) = \mathbb{P}(X \leq t).$$

This cumulative distribution function captures the behavior of X, in the sense that for any measurable set $B \in \mathcal{B}$, the value of $\mathbb{P}(X \in B)$ may be recovered from F_X, as $\mathcal{B}$ is also generated as a σ-algebra by sets of the form $(-\infty, t]$ for $t \in \mathbb{R}$.

As a converse to the above consideration, let $F : \mathbb{R} \to [0,1]$ be a function which satisfies the following three properties:

i) F is non-decreasing.

ii) $\lim_{x\to\infty} F(x) = 1$, $\lim_{x\to-\infty} F(x) = 0$

iii) F is right continuous.

We then have the following standard proposition.

Proposition 2.1.4 ([Dur19], Theorem 1.2.1)**.** For $F : \mathbb{R} \to [0,1]$ which satisfies conditions **i)-iii)** above, there is a probability space $(\Omega, \Sigma, \mathbb{P})$ and a random variable X for which $F = F_X$.

Given two random variables X, Y, we say that they are *equal in distribution* if $F_X = F_Y$. We refer to a function F as in **Proposition** 2.1.4, slightly abusing terminology, as a *distribution*, and whenever F is continuous, we say that the distribution is continuous. In **Chapter** 5 we will consider a countable number of random variables $\{X_i\}_{i\in\mathbb{N}}$ which are independent of one another and which are all equal in distribution with F, and say that such a collection of random variables is *drawn independently* from F. Such a collection of random variables exists by the following construction:

For a distribution F, by **Proposition** 2.1.4 let $(\Omega, \Sigma, \mathbb{Q})$ be a probability space such that $X : \Omega \to \mathbb{R}$ is a random variable with $F_X = F$. Consider the product probability space $\left(\prod_{j\in\mathbb{N}}\Omega, \bigotimes_{j\in\mathbb{N}}\Sigma, \mathbb{P}\right)$ given by **Proposition** 2.1.3. For each $i \in \mathbb{N}$, let the random variable $X_i : \prod_{j\in\mathbb{N}}\Omega \to \mathbb{R}$ be given by

$$X_i(\{\omega_j\}_{j\in\mathbb{N}}) = X(\omega_i).$$

To see that X_i is indeed a random variable, we have for $B \in \mathcal{B}$,

$$\begin{aligned} X_i^{-1}(B) &= \{\{\omega_j\}_{j\in\mathbb{N}} : X(\omega_i) \in B\} \\ &= X^{-1}(B) \times \prod_{j\neq i}\Omega, \end{aligned}$$

which is measurable. Additionally, for any $n \in \mathbb{N}$, $X_1, \ldots, X_n$ are independent, as for any $B_1, \ldots, B_n \in \mathcal{B}$,

$$\begin{aligned} \mathbb{P}(X_i \in B_i, \forall i \in \{1, \ldots, n\}) &= \mathbb{P}\left\{\{\omega_j\}_{j\in\mathbb{N}} : X(\omega_i) \in B_i, \forall i \in \{1, \ldots, n\}\right\} \\ &= \mathbb{P}\left(\prod_{i=1}^{n} X^{-1}(B_i) \times \prod_{i>n}\Omega\right) \\ &= \prod_{i=1}^{n} \mathbb{Q}(X^{-1}(B_i)) \\ &= \prod_{i=1}^{n} \mathbb{P}\{X_i \in B_i\}. \end{aligned}$$

Finally, to see that for all $n, m \in \mathbb{N}$, X_n, X_m are equal in distribution, we have

$$\mathbb{P}\{X_n \leq t\} = \mathbb{Q}\{X \leq t\} = \mathbb{P}\{X_m \leq t\}.$$

2.2 Graph Theory

The following section follows Graph Theory by Diestel [Die24]. A graph is a pair $G = (V, E)$, where V is a set of *vertices* and E is a set containing two element subsets of V called *edges*. For $x, y \in V$, the edge $\{x, y\}$ is typically referred to as xy, and for an edge $e = xy$ we say that e is *incident* to the vertices x, y. We say that two vertices x, y are *adjacent* whenever there is an edge $xy \in E$, and similarly two edges e, e' are adjacent, denoted $e \sim e'$, whenever they are incident to a common vertex. A subgraph $G' = (V', E')$ of G is a graph with $V' \subset V$ and $E' \subset E$ where each $e' \in E'$ is of the form $e' = xy$ for some $x, y \in V'$.

A path γ in G is a subgraph with vertices $V' = \{x_0, x_1, \ldots, x_n\}$ and edges $E' = \{x_0x_1, x_1x_2, \ldots, x_{n-1}x_n\}$. We call x_0, x_n the *endpoints* of γ, and say that γ is a path from x_0 to x_n. The length of a path, denoted by $|\gamma|$, is given by the number of edges in γ. We call a graph *connected* if for any two vertices x, y there exists a path in G from x to y. It is possible to uniquely partition any graph G into connected subgraphs, which are termed the *connected components* of G. The connected component of a vertex $x \in V$ is then the connected component to which it belongs, and we call this connected component the *cluster* of x.

On connected graphs, there is a natural metric, called the graph distance, on V given by

$$d(x, y) = \min\{|\gamma| : \gamma \text{ a path from } x \text{ to } y\}.$$

Closed balls in this metric are given by

$$B_r(x) = \{y \in V : d(x, y) \leq r\}$$

and we denote the set of edges in this ball by

$$E_r(x) = \{e \in E : e = yz \text{ for } y, z \in B_r(x)\}.$$

Given a set of vertices, $X \subset V$, we define the *vertex (inner) boundary* of X by

$$\partial X = \{x \in X : \exists y \notin X, y \text{ adjacent to } x\}$$

In the case of the closed ball,

$$\partial B_r(x) = \{y \in V : d(x, y) = r\}.$$

We also define the edge boundary of X to be

$$\partial_e X = \{e \in E : e = xy, x \in X, y \notin X\}.$$

One collection of graphs which will be the exclusive focus of this thesis are the d-dimensional nearest neighbor lattices, where for each $d \in \mathbb{N}$,
$\mathbb{L}^d := (\mathbb{Z}^d, E)$, with

$$E = \{xy : x, y \in \mathbb{Z}^d, \|x - y\|_1 = 1\}.$$

Chapter 3

Bernoulli Bond Percolation

The following chapter provides a brief overview of bond percolation, drawing upon [DC18, Gri99, Ste09]. Bond percolation on the lattice $\mathbb{L}^d$ with parameter $p \in [0,1]$ is a process of producing a random subgraph of $\mathbb{L}^d$ where each edge is included according to an independent Bernoulli random variable with parameter p. We might imagine that for $p \approx 0$, this subgraph consists mostly of small, disconnected clusters, whereas for $p \approx 1$, it consists primarily of one large component. This intuition is supported by the simulations of bond percolation in $\mathbb{L}^2$ with parameters $p = 0.2, 0.49, 0.51, 0.8$ given in **Figure** 3.1. In what follows, we give a slightly nonstandard definition of bond percolation, motivated by a subsequent ease of coupling with invasion percolation.

Formally, given the lattice $\mathbb{L}^d$ with nearest neighbor edge set E, we define a product probability space $([0,1]^E, \Sigma, \mathbb{P})$, where Σ is the σ-algebra generated by the set of events which depend on finitely many edges $e \in E$, and $\mathbb{P}$ is the product measure with each coordinate $\mathbb{P}_e$ given by the Lebesgue measure on $[0,1]$. For a *configuration* $\omega \in [0,1]^E$, an edge $e \in E$ is said to be *open* in Bernoulli bond percolation with parameter $p \in [0,1]$ if $\omega(e) < p$, otherwise it is said to be *closed.* For fixed p, this is identical to the standard definition of bond percolation, where we work in the probability space $(\{0,1\}^E, \Sigma, \mathbb{P})$ with each coordinate of $\mathbb{P}$ a Bernoulli distribution of parameter p, and for a configuration $\nu \in \{0,1\}^E$, edges with $\nu(e) = 1$ are declared to be open.

For each $\omega \in [0,1]^E$, this process defines a subgraph of $\mathbb{L}^d$, with edge set given by the open edges. We will denote the cluster of a vertex $v \in \mathbb{L}^d$ in this subgraph $C_p(v)$, which will depend on the configuration ω. It is clear that for a fixed configuration ω and $p < p'$, $C_p(v) \subset C_{p'}(v)$, meaning that taking larger values of p will only serve to increase the size of the cluster of a vertex. Because we hope to study the size of clusters in bond percolation, we consider the set of configurations where the cluster of a vertex is infinite. The following proposition ensures that this is an event in Σ, and so it is meaningful to speak of its probability.

Proposition 3.0.1. Given some $v \in \mathbb{Z}^d$, the event

$$\mathcal{E} = \{\omega : |C_p(v)| = \infty\}$$

is measurable.

Proof. For each $n \in \mathbb{N}$, consider the event

$$\mathcal{E}_n = \{\omega : C_p(v) \cap \partial B_n(v) \neq \emptyset\}.$$

Because $\mathcal{E}_n$ depends only on the finitely many edges in $E_n(v)$, $\mathcal{E}_n \in \Sigma$, and thus

$$\mathcal{E} = \bigcap_{n=0}^{\infty} \mathcal{E}_n \in \Sigma.$$

□

By the above proposition, we may define the following function,

$$\theta(p) := \mathbb{P}\{|C_p(0)| = \infty\}$$

which describes the probability that the origin is in an infinite connected component in p-bond percolation. Because increasing p only serves to enlarge $|C_p(0)|$, $\theta(p)$ is a non-decreasing function.

The behavior of $\theta(p)$ has been studied since the introduction of bond percolation in [BH57], and the case $d = 1$ may be answered immediately.

Proposition 3.0.2. For $d = 1$,

$$\theta(p) = \begin{cases} 0, & p \in [0,1) \\ 1, & p = 1 \end{cases}$$

Proof. It is clear that $\theta(1) = 1$. For $p \in [0,1)$, we note that $C_p(0)$ is infinite in $\mathbb{L}$ if and only if all edges between non-negative integers or all edges between non-positive integers are open. We then have for each $n \in \mathbb{N}$,

$$\begin{aligned} \theta(p) &= \mathbb{P}\{|C_p(0)| = \infty\} \\ &\leq \mathbb{P}\{\text{All edges in } [\![0,n]\!] \text{ are open}\} + \mathbb{P}\{\text{All edges in } [\![-n,0]\!] \text{ are open}\} \\ &= 2p^n \end{aligned}$$

which tends to 0 as $n \to \infty$. $\square$

We also define a critical value,

$$p_c(d) = \inf\{p : \theta(p) > 0\}, \tag{3.1}$$

which will correspond to a phase transition in the bond percolation model, where for values $p < p_c(d)$, bond percolation almost surely consists of only finite components, and for $p > p_c(d)$, bond percolation almost surely has exactly one infinite component. This claim is formalized by the following standard propositions, for the first of which we will provide a proof following [DC18].

Proposition 3.0.3. For any $d \in \mathbb{N}$, if $\theta(p) = 0$, then

$$\mathbb{P}\{\exists v \in \mathbb{Z}^d \text{ with } |C_p(v)| = \infty\} = 0.$$

Proposition 3.0.4 ([Gri99], Theorem 8.1)**.** For any $d \in \mathbb{N}$, if $\theta(p) > 0$,

$$\mathbb{P}\{\exists! \text{ Infinite component of } p\text{-bond percolation}\} = 1.$$

To prove **Proposition** 3.0.3, we must first discuss the invariance of $\mathbb{P}$ under translations. For any vertex $v \in \mathbb{Z}^d$, let τ_v denote the translation mapping vertices x to $x+v$ and edges between vertices x, y to the edges between $x+v, y+v$. For an event $\mathcal{E} \in \Sigma$, we define the translation

$$\tau_v \mathcal{E} = \{\omega \in [0,1]^E \ : \ \omega \circ \tau_v^{-1} \in \mathcal{E}\}$$

Because the graph $\mathbb{L}^d$ is translation invariant, the probability measure $\mathbb{P}$ is also invariant with respect to translations, as given by the following lemma.

Lemma 3.0.5. For any event $\mathcal{E} \in \Sigma$,

$$\mathbb{P}(\mathcal{E}) = \mathbb{P}(\tau_v \mathcal{E}).$$

Proof. For any finite collection of edges $A \subset E$, events

$$\mathcal{E} = \prod_{e \ \in A} \mathcal{E}_e \times \prod_{e' \notin A} [0,1] \in \Sigma$$

are such that

$$\begin{aligned}
\mathbb{P}(\tau_v \mathcal{E}) &= \mathbb{P}\{\omega \in [0,1]^E : \omega(\tau_v^{-1}(e)) \in \mathcal{E}_e, \forall e \in A\} \\
&= \prod_{e \in A} \mathbb{P}_{e+v}(\mathcal{E}_e) \\
&= \prod_{e \in A} \mathbb{P}_e(\mathcal{E}_e) \\
&= \mathbb{P}(\mathcal{E})
\end{aligned}$$

as every coordinate of $\mathbb{P}$ is the same measure. We note $\tau_v(\mathcal{E}^c) = (\tau_v \mathcal{E})^c$, and because τ_v is injective, $\tau_v(\mathcal{E} \sqcup \mathcal{F}) = \tau_v(\mathcal{E}) \sqcup \tau_v(\mathcal{F})$, giving us that this equality holds for finite unions and compliments of the above events, and hence we may apply **Proposition** 2.1.1 twice to extend this equality to all events $\mathcal{E} \in \Sigma$. $\square$

Lemma 3.0.5 now allows us to prove **Proposition** 3.0.3.

Proof of ***Proposition*** *3.0.3.* By definition, for $\theta(p) = 0$,

$$\mathbb{P}\{|C_p(0)| = \infty\} = 0.$$

Utilizing **Lemma** 3.0.5, for any vertex $v \in \mathbb{Z}^d$,

$$\begin{aligned}\mathbb{P}\{|C_p(v)| = \infty\} &= \mathbb{P}(\tau_v\{|C_p(0)| = \infty\}) \\ &= \mathbb{P}\{|C_p(0)| = \infty\} \\ &= 0,\end{aligned}$$

and hence

$$\begin{aligned}\mathbb{P}\{\exists v \in \mathbb{Z}^d \text{ with } |C_p(v)| = \infty\} &= \mathbb{P}\left(\bigcup_{v\in\mathbb{Z}^d}\{|C_p(v)| = \infty\}\right) \\ &\le \sum_{v\in\mathbb{Z}^d}\mathbb{P}\{|C_p(v)| = \infty\} \\ &= 0.\end{aligned}$$

□

For all $d \ge 1$, trivially $\theta(0) = 0$, $\theta(1) = 1$, and so it is natural to ask when $p_c(d) \in (0, 1)$. For $d = 1$, by **Proposition** 3.0.2, we see $\theta(p) = 0$ for $p < 1$, and so $p_c(1) = 1$. For $d = 2$ the simulations presented in **Figure** 3.1 suggest that $p_c(2) \approx 1/2$, and as we may embed $\mathbb{L}^2$ into $\mathbb{L}^d$ for $d > 2$, we are led to conjecture $p_c(d) \in (0, 1)$ for $d \ge 2$. This intuition is confirmed by the following proposition, with proof following [DC18, Ste09].

Proposition 3.0.6 ([DC18], Theorem 1.1.)**.** For $d \ge 2$, $0 < p_c(d) < 1$.

Proof. For $n \in \mathbb{N}$, let Ω_n denote the set of all self-avoiding paths of length n beginning at the origin. We may naively bound

$$|\Omega_n| \le (2d)^n, \tag{3.2}$$

as any self-avoiding path may be constructed by choosing a sequence of neighboring vertices, beginning with the origin, where at each step there are $2d$ choices of neighboring vertices.

For any fixed $n > 0$ the event $\{\omega : |C_p(0)| = \infty\}$ is contained within the event that there exists a path $\gamma \in \Omega_n$ contained within $C_p(0)$, and hence by a union bound we have

$$\begin{aligned}\mathbb{P}\{|C_p(0)| = \infty\} &\leq \mathbb{P}\{\exists \gamma \in \Omega_n : \gamma \text{ is open}\} \\ &\leq \sum_{\gamma \in \Omega_n} \mathbb{P}\{\gamma \text{ is open}\} \\ &= \sum_{\gamma \in \Omega_n} p^n \\ &\leq (2d)^n p^n.\end{aligned}$$

When p is small enough this tends to 0 as $n \to \infty$, and hence $p_c(d) > 0$.

For all $d > 2$, $\mathbb{L}^2$ may be embedded in $\mathbb{L}^d$ as a plane containing the origin, meaning that if an infinite open cluster about 0 exists with positive probability in this copy of $\mathbb{L}^2$, it also exists within $\mathbb{L}^d$. Thus it suffices to show $p_c(2) < 1$.

The proof of the $d = 2$ case utilizes *Peierls' argument*, leveraging the self duality of $\mathbb{L}^2$. Specifically, consider the dual graph $(\mathbb{L}^2)^*$ whose vertices are given by $\mathbb{Z}^2 + (1/2, 1/2)$ with nearest neighbor edges, see **Figure** 3.2. The edges of $(\mathbb{L}^2)^*$ and $\mathbb{L}^2$ may be put in bijection, where an edge in $\mathbb{L}^2$ corresponds to the unique edge in $(\mathbb{L}^2)^*$ which it intersects when viewed as line segments in $\mathbb{R}^2$. We will call such a pair of edges dual to one another.

Given a configuration $\omega \in [0,1]^E$, we will declare an edge $e' \in (\mathbb{L}^2)^*$ to be open whenever its dual edge is closed, and vice versa. This then gives a coupling between p-bond percolation on $\mathbb{L}^2$ and $(1-p)$-bond percolation on $(\mathbb{L}^2)^*$ when viewed as a copy of $\mathbb{L}^2$. Within this coupling, $|C_p(0)| < \infty$ if and only if there is an open, simple loop in $(\mathbb{L}^2)^*$ which contains the origin, see **Figure** 3.3. Let Γ_n be the set of all simple loops of length n in $(\mathbb{L}^2)^*$ containing the origin in their interior. We note that any loop in Γ_n can be based at its right most intersection point with the line $y = 1/2$, which, because the loop must contain the origin in its interior, is a point of the form $(i + 1/2, 1/2)$ for some $0 \le i < n$. There is an injection from Γ_n to $\Omega_{n-1} \times \{1/2, \dots, n - 1/2\}$ given by mapping a loop to the pair consisting of the first, $n - 1$ edges, walking counterclockwise, translated to begin at the origin and the x coordinate of the base point, see **Figure** 3.4, and so

$$|\Gamma_n| \le n \cdot |\Omega_{n-1}| \le n 4^{n-1},$$

where the final inequality follows from (3.2).

Since the cluster of the origin is finite in bond percolation on $\mathbb{L}^2$ if and only if a loop in $\Gamma = \cup_n \Gamma_n$ is open in the dual graph, we have,

$$\begin{aligned}
\mathbb{P}\{|C_p(0)| < \infty\} &= \mathbb{P}\{\exists \gamma \in \Gamma \text{ open}\} \\
&\le \sum_{n=1}^{\infty} \sum_{\gamma \in \Gamma_n} \mathbb{P}\{\gamma \text{ open}\} \\
&\le \sum_{n=1}^{\infty} n 4^{n-1} (1-p)^n.
\end{aligned}$$

For p close enough to 1 this sum is strictly less than 1, and hence

$$\theta(p) = 1 - \mathbb{P}\{|C_p(0)| < \infty\} > 0.$$

$\square$

It is also natural to ask if θ is a continuous function on $[0, 1]$. This is partially answered in the following two propositions.

Proposition 3.0.7 ([Gri99], Lemma 8.9.)**.** For all $d \ge 1$, $\theta(p)$ is right continuous on $[0, 1]$,

Proposition 3.0.8 ([Gri99], Lemma 8.10.)**.** For all $d \geq 1$, $\theta(p)$ is left continuous on $(p_c(d), 1]$.

As $\theta(p) \equiv 0$ on $[0, p_c(d))$, these propositions allow us to say $\theta(p)$ is continuous on $[0, p_c(d)) \cup (p_c(d), 1]$.

Propositions 3.0.3 and 3.0.4 tell us that bond percolation undergoes a phase transition at $p_c(d)$, with the sub-critical $p < p_c(d)$ regime characterized by the almost sure lack of an infinite cluster and the super-critical $p > p_c(d)$ regime characterized by the almost sure existence of a unique infinite cluster. This leaves the question of the behavior of bond percolation exactly at the critical value $p = p_c(d)$. It is widely believed that for all $d \geq 2$, there almost surely does not exist an infinite cluster exactly at $p_c(d)$, expressed in the following conjecture.

Conjecture 3.0.9. $\theta(p_c(d)) = 0$ for all $d \geq 2$.

By **Propositions** 3.0.7 and 3.0.8, this conjecture is equivalent to stating that θ is continuous on $[0, 1]$.

Some progress has been made in resolving this conjecture. The $d = 2$ and $d \geq 11$ cases were proved in 1980 by Kesten [Kes80], and in 2017, Fitzner and Hofstad [FH17] respectively. **Chapter** 4 explores the model of invasion percolation with a focus towards its connection with **Conjecture** 3.0.9.

In what follows, it will be convenient to utilize the notation $x \leftrightarrow y$ to refer to the event that there exists an open path in bond percolation between vertices x, y, and $x \leftrightarrow \infty$ to refer to the event that for any $n \in \mathbb{N}$ there exists an open self avoiding path of length n beginning at x.

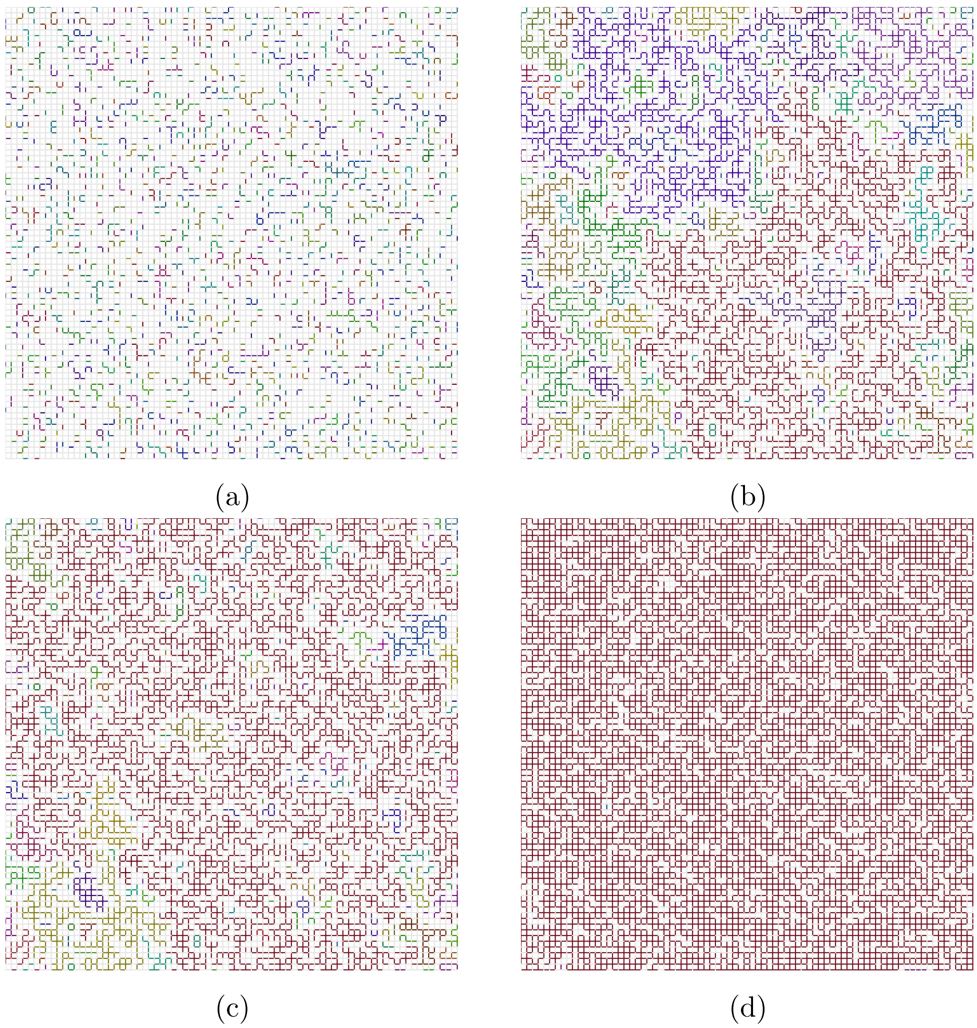


Figure 3.1: Four example simulations of bond percolation for a configuration on the finite two dimensional lattice of side length 100 with $p = 0.2$, 0.49, 0.51, 0.8 respectively. Connected components of bond percolation are given by the colored edges.

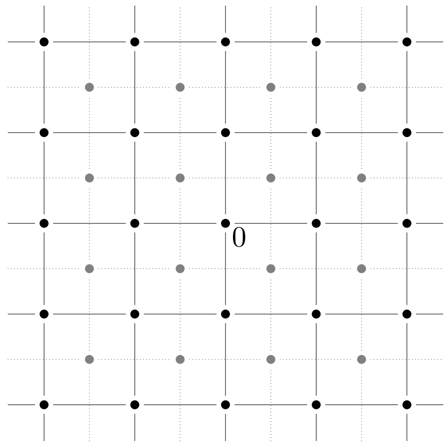


Figure 3.2: The lattice $\mathbb{L}^2$ in solid black, and the dual lattice $(\mathbb{L}^2)^*$ in dotted gray.

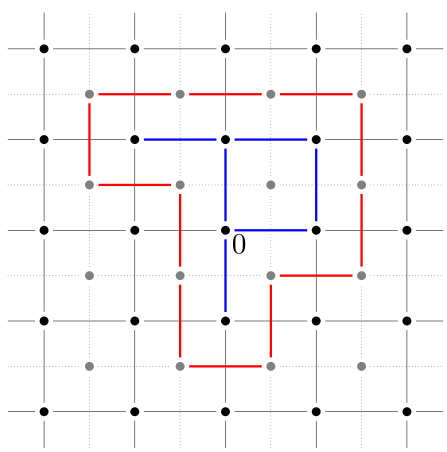


Figure 3.3: An example finite component of 0 in bond percolation on $\mathbb{L}^2$ in blue, along with a corresponding simple dual loop containing the origin in red.

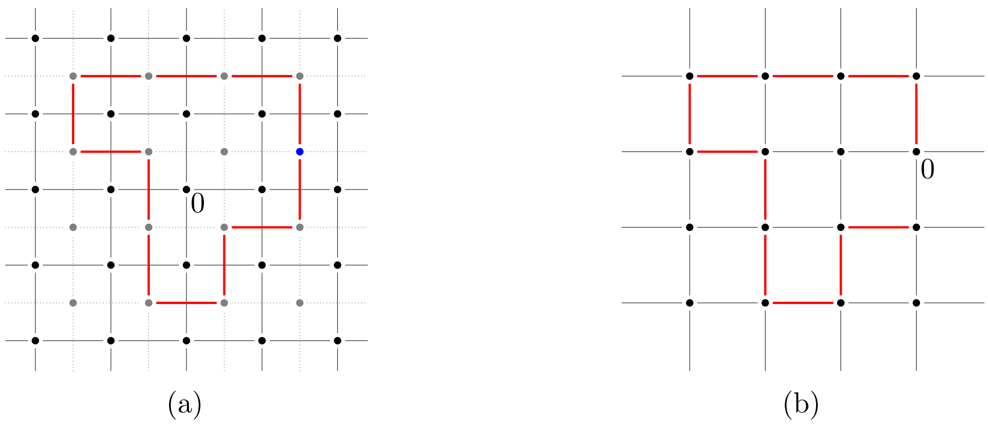


Figure 3.4: On the left, in red, an example simple loop in the dual lattice $(\mathbb{L}^2)^*$ of length 12 containing 0, and in blue, the right most intersection point with the line $y = 1/2$. On the right, in red, the corresponding self avoiding path in $\mathbb{L}^2$ beginning at 0 of length 11.

Chapter 4

Invasion Percolation

In this chapter, we provide an introduction to invasion percolation, drawing upon [Kes87, CCN85]. On the lattice $\mathbb{L}^d$, we assign to each edge e a random weight uniformly in $[0,1]$. Invasion percolation is a process of sequentially forming a random subgraph of this lattice, where at each step, beginning by invading the origin, the uninvaded edge of least weight in the edge boundary of the invaded component is chosen to be invaded along with its uninvaded vertex. Interestingly, invasion percolation imitates the behavior of bond percolation at $p_c(d)$.

Formally, on the lattice $\mathbb{L}^d$ with nearest neighbor edge set E, we work, as in bond percolation, with the probability space $([0,1]^E, \Sigma, \mathbb{P})$ where Σ is the σ-algebra generated by events which depend on finitely many edges $e \in E$ and $\mathbb{P}$ is the product measure with each coordinate $\mathbb{P}_e$ given by the Lebesgue measure on $[0,1]$. In what follows, we restrict ourselves to the subset of configurations $\omega \in [0,1]^E$ for which $\omega(e) \neq \omega(e')$ whenever $e \neq e'$, which is measurable and has probability 1, and hence will not change our analysis.

Invasion percolation for a configuration ω will consist of a subgraph $I = (\{x_n\}_{n=0}^{\infty}, \{e_n\}_{n=1}^{\infty})$ of $\mathbb{L}^d$ given by two sequences $\{x_n\}_{n=0}^{\infty}$, $\{e_n\}_{n=1}^{\infty}$, of invaded vertices and invaded edges respectively, with $x_0 = 0$. Given the initial terms $\{x_n\}_{n=0}^{N}$ $\{e_n\}_{n=1}^{N}$, the subsequent invaded edge and vertex are

$$e_{N+1} = \arg \min_{\substack{x \in \{x_n\}_{n=0}^N \\ y \notin \{x_n\}_{n=0}^N}} \omega(xy)$$

$$x_{N+1} = y.$$

Due to our stipulation that $\omega(e) \neq \omega(e')$ when $e \neq e'$, there will never be any ambiguity in which edge should be taken to be e_{N+1}.

Note that as at each step of our invasion process we are only concerned with the ordering of the $\omega(xy)$ for edges incident to vertices in the invaded cluster, rather than the values of $\omega(xy)$, the same behavior of invasion percolation may be replicated by instead comparing $f(\omega(xy))$ for f an increasing function, or more generally, drawing the values of ω independently from a continuous distribution.

Later, it will be convenient for us to use a slightly modified invasion percolation model allowing the possibility of invading edges with both incident vertices already having been invaded. Because there are only finitely many of these edges at any given step of the invasion process, and because the invasion of these edges will not allow any new edges to be invaded, the invasion of edges between invaded vertices will not essentially impact the behavior of the invasion percolation model, in the sense that the set of vertices invaded will be identical in both scenarios, and any edge that would have been invaded in the invasion percolation model above will also be invaded in this modified model.

In this modified model, for each configuration ω we will assign a total ordering, denoted $\leq_{\texttt{IP}}$, to the vertices and edges that are invaded according to the step in which they were invaded. Whenever an edge e and a vertex v were invaded in the same step, we take $e <_{\texttt{IP}} v$. See **Figure** 4.1 for the ordering of the first few steps of an example modified invasion process. In this example, our ordering is given by

$$0 <_{\texttt{IP}} e_1 <_{\texttt{IP}} v_1 <_{\texttt{IP}} e_2 <_{\texttt{IP}} v_2 <_{\texttt{IP}} e_3 <_{\texttt{IP}} v_3 <_{\texttt{IP}} e_4 <_{\texttt{IP}} e_5 <_{\texttt{IP}} v_5 <_{\texttt{IP}} \cdots .$$

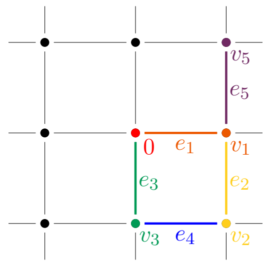


Figure 4.1: An example of the ordering of invaded vertices and edges in the first 6 steps of the modified invasion process. Each step is colored in rainbow order.

4.1 Connections between Bond and Invasion Percolation

Recall that in p-bond percolation, an edge is open if $\omega(e) < p$, and these open edges form a subgraph of $\mathbb{L}^d$. Given our definitions of bond and invasion percolation based on configurations in $[0,1]^E$, we may draw some immediate connections between the two models. One of these is the tendency of the invasion process to consume connected components in p-bond percolation, 'jumping' to another component once it has been entirely invaded, see **Figure** 4.2. This is because the edge boundary of a connected component must consist only of edges e with $\omega(e) > p$, whereas the edges within this connected component all are of the form $\omega(e) \leq p$ and hence the invasion process will choose to invade every edge within this component before it could possibly invaded any edge within the edge boundary. Thus we see that whenever the invasion process first invades an edge incident to a vertex in the infinite connected component in bond percolation, it will become trapped in this component forever.

Because we may interpret this invasion process through the lens of bond percolation for any value $p \in [0,1]$, intuitively this invasion process should eventually become trapped in smaller and smaller infinite components in bond percolation, which, by the analysis in **Chapter** 3, suggests taking values of $p \to p_c^+$.

More concretely, for each $n \in \mathbb{N}$ let

$$p_n := \sup\{\omega(e) : e \in \{e_k\}_{k \geq n}\}$$

be the largest weight of an edge invaded on or after step n. We have the following proposition and proof essentially from [CCN85],

Proposition 4.1.1 ([CCN85], Theorem 3.2)**.** $p_n \searrow p_c$ a.s.

Proof. It is clear that p_n is a non-increasing sequence, and that for all $n \in \mathbb{N}$, $p_n \geq p_c$ a.s., as otherwise with positive probability part of the invaded component would constitute an infinite connected component in bond percolation with parameter $p_n < p_c(d)$. We will now show for any fixed $p > p_c$, $p_n \leq p$ for large n a.s.

Fixing $p > p_c$, let Λ_m denote the vertex cube of side length $2m$ centered at 0,

$$\Lambda_m = \{x \in \mathbb{Z}^d : \|x\|_\infty \leq m\},$$

and $\partial\Lambda_m$ be its boundary,

$$\partial\Lambda_m = \{x \in \mathbb{Z}^d : \|x\|_\infty = m\}.$$

Let n_{Λ_m} be the first step at which $\partial\Lambda_m$ is invaded, and let $\mathcal{S}_x$ be the event that at step n_{Λ_m}, the invaded vertex is $x \in \partial\Lambda_m$. $\mathcal{S}_x$ depends only on the edges between vertices in Λ_m, and hence is independent of the event

$$\mathcal{E}_x = \{x \leftrightarrow \infty \text{ by an open path } \gamma \subset \mathbb{L}^d \setminus \Lambda_m^\circ \text{ in } p\text{-bond percolation}\}.$$

Without loss of generality, assume that the first coordinate of x is m. As any path from x to ∞ which does not intersect the hyper-plane $x_1 = m - 1$ is also a path outside of Λ_m°, we may give an injection from the paths from 0 to ∞ in the half space $\mathbb{Z}^{d-1} \times \mathbb{Z}_{\geq 0}$ and the paths from x to ∞ in $\mathbb{L}^d \setminus \Lambda_m^\circ$. Identifying events via this injection, $\mathcal{E}_x$ contains the event that there is an open path from 0 to ∞ when invasion percolation is naturally coupled with bond percolation on the half space, whose theory may be analogously developed to that of **Chapter 3**. Therefore

$$\mathbb{P}(\mathcal{E}_x) \geq \mathbb{P}\{0 \leftrightarrow \infty \text{ in half space percolation}\}.$$

Denoting

$$\overline{\theta}(p) = \mathbb{P}\{0 \leftrightarrow \infty \text{ in half space percolation}\}$$

and

$$\overline{p_c}(d) = \inf\{p : \overline{\theta}(p) > 0\},$$

in [GM90], Grimmett and Marstrand proved for $d \geq 3$,

$$\overline{p_c}(d) = p_c(d).$$

This result also holds for $d = 2$, see [Gri99, Section 7.3], and thus for $p > p_c$, we have

$$\mathbb{P}\mathcal{E}_x \geq \overline{\theta}(p) > 0.$$

Let

$$M_n(p) = \sum_{j=1}^{n} \mathbf{1}_{w(e_n) \geq p}$$

be the number of edges invaded in the first n steps with weight at least p. Conditioning on $\mathcal{S}_x$, if $M_\infty(p) = \infty$, we know that the event $\mathcal{E}_x$ did not occur, as otherwise there would have existed an infinite path of edges of weight at most p beginning with the invaded vertex y, precluding the invasion process from ever invading an edge of weight greater than p again. Therefore for all k we have

$$\mathbb{P}(M_\infty(p) = \infty | M_{n_{\Lambda_m}} \geq k) \leq \mathbb{P}(\mathcal{E}_x^c) \leq 1 - \overline{\theta}(p). \tag{4.1}$$

By definition,

$$\mathbb{P}(M_\infty(p) = \infty | M_{n_{\Lambda_m}}(p) \geq k) = \frac{\mathbb{P}(\{M_\infty(p) = \infty\} \cap \{M_{n_{\Lambda_m}}(p) \geq k\})}{\mathbb{P}\{M_{n_{\Lambda_m}}(p) \geq k\}}$$

and so by taking $m \to \infty$ we have

$$\begin{aligned}\mathbb{P}(M_\infty(p) = \infty | M_\infty(p) \geq k) &= \frac{\mathbb{P}(\{M_\infty(p) = \infty\} \cap \{M_\infty(p) \geq k\})}{\mathbb{P}\{M_\infty(p) \geq k\}} \\ &= \frac{\mathbb{P}\{M_\infty(p) = \infty\}}{\mathbb{P}\{M_\infty(p) \geq k\}}\end{aligned}$$

and hence (4.1) gives

$$\mathbb{P}\{M_\infty(p) = \infty\} \leq \mathbb{P}\{M_\infty(p) \geq k\}(1 - \overline{\theta}(p)).$$

Taking $k \to \infty$ gives

$$\mathbb{P}\{M_\infty(p) = \infty\} \leq \mathbb{P}\{M_\infty(p) = \infty\}(1 - \overline{\theta}(p)),$$

and so

$$\mathbb{P}\{M_\infty(p) = \infty\} = 0.$$

Thus for any fixed $p > p_c(d)$, $p_n \leq p$ a.s. for large n.

Let $\{q_k\}$ be a sequence such that $q_k \searrow p_c(d)$, and $\mathcal{A}_k$ be the event that $p_n \leq q_k$ for large n. Then $\mathbb{P}(\mathcal{A}_k) = 1$ for all k, and so

$$\begin{aligned}\mathbb{P}\{p_n \searrow p_c(d)\} &= \mathbb{P}\left(\cap_k \mathcal{A}_k\right) \\ &= 1 - \mathbb{P}\left(\cup_k \mathcal{A}_k^c\right) \\ &= 1.\end{aligned}$$

□

Relations like the one above may be utilized to connect the behavior of invasion percolation to critical bond percolation. One connection relevant to this thesis is given in the following equivalence to **Conjecture** 3.0.9. Let $\mathcal{I}_x$ denote the event that the vertex $x \in \mathbb{Z}^d$ is eventually invaded.

Proposition 4.1.2 ([CCN85],[New23])**.**

$$\inf_{x \in \mathbb{Z}^d} \mathbb{P}(\mathcal{I}_x) = 0 \iff \theta(p_c(d)) = 0. \tag{4.2}$$

To prove the above statement, we require the following two propositions.

Proposition 4.1.3. For any $x \in \mathbb{Z}^d$, if x and 0 are in the infinite cluster $\mathcal{C}$ of critical bond percolation, then x is eventually invaded almost surely.

Proof. It will be more convenient to work with the modified invasion percolation model described in the previous section, where edges between invaded vertices may be invaded. This will prove our result, as the set of vertices eventually invaded in both models is identical. Discarding the null events

$$\{\omega : \exists e \in \mathbb{L}^d \text{ such that } \omega(e) = p_c\}$$

and

$$\{\omega : \exists p < p_c \text{ such that } \omega \text{ has an infinite cluster in } p\text{-bond percolation}\},$$

if $0, x \in \mathcal{C}$ then there exists a path γ from 0 to x whose edges are such that $\omega(e) < p_c$. Enumerating the edges e_i, and their values $p_i = \omega(e_i)$, then by induction there must exist a step where e_i is incident to an invaded vertex and p_i is minimal among all values of uninvaded edges incident to invaded vertices, and hence is invaded, as otherwise there would exist an infinite connected component in p_i-bond percolation. Thus all edges in γ are eventually invaded, and so x is eventually invaded. $\square$

Proposition 4.1.4 ([CCN85], Theorem 5.2)**.** For $e \in \mathbb{L}^d$, denote Θ_e to be the event that e is eventually invaded. Then the volume fraction

$$\mathscr{V} := \limsup_{m\to\infty} \frac{1}{|\Lambda_m|} \sum_{e\in\Lambda_m} \mathbf{1}_{\Theta_e}$$

satisfies

$$\mathscr{V} \leq \theta(p_c) \quad a.s.$$

Proof of ***Proposition*** *4.1.2.* For the forward direction, assume $\theta(p_c) > 0$. By **Proposition** 4.1.3,

$$\mathbb{P}(\mathcal{I}_x) \geq \mathbb{P}\{0, x \in \mathcal{C}\} \geq \mathbb{P}\{0 \in \mathcal{C}\}\mathbb{P}\{x \in \mathcal{C}\} = [\mathbb{P}\{0 \in \mathcal{C}\}]^2 > 0, \quad \forall x \in \mathbb{Z}^d,$$

where the second step utilizes the FKG-inequality, see [Gri99, Section 2.1].

For the reverse direction, assume $\inf_{x \in \mathbb{Z}^d} \mathbb{P}(\mathcal{I}_x) = \epsilon > 0$. Because any invaded vertex in $\Lambda_{m-1} \setminus \{0\}$ may be uniquely associated with the edge in Λ_m invaded in the same step, we have

$$\limsup_{m \to \infty} \frac{1}{|\Lambda_m|} \sum_{x \in \Lambda_{m-1} \setminus \{0\}} \mathbf{1}_{\mathcal{I}_x} \leq \limsup_{m \to \infty} \frac{1}{|\Lambda_m|} \sum_{e \in \Lambda_m} \mathbf{1}_{\Theta_e} < \theta(p_c), \quad a.s.$$

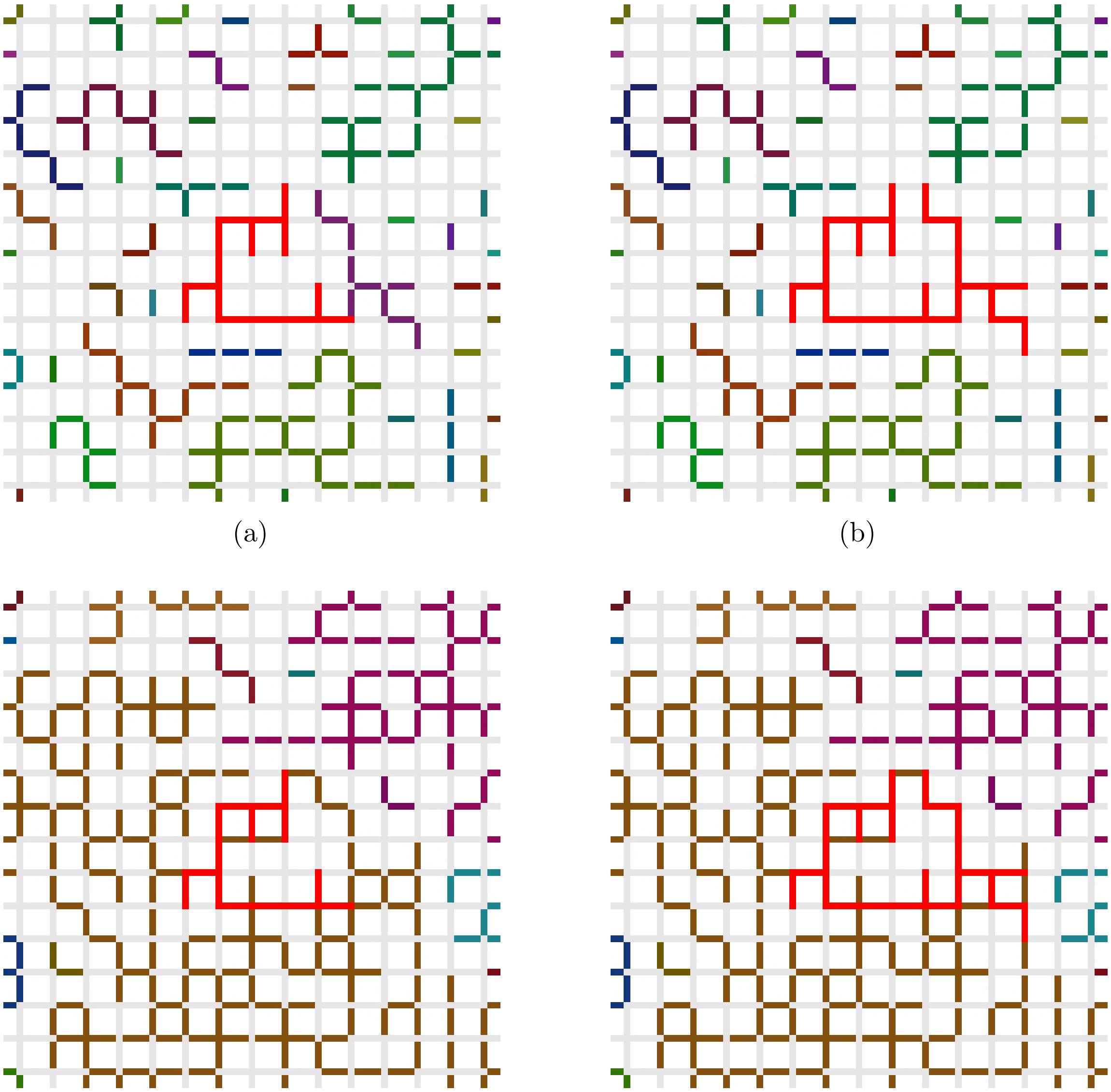
Taking expected values and utilizing the boundedness of each term gives, by **Proposition** 4.1.4

$$\limsup_{m \to \infty} \frac{|\Lambda_{m-1}| - 1}{|\Lambda_m|} \epsilon \leq \limsup_{m \to \infty} \frac{1}{|\Lambda_m|} \sum_{x \in \Lambda_{m-1} \setminus \{0\}} \mathbb{P}(\mathcal{I}_x) \leq \theta(p_c),$$

and as $|\Lambda_{m-1}|/|\Lambda_m| \to 1$,

$$0 < \epsilon \leq \theta(p_c).$$

$\square$



(a) (b)

(c) (d)

Figure 4.2: Examples of one configuration interpreted through the lens of bond percolation with parameter $p = 0.3$ above, and $p = 0.49$ below, along with the invaded component in red after 15 steps left and 25 steps right. Gifs of the invasion process on a finite grid in two different lenses are available in this GitHub repository.

Chapter 5

First Passage Percolation

We begin this chapter by providing a brief introduction to first passage percolation, following [ADH16]. First passage percolation on the lattice $\mathbb{L}^d$ may be thought of as a process where a fluid continuously emanating from the origin flows through channels (the edges of $\mathbb{L}^d$) at a rate determined by their randomly chosen passage time. Given some distribution of passage times, it is natural to ask which edges and vertices have been reached by this fluid by some finite time T, and how the shape the fluid traces out changes as T increases.

Formally, to each edge e of $\mathbb{L}^d$ we associate a *passage time* τ_e, each drawn independently from a distribution F. Given a path γ in $\mathbb{L}^d$, the passage time of γ is defined to be

$$T(\gamma) = \sum_{e \in \gamma} \tau_e.$$

This then induces a random pseudo-metric on $\mathbb{L}^d$ given by

$$T(x, y) = \inf_{\gamma} T(\gamma),$$

where the infimum is taken over all paths γ from x to y. If the distribution F of passage times is such that $F(0) = 0$, then T is almost surely a metric.

One of the fundamental objects of study in first passage percolation is the ball about the origin in this pseudo-metric. Let

$$B(t) = \{x \in \mathbb{Z}^d : T(0, x) < t\}.$$

In one simple case of F, this ball may be identified with the connected component of 0 in bond percolation. In analogy with p-bond percolation, let F be such that τ_e is 0 with probability p and ∞ with probability $1 - p$. By coupling this first passage process with bond percolation, we then see that for any $t > 0$,

$$B(t) = C_p(0).$$

The Cox-Durrett Shape Theorem, which we reproduce below, concerns the shape of $B(t)$ for first passage percolation with certain passage time distributions F. Let F be a distribution such that $F(0) < p_c(d)$, recall (3.1), and for $t_1, \ldots, t_{2d}$ independently and identically distributed according to F

$$\mathbb{E} \min(t_1, \ldots, t_{2d}) < \infty.$$

We then have the following,

Proposition 5.0.1 ([CD81]). There exists a deterministic, convex, compact set $\mathbf{B} \subset \mathbb{R}^d$ (depending on F) with non-empty interior, such that for each $\epsilon > 0$,

$$\mathbb{P}\left((1-\epsilon)\mathbf{B} \subset \frac{B(t)}{t} \subset (1+\epsilon)\mathbf{B}, \text{ for all large } t \right) = 1.$$

Here, $(1-\epsilon)\mathbf{B} = \{(1-\epsilon)x : x \in \mathbf{B}\}$, and $(1+\epsilon)\mathbf{B}$ is defined similarly.

By the above consideration, we see immediately that the condition on $F(0) < p_c(d)$ could not be substantially loosened, as otherwise there would be a positive probability that $|B(t)| = \infty$ for any positive t.

5.1 Log-Uniform First Passage Percolation

The remainder of this thesis will concern first passage percolation with passage time distribution $\mathrm{Law}(e^{Kt})$, where t is distributed as a uniform random variable on $[0,1]$, denoted by $t \sim U(0,1)$, and $K > 0$ is a parameter, and its connection with invasion percolation. We denote the passage time of an edge e drawn from this distribution by $\tau_{K,e}$, and the induced random metric $T_K(x,y)$.

For fixed $R, K \geq 0$, we define the passage time to ∂B_R by

$$T_K(0, \partial B_R) := \min_{v \in \partial B_R} T_K(0, v),$$

and the passage time from the origin to an edge $e \in E$ by

$$T_K(0, e) := \min(T_K(0, v_1), T_K(0, v_2)) + \tau_{K,e},$$

where v_1, v_2 are the vertices incident to e. For a vertex $x \in B_R \setminus \partial B_R$ and path γ from 0 to x such that $T_K(\gamma) = T_K(0,x)$, by the minimality of γ,

$$T_K(0, x) = T_K(0, e) \tag{5.1}$$

for the edge $e \in \gamma$ incident to x. We assign a total ordering $\leq_{\mathtt{FPP}}$ to the edges in E_R such that $e <_{\mathtt{FPP}} e'$ whenever $T_K(0,e) < T_K(0,e')$.

We claim that when K is large, the first passage process is often identical to the modified invasion process described in **Chapter** 4, which will be shown rigorously in **Chapter** 6. To get a sense of why this should be the case, begin by coupling a first passage process with the invasion process, where given a configuration $\omega \in [0,1]^E$, the passage time of an edge e is $\tau_e = f(\omega(e))$ for some increasing function $f : [0,1] \to \mathbb{R}_{\geq 0}$. If this function f were to grow rapidly, small changes in the value of $\omega(e)$ would produce large changes in the value of τ_e, and hence geodesic paths will tend towards minimizing the passage time of their maximal edge whenever the values of $\omega(e)$ are sufficiently separated. Given a finite region Λ, if f were such that $f(w+\delta) \geq |\Lambda| f(w)$, then the passage time of an edge e' with $\omega(e') \geq \omega(e) + \delta$ is greater than the sum of the passage times of all edges $\tilde{e} \in \Lambda$ such that $\omega(\tilde{e}) \leq \omega(e)$. As such, if no two values of ω in Λ are within δ of one another, a path of minimal passage time contained within Λ will necessarily minimize its maximum passage time edge, and so for small times t, the set $B(t)$ evolves identically to the invaded component of invasion percolation, in the sense that the orderings $<_{\texttt{IP}}$ and $<_{\texttt{FPP}}$ are identical.

If we wish to enforce the condition $f(w+\delta) \geq |\Lambda| f(w)$ for $w \in [0,1]$, a natural choice of f is given by

$$f(w) = \exp(\log|\Lambda| w/\delta),$$

motivating our choice of passage time distribution e^{Kt} with $t \sim U(0,1)$ with K large. Thus by taking $K = \log|\Lambda|/\delta$, so long as no two values of $\omega(e)$ for edges $e \in \Lambda$ are within δ of one another, we may guarantee that the orderings $<_{\texttt{IP}}$ and $<_{\texttt{FPP}}$ are identical.

As mentioned in **Chapter 4**, invasion percolation may be equivalently defined with the values of ω being independently drawn from any fixed continuous distribution. At first impression this seems to help with the issue of large difference in magnitude of the passage times relative to one another, as we might now couple our first passage process to invasion percolation where the values of ω are independently drawn from $\mathrm{Law}(\log(t))$, where $t \sim U(0,1)$, and hence our coupled passage times are distributed as t^K for $t \sim U(0,1)$. This however does not solve our problem, as now if we hope to have the coupled models behave identically with high probability, we must choose a value of δ much smaller than the value needed when $\omega(e) \sim U(0,1)$.

Chapter 6

Coupling Invasion and First Passage Percolation

In the preceding chapters we have explored three percolation models and connections between them, which have allowed for the study of one model with the tools of another. We ended this exploration with an outline of how the behavior of invasion percolation might be replicated in log-uniform first passage percolation, which we make precise in this chapter. This connection is the main contribution of this thesis, and the text of the following chapters is reproduced in great part from the manuscript [Mor25]. As this log-uniform distribution is both explicitly defined and continuous, it is our hope that it might be approachable with the tools of first passage percolation. For ease of repetition, in this chapter we shall abbreviate "invasion percolation" by "IP", and "first passage percolation" by "FPP".

Working in the probability space $([0,1]^E, \Sigma, \mathbb{P})$ as in **Chapter 4**, we will couple IP and FPP with passage time distribution e^{Kt} for $t \sim U(0,1)$ by taking $\tau_{K,e} = e^{K\omega(e)}$ on a configuration $\omega \in [0,1]^E$. For large K, the behavior of the two coupled models is often identical for vertices close to the origin.

More specifically, we say that, when coupled on a configuration $\omega \in [0,1]^E$, IP *contains* (K,R) log-uniform FPP on a set of vertices V if any vertex $v \in V$,

$$T_K(0,v) < T_K(0,\partial B_R) \implies v \text{ eventually invaded,}$$

and similarly we say (K,R) log-uniform FPP *contains* IP on V if for any $v \in V$

$$v \text{ eventually invaded} \implies T_K(0,v) < T_K(0,\partial B_R)$$

Letting

$$\delta(R,\epsilon) = \frac{1-(1-\epsilon)^{\frac{1}{|E_R|}}}{|E_R|-1},$$

we have the following theorem.

Theorem 6.0.1. Fix $d \geq 2$. For any $\epsilon > 0$, $r \geq 0$, there exists an $R_0 = R_0(\epsilon, r)$ such that for all $R \geq R_0$

$$\mathbb{P}[\text{IP contains } (K,R) \text{ log-uniform FPP on } B_R] \geq 1-\epsilon, \tag{6.1}$$

and

$$\mathbb{P}[(K,R) \text{ log-uniform FPP contains IP on } B_r] \geq 1-\epsilon \tag{6.2}$$

where $K = K(R,\epsilon/2) := \frac{\log |E_R|}{\delta(R,\epsilon/2)} = O(\epsilon^{-1}R^{4d}\log R)$ as $R \to \infty, \epsilon \to 0^+$.

Theorem 6.0.1 gives us a corollary regarding $\theta(p_c(d))$.

Corollary 6.0.2. Fix $d \geq 2$. For $K = K(R,\epsilon/2)$ as in **Theorem** 6.0.1,

$$\inf_{x\in\mathbb{Z}^d} \liminf_{R\to\infty} \mathbb{P}[T_K(0,x) < T_K(0,\partial B_R)] = 0 \iff \theta(p_c(d)) = 0.$$

To see how **Corollary** 6.0.2 follows from **Theorem** 6.0.1, by taking R and K large enough, we may guarantee with arbitrarily high probability that the behavior of the invasion percolation and short term log-uniform FPP models at any specific vertex are identical, in the sense that a vertex is invaded exactly when its passage time is less than that of ∂B_R. In such cases, we may replace the condition on vertices being invaded in Equation (4.2) with the condition that vertices have passage time less than ∂B_R, leading to **Corollary** 6.0.2.

Proof Sketch of Theorem 6.0.1

By first establishing that on the event that no two weights of edges inside E_R are within some small δ of one another, the orders $<_{\mathtt{IP}}$ and $<_{\mathtt{FPP}}$ are identical up until the first invaded vertex in ∂B_R, we see that with high probability any vertex which has passage time less than $T_K(0, \partial B_R)$ must have been invaded, giving (6.1). Additionally, because the probability that any vertex close to the origin is invaded after a vertex in ∂B_R may be made arbitrarily small by taking R large enough, the reverse holds with high probability close to the origin, giving (6.2).

6.1 Proof of Theorem 6.0.1

Before giving the proof of **Theorem** 6.0.1, we state the following lemmas, which are proven in **Section** 6.2.

Lemma 6.1.1. Let $v_R^{\mathtt{IP}}$ denote the first vertex in ∂B_R to be invaded and $\mathcal{T}_\delta$ denote the event that no two edge weights in E_R are within δ of one another. With $K(R, \epsilon)$ as in **Theorem** 6.0.1, on the event $\mathcal{T}_\delta$, for all edges e, e' invaded before $v_R^{\mathtt{IP}}$, $e <_{\mathtt{IP}} e'$ if and only if $e <_{\mathtt{FPP}} e'$.

Lemma 6.1.2. Let $\mathcal{B}_{r,R}$ denote the event that there is a vertex in B_r which is invaded after $v_R^{\mathtt{IP}}$. For any $\epsilon > 0$ and $r \geq 0$ there exists $R_0(r, \epsilon) \in \mathbb{N}$ such that for all $R \geq R_0$,

$$\mathbb{P}[\mathcal{B}_{r,R}] < \epsilon.$$

*Proof of **Theorem** 6.0.1.* Observe that, for

$$\delta(R, \epsilon) = \frac{1 - (1 - \epsilon)^{\frac{1}{|E_R|}}}{|E_R| - 1}$$

$\mathbb{P}[\mathcal{T}_\delta] = 1 - \epsilon$. This follows from calculating the probability that the minimum distance between $|E_R|$ independent uniform random variables on $[0, 1]$ is at least δ (See for instance [Pyk65, Section 2.1]).

Let $\epsilon > 0, r \geq 0$ be given. By **Lemma** 6.1.1 and (5.1), on the event $\mathcal{T}_\delta$ any vertex $v \in B_R$ with $T_K(O, v) < T_K(0, \partial B_R)$ must have been invaded, and hence

$$\begin{aligned}\mathbb{P}[\text{IP contains } (K,R) \text{ log-uniform FPP on } B_R] &\geq \mathbb{P}[\mathcal{T}_{\delta(R,\epsilon)}] \\ &= 1-\epsilon.\end{aligned}$$

Similarly, by **Lemma** 6.1.1, on the event $\mathcal{T}_\delta$, for there to be an invaded vertex $v \in B_r$ with $T_K(O, v) \geq T_K(0, \partial B_R)$, v must have been invaded after $v_R^{\mathtt{IP}}$. By **Lemma** 6.1.2, let R_0 be large enough so that for $R \geq R_0$ the probability that there is a vertex in B_r invaded after $v_R^{\mathtt{IP}}$ is at most $\epsilon/2$, and so for $R \geq R_0$,

$$\begin{aligned}\mathbb{P}[(K,R) \text{ log-uniform FPP contains IP on } B_r] &\geq 1-\mathbb{P}[\mathcal{T}^c_{\delta(R,\epsilon/2)} \cup \mathcal{B}_{r,R}] \\ &\geq 1-\mathbb{P}[\mathcal{T}^c_{\delta(R,\epsilon/2)}]-\mathbb{P}[\mathcal{B}_{r,R}] \\ &\geq 1-\epsilon.\end{aligned}$$

□

It is now straightforward to prove **Corollary** 6.0.2. First, assume that

$$\inf_{x \in \mathbb{Z}^d} \liminf_{R \to \infty} \mathbb{P}[T_K(0,x) < T_K(0, \partial B_R)] = 0,$$

and let $\epsilon > 0$ be given. Fix $x_0 \in \mathbb{Z}^d$ such that

$$\liminf_{R \to \infty} \mathbb{P}[T_K(0,x_0) < T_K(0, \partial B_R)] < \epsilon/2$$

and choose $r \geq 0$ such that $x_0 \in B_r$. By **Theorem** 6.0.1, there is an R_0 be such that for $R \geq R_0$,

$$\mathbb{P}[(K,R) \text{ log-uniform FPP contains IP on } B_r] \geq 1-\epsilon/2.$$

Hence, for all $R \geq R_0$,

$$\mathbb{P}[\mathcal{I}_{x_0}] \leq \mathbb{P}[T_K(0,x_0) < T_K(0, \partial B_R)] + \epsilon/2$$

and because $\liminf_{R\to\infty} \mathbb{P}[T_K(0, x_0) < T_K(0, \partial B_R)] < \epsilon/2$, we may find an $R_1 \geq R_0$ such that $\mathbb{P}[T_K(0, x_0) < T_K(0, \partial B_{R_1})] < \epsilon/2$. Thus

$$\mathbb{P}[\mathcal{I}_{x_0}] \leq \epsilon,$$

and so

$$\inf_{x\in\mathbb{Z}^d} \mathbb{P}[\mathcal{I}_x] = 0,$$

giving, by (4.2),

$$\theta(p_c(d)) = 0.$$

Now assume that $\theta(p_c(d)) = 0$, meaning $\inf_{x\in\mathbb{Z}^d} \mathbb{P}[\mathcal{I}_x] = 0$ by (4.2). Given $\epsilon > 0$, fix $x_0 \in \mathbb{Z}^d$ such that $\mathbb{P}[\mathcal{I}_{x_0}] < \epsilon/2$ and note by **Theorem** 6.0.1, for large R,

$$\begin{aligned}\mathbb{P}[T_K(0, x_0) < T_K(0, \partial B_R)] &< \mathbb{P}[\mathcal{I}_{x_0}] + \epsilon/2\\ &< \epsilon.\end{aligned}$$

As this holds for all large R, $\liminf_{R\to\infty} \mathbb{P}[T_K(0, x_0) < T_K(0, \partial B_R)] < \epsilon$, and hence

$$\inf_{x\in\mathbb{Z}^d} \liminf_{R\to\infty} \mathbb{P}[T_K(0, x) < T_K(0, \partial B_R)] = 0.$$

6.2 Proofs of Lemmas

To prove **Lemma** 6.1.1, we require the following lemma.

Lemma 6.2.1. For any edge e not incident to the origin,

$$T_K(0, e) = \min_{f\sim e} T_K(0, f) + \tau_{K,e}.$$

Proof. Without loss of generality, for vertices v_1, v_2 of e, neither of which are 0, assume that $T_K(0, v_1) < T_K(0, v_2)$, and let γ be the non-empty path from 0 to v_1 for which $T_K(\gamma) = T_K(0, v_1)$. Because the passage time to v_1 is less than that of v_2, we know that $T_K(\gamma) \leq T_K(\pi)$ for any path π from 0 to either v_1 or v_2.

There is a unique edge $e' \in \gamma$ adjacent to e, because if another were to be incident to v_1, then we could cut off a portion of the path, contradicting the minimality of γ, and if another were to be incident to v_2, then $T_K(0, v_2) \leq T_K(0, v_1)$. Let v_1' be the second vertex of e' along with v_1. We claim that $T_K(0, e') = T_K(0, v_1') + \tau_{K,e'}$. Indeed as there is a path $\gamma \setminus \{e'\}$ from 0 to v_1' such that $T_K(\gamma \setminus \{e'\}) < T_K(\gamma)$, we know that $T_K(0, v_1') < T_K(0, v_1)$.

We also claim that $T_K(0, v_1') = T_K(\gamma \setminus \{e'\})$. If there were to exist another path γ' from 0 to v_1' with $T_K(\gamma') < T_K(\gamma \setminus \{e'\})$, then

$$\begin{aligned} T_K(\gamma' \cup \{e'\}) &= T_K(\gamma') + \tau_{K,e'} \\ &< T_K(\gamma \setminus \{e'\}) + \tau_{K,e'} \\ &= T_K(\gamma). \end{aligned}$$

Because $\gamma' \cup \{e'\}$ forms a path from 0 to v_1, this contradicts the minimality of γ. Thus we have that $T_K(0, v_1') = T_K(\gamma \setminus \{e'\})$

Finally, we claim that $T_K(0, e) = T_K(0, e') + \tau_{K,e}$, and that $T_K(0, e') \leq T_K(0, f)$ for all f adjacent to e. Indeed, we know that

$$\begin{aligned} T_K(0, v_1) &= T_K(\gamma) \\ &= T_K(\gamma \setminus \{e'\}) + \tau_{K,e'} \\ &= T_K(0, v_1') + \tau_{K,e'} \\ &= T_K(0, e'), \end{aligned}$$

and hence $T_K(0, e) = T_K(0, e') + \tau_{K,e}$. Moreover, if there exists some f adjacent to e such that $T_K(0, f) < T_K(0, e')$, then we know by an identical argument to the one above that a vertex $x \neq v_1, v_2$ of f must be such that $T_K(0, f) = T_K(0, x) + \tau_{K,f}$. But then a path π from 0 to x is such that $T_K(\pi \cup \{f\}) < T_K(\gamma)$, contradicting the minimality of γ over all paths from 0 to either v_1 or v_2. □

Lemma 6.1.1. Let $v_R^{\mathtt{IP}}$ denote the first vertex in ∂B_R to be invaded and $\mathcal{T}_\delta$ denote the event that no two edge weights in E_R are within δ of one another. With $K(R, \epsilon)$ as in **Theorem** 6.0.1, on the event $\mathcal{T}_\delta$, for all edges e, e' invaded before $v_R^{\mathtt{IP}}$, $e <_{\mathtt{IP}} e'$ if and only if $e <_{\mathtt{FPP}} e'$.

Proof. We will prove this lemma by building up the orders $<_{\mathtt{IP}}$ and $<_{\mathtt{FPP}}$ from their minimal elements, and confirm that they are identical for all edges $e, e' <_{\mathtt{IP}} v_R^{\mathtt{IP}}$. Indeed, clearly if e_0 is the edge incident to 0 of minimal weight, then e_0 is the minimum edge with respect to $<_{\mathtt{IP}}$. It is also the minimum edge with respect to $<_{\mathtt{FPP}}$, as for any vertex x, $T_K(0, x) \geq \min_{1 \leq k \leq d}\{T_K(0, \pm v_k)\}$, where v_k denote the standard basis vectors of $\mathbb{R}^d$. Proceeding via induction, assume that the two orderings are identical up to some edge e_k, and let e_{k+1} be an edge such that there is no edge e' with $e_k <_{\mathtt{IP}} e' <_{\mathtt{IP}} e_{k+1}$. First, as the orderings are identical up to e_k, we have $e_k <_{\mathtt{FPP}} e_{k+1}$ as well. Assume for the sake of contradiction that there is an e' such that $e_k <_{\mathtt{FPP}} e' <_{\mathtt{FPP}} e_{k+1}$, and take e' to be minimal in this respect.

First, e' must have been adjacent to some edge $e'_{\mathtt{adj}} \leq_{\mathtt{FPP}} e_k$. If e' is incident to the origin, then we may take $e'_{\mathtt{adj}} = e_0 \leq_{\mathtt{FPP}} e_k$. If not, by **Lemma** 6.2.1 take $e'_{\mathtt{adj}}$ adjacent to e' such that

$$T_K(0, e') = T_K(0, e'_{\mathtt{adj}}) + \tau_{e',K}$$

and thus $e'_{\mathtt{adj}} <_{\mathtt{FPP}} e'$, meaning $e'_{\mathtt{adj}} \leq_{\mathtt{FPP}} e_k$ by the minimality of e'.

By **Lemma** 6.2.1, take $e_{\mathtt{adj}}$ adjacent to e_{k+1} such that $e_{\mathtt{adj}} \leq_{\mathtt{IP}} e_k$ and $T_K(0, e_{k+1}) = T_K(0, e_{\mathtt{adj}}) + \tau_{K,e_{k+1}}$ if e_{k+1} is not incident to the origin, and $e_{\mathtt{adj}} = e_0$ if it is. With $e_{\mathtt{adj}}^{(0)} := e_{\mathtt{adj}}$, we may recursively find an edge $e_{\mathtt{adj}}^{(n+1)}$ adjacent to $e_{\mathtt{adj}}^{(n)}$ such that $e_{\mathtt{adj}}^{(n+1)} <_{\mathtt{IP}} e_{\mathtt{adj}}^{(n)}$ and has the above property, stopping this procedure with the first

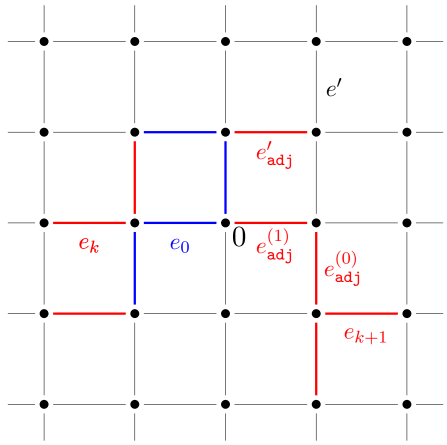


Figure 6.1: An example configuration in the coupled process in $\mathbb{L}^2$. Edges f drawn in red are such that $e'_{\mathtt{adj}} \leq_{\mathtt{IP}} f <_{\mathtt{IP}} e'$, edges g drawn in blue are such that $g <_{\mathtt{IP}} e'_{\mathtt{adj}}$. In this case, as $e^{(1)}_{\mathtt{adj}}$ is incident to the origin and $e^{(1)}_{\mathtt{adj}} >_{\mathtt{IP}} e'_{\mathtt{adj}}$, we stop our process at $e^{(2)}_{\mathtt{adj}} = e_0$.

(potentially 0) N such that $e^{(N)}_{\mathtt{adj}} \leq_{\mathtt{IP}} e'_{\mathtt{adj}}$, see **Figure** 6.1. We note $N+1 < |E_R|$, as for $e_k, e_{k+1} <_{\mathtt{IP}} v^{\mathtt{IP}}_R$, all $e^{(i)}_{\mathtt{adj}}$ are contained within E_R, and that there is no edge $e'_{\mathtt{adj}} <_{\mathtt{IP}} \tilde{e} \leq_{\mathtt{IP}} e_{k+1}$ such that $w(\tilde{e}) > w(e')$, as e' was adjacent to an already invaded edge and hence would have been chosen to be invaded before $\tilde{e}$. We let $e_{\max}$ be the edge of largest weight in $\{f : e'_{\mathtt{adj}} <_{\mathtt{IP}} f \leq_{\mathtt{IP}} e_{k+1}\}$, and note that by the above, $w(e') > w(e_{\max})$. On the event $\mathcal{T}_\delta$, we must have that $w(e') \geq w(e_{\max}) + \delta$, and hence

$$\tau_{K,e'} = e^{K(R)w(e')} \geq e^{K(R)\delta} e^{K(R)w(e_{\max})} = |E_R| \tau_{K,e_{\max}}.$$

Thus

$$\begin{aligned}
T_K(0, e_{k+1}) &\leq T_K(0, e^{(N)}_{\mathtt{adj}}) + \tau_{K,e^{(N-1)}_{\mathtt{adj}}} + \cdots + \tau_{K,e^{(0)}_{\mathtt{adj}}} + \tau_{K,e_{k+1}} \\
&\leq T_K(0, e'_{\mathtt{adj}}) + (N+1)\tau_{K,e_{\max}} \\
&\leq T_K(0, e'_{\mathtt{adj}}) + |E_R| \tau_{K,e_{\max}} \\
&\leq T_K(0, e'_{\mathtt{adj}}) + \tau_{K,e'} \\
&= T_K(0, e'),
\end{aligned}$$

giving a contradiction to $e' <_{\mathtt{FPP}} e_{k+1}$. $\square$

Lemma 6.1.2. Let $\mathcal{B}_{r,R}$ denote the event that there is a vertex in B_r which is invaded after $v_R^{\mathtt{IP}}$. For any $\epsilon > 0$ and $r \geq 0$ there exists $R_0(r, \epsilon) \in \mathbb{N}$ such that for all $R \geq R_0$,

$$\mathbb{P}[\mathcal{B}_{r,R}] < \epsilon.$$

Proof. Assume to the contrary that there exists some $\epsilon > 0$ and $r \geq 0$ such that for any $R_0 \in \mathbb{N}$ there exists an $R \geq R_0$ such that $\mathbb{P}[\mathcal{B}_{r,R}] \geq \epsilon$.

Because $\mathcal{B}_{r,R+1} \subset \mathcal{B}_{r,R}$, we get that $\mathbb{P}[\mathcal{B}_{r,R+1}] \leq \mathbb{P}[\mathcal{B}_{r,R}]$ for all R. Our assumption gives us that there exists a sequence of R_k's such that $\mathbb{P}[\mathcal{B}_{r,R_k}] \geq \epsilon$, and hence $\mathbb{P}[\mathcal{B}_{r,R}] \geq \epsilon$ for all $R \geq R_0$.

Let $\mathcal{B}_r = \cap_{R \geq R_0} \mathcal{B}_{r,R}$. Then because $\mathcal{B}_{r,R}$ is a decreasing sequence of events,

$$\mathbb{P}[\mathcal{B}_r] = \lim_{R \to \infty} \mathbb{P}[\mathcal{B}_{r,R}] \geq \epsilon.$$

We claim that $\mathcal{B}_r = \emptyset$. Assume for the sake of contradiction that there exists a configuration $\omega \in \mathcal{B}_r$. We have that $\omega \in \mathcal{B}_{r,R}$ for all $R \geq R_0$, and so there is a vertex in B_r which is invaded after $v_R^{\mathtt{IP}}$ for all $R \geq R_0$. Because there are a finite number of vertices in B_r, we may let step $n \geq R_0$ be the step when the final vertex $v \in B_r$ is invaded for the configuration ω. But then $v_{n+1}^{\mathtt{IP}}$ could not have been invaded before v as it would take at least $n+1$ steps for the invasion process to reach ∂B_{n+1}, contradicting $\omega \in \mathcal{B}_{r,n+1}$. Therefore there can be no $\omega \in \mathcal{B}_r$.

This gives us

$$0 = \mathbb{P}[\emptyset] = \mathbb{P}[\mathcal{B}_r] \geq \epsilon > 0,$$

a contradiction. $\square$

Chapter 7

Simulation of First Passage Percolation

We end this thesis with some simulations of the behavior of log-uniform first passage percolation in $\mathbb{L}^2$ for $K(R, 0.01)$ as defined in **Theorem** 6.0.1, the purpose of which were to examine the quantity

$$s(x) := \mathbb{P}[T_K(0,x) < T_K(0, \partial B_R)] \tag{7.1}$$

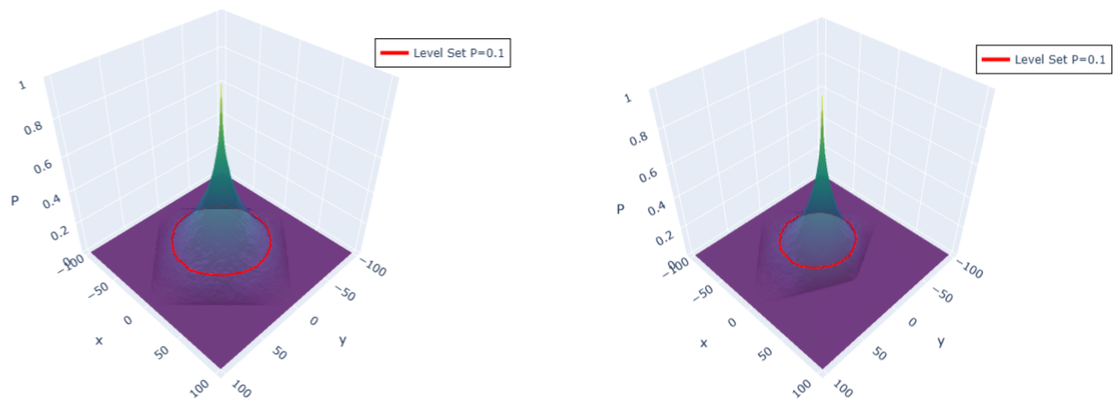
as in **Corollary** 6.0.2. Based on the data gathered, we give the following two observations.

The first is that for fixed R, this quantity seems to depend roughly only on the Euclidean norm of x for vertices away from ∂B_R, and so s is approximately radially symmetric. Additionally, when ∂B_R in (7.1) was replaced by the set of integral vertices on the kite $-x_1 + |x_2| = 100$ for $x_1 \leq 0$ and $2x_1 + |x_2| = 100$ for $x_1 > 0$, this approximate radial behavior was still present, indicating that this dependence does not rely on the shape of the boundary.

The second is the scaled slices $s(Rx_1, 0)$ seem to follow an approximate power law behavior of the form $s(Rx_1, 0) \approx 1 - |x_1|^{\alpha(R)}$, which under the assumption of radial symmetry indicates $s(x) \approx 1 - \|x\|_2^{\alpha(R)}$, giving an explicit description of the behavior of $s(x)$.

7.1 Experimental Results

Trials ($n = 10\,000$) of the first passage process were ran until time $t = T_K(0, \partial B_R)$, recording for each $x \in B_R$ the proportion of trials where the event $T_K(0, x) < T_K(0, \partial B_R)$ occurred. A plot of these proportions for $R = 100$ is included in **Figure** 7.1(a). The level sets of values away from 0 appear to be circular although the boundary being compared to was the ℓ_1 ball.



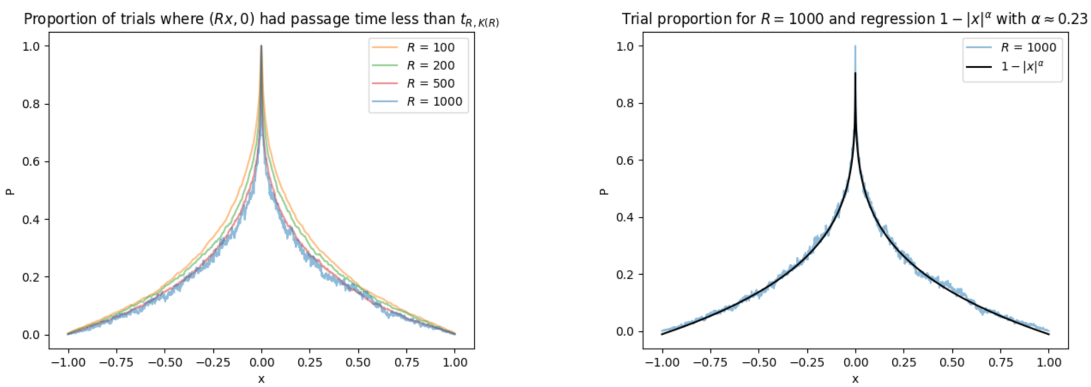
(a) (b)

Figure 7.1: Simulations recording the number P of trials ($n = 10\,000$) for which the vertex (x_1, x_2) has passage time less than the passage time to the indicated boundary. (a) Boundary ∂B_{100}. (b) Boundary $-x_1 + |x_2| = 100$ for $x_1 \leq 0$ and $2x_1 + |x_2| = 100$ for $x_1 > 0$. The level curve $P = 0.1$ is shown in both figures by a solid red line. The code used to run these simulations is available in this GitHub Repository.

Simulations were performed with lopsided boundary conditions appear to show similar results, as demonstrated in **Figure** 7.1(b).

Restricting to the slice $x_2 = 0$, **Figure** 7.2(a) shows the proportions of vertices $(x_1, 0)$ with passage times less than $T_K(0, \partial B_R)$ for $R = 100, 200, 500, 1000$, where

$[-R, R]$ has been scaled to $[-1, 1]$. The resulting slice resembles a curve of the form $1 - |x_1|^{\alpha(R)}$ for some $0 < \alpha(R) < 1$. A regression of this form for $R = 1000$ is given in **Figure** 7.2(b), which was accomplished by finding the least squares solution α to the system $\alpha \log |x_1| = \log(1 - P)$. We note that as a consequence of **Theorem** 6.0.1, because every vertex aside from the origin has probability strictly less than 1 of being invaded, if such an $\alpha(R)$ exists, then $\alpha(R) \to 0$ as $R \to \infty$. If the overall distribution is approximately radially symmetric regardless of boundary, this restriction to $x_2 = 0$ is representative of all directions away from the boundary.



(a) (b)

Figure 7.2: (a) Proportion of trials for which the vertices $(Rx_1, 0)$ with $x_1 = k/R$ and $k \in \{-R, \ldots, R\}$ satisfy $T((Rx_1, 0)) \leq T_K(0, \partial B_R)$ for $R = 100, 200, 500, 1000$. (b) Proportion of trials as in (a) with $R = 1000$ along with a solid curve depicting a regression of the form $1 - |x_1|^\alpha$ with $\alpha \approx 0.23$ and Pearson correlation coefficient $r = 0.998$

One possible direction for further experimentation would be the simulation of (K, R) log-uniform first passage percolation for time scales much longer than that of e^K, to observe an approximate limit shape as in **Proposition** 5.0.1. It may also be of interest to compare how these approximate shapes change as K grows. Additionally, more simulations may be ran to further explore the dependence of s on the shape of the chosen boundary.

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