Bilinear pseudodifferential operators: the Coifman-Meyer class and beyond

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A toy example

Consider

$$T(f,g) = f \cdot g$$

- *bilinear*...$T(af_1 + bf_2, g) = aT(f_1, g) + bT(f_2, g)$

- *translation invariant*...$T(\tau_h f, \tau_h g)(x) = \tau_h T(f,g)(x)$,
where $\tau_h f(x) = f(x-h)$

By Hölder's inequality

$$\|f \cdot g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}$$

with $1/r = 1/p + 1/q$, $1 \leq p, q < \infty$So...$T : L^p \times L^q \to L^r$

Write now the product $f \cdot g$ in *multiplier form*

$$(f \cdot g)(x) = (2\pi)^{-2n} \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{ix\cdot\xi}\, d\xi \cdot \int_{\mathbb{R}^n} \widehat{g}(\eta) e^{ix\cdot\eta}\, d\eta$$

$$= (2\pi)^{-2n} \int_{\mathbb{R}^{2n}} \widehat{f}(\xi)\widehat{g}(\eta) e^{ix\cdot(\xi+\eta)}\, d\xi d\eta$$

$$= \int_{\mathbb{R}^{2n}} m(\xi,\eta) \widehat{f}(\xi)\widehat{g}(\eta) e^{ix\cdot(\xi+\eta)}\, d\xi d\eta,$$

where $m(\xi, \eta)= (2\pi)^{-2n}$ *is independent of* x

Bilinear multipliers

In general, any translation invariant operator T can be represented as

$$T(f,g)(x) = \int m(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix\cdot(\xi+\eta)}\,d\xi d\eta,$$

where $m = m(\xi,\eta)$ is its *multiplier*...$T = T_m$

Equivalently, if its *kernel* $K(u,v) = \mathcal{F}_{2n}^{-1}(m(\cdot,\cdot))(u,v)$,

$$T(f,g)(x) = \int K(x-y,x-z)f(y)g(z)\,dydz$$

Question: What conditions are needed for $T_m : L^p \times L^q \longrightarrow L^r$?

Simple remarks

• *Scaling/Homogeneity*Assume T_m also commutes with simultaneous dilations,equivalently, m is homogenous of degree 0,

$$m(\lambda\xi, \lambda\eta) = m(\xi, \eta), \lambda > 0$$

Then (p, q, r) satisfies the Hölder condition

$$1/p + 1/q = 1/r,\ p, q > 1$$

• *Necessary condition* (Coifman-Meyer?):

$$|m(\xi,\eta)| \le C, \text{ for all } \xi, \eta$$

• *But not sufficient!* For $m(\xi,\eta) = \text{sgn}\,(\xi+\eta), \xi, \eta \in \mathbb{R}$,we easily compute

$$T_m(f,g) = H(f \cdot g),$$

where $H(f) = p.v.(x^{-1} * f)$ is the Hilbert transform

It is known that $H : L^1 \not\to L^1$

...so $T_m : L^2 \times L^2 \not\to L^1$

Coifman-Meyer bilinear multipliers

Let m be bounded such that

$$|\partial_\xi^\alpha \partial_\eta^\beta m(\xi,\eta)| \leq C_{\alpha\beta}(|\xi|+|\eta|)^{-|\alpha|-|\beta|}$$

We have

$$\|T_m(f,g)\|_{L^r} \leq C\|f\|_{L^p}\|g\|_{L^q}$$

for $1/p + 1/q = 1/r, 1 < p, q < \infty$ and $1/2 < r < \infty$ (as well as appropriate end-point results)

- Coifman-Meyer (1978): $r > 1$

- Grafakos-Torres, Kenig-Stein (1999): $r > 1/2$ (optimal)

Application to fractional differentiation

Define

$$\widehat{|\nabla|^s f}(\xi) = |\xi|^s \widehat{f}(\xi), s > 0$$

$$|\nabla|^s(f \cdot g)(x) = c_n \int |\xi + \eta|^s \widehat{f}(\xi)\widehat{g}(\eta) e^{ix\cdot(\xi+\eta)}\, d\xi d\eta$$

$$= c_n \int_{|\xi|>|\eta|} \frac{|\xi + \eta|^s}{|\xi|^s} |\xi|^s \widehat{f}(\xi)\widehat{g}(\eta) e^{ix\cdot(\xi+\eta)}\, d\xi d\eta$$

$$+ c_n \int_{|\xi|<|\eta|} \frac{|\xi + \eta|^s}{|\eta|^s} \widehat{f}(\xi)|\eta|^s\widehat{g}(\eta) e^{ix\cdot(\xi+\eta)}\, d\xi d\eta$$

If we do the split in a *smooth way*, we actually get

$$|\nabla|^s(f \cdot g)(x) = T_{m_1}(|\nabla|^s f, g) + T_{m_2}(f, |\nabla|^s g)$$

with m_1 and m_2 Coifman-Meyer bilinear multipliers.

Then Leibniz's rule for fractional derivatives follows:

$$\||\nabla|^s(f \cdot g)\|_{L^r} \lesssim \||\nabla|^s f\|_{L^p}\|g\|_{L^q} + \|f\|_{L^p}\||\nabla|^s g\|_{L^q}$$

Another example

The Riesz transforms in $\mathbb{R}^2$ can be seen as bilinear multipliers on $\mathbb{R} \times \mathbb{R}$, e.g.

$$R_1(f,g)(x) = \text{p.v.} \int_{\mathbb{R}^2} K_1(x-y, x-z) f(y) g(z)\, dy dz,$$

where

$$K_1(y,z) = \frac{y}{|(y,z)|^3}$$

Note that K_1 is a Calderón-Zygmund kernel

$$|\partial_y^\alpha \partial_z^\beta K_1(y,z)| \leq C_{\alpha\beta}(|y| + |z|)^{-2-|\alpha|-|\beta|}$$

R_1 corresponds to a Coifman-Meyer bilinear multiplier; $R_1 : L^p \times L^q \longrightarrow L^r$

Variable coefficient bilinear operator

...or non-translation invariant bilinear operator ...associated to an x-dependent symbol $\sigma(x,\xi,\eta)$, or a kernel $\tilde{K}(x,u,v) = \mathcal{F}_{2n}^{-1}(\sigma(x,\cdot,\cdot))(u,v)$

$$T_\sigma(f,g)(x) = \int_{\mathbb{R}^{2n}} \sigma(x,\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix\cdot(\xi+\eta)}\,d\xi d\eta$$

$$= \int_{\mathbb{R}^{2n}} \tilde{K}(x,x-y,x-z)f(y)g(z)\,dydz$$

$$= \int_{\mathbb{R}^{2n}} K(x,y,z)f(y)g(z)\,dydz$$

Bilinear $T(1)$ theorem

Consider a bilinear Calderón-Zygmund operator

$$T : \mathcal{S} \times S \to \mathcal{S}'$$

$$\langle T(f_1, f_2), f_3\rangle = \langle K, f_1 \otimes f_2 \otimes f_3\rangle$$

$$|\partial^\alpha K(y_0, y_1, y_2)| \lesssim (\sum |y_j - y_k|)^{-2n-|\alpha|}, |\alpha| \leq 1$$

• Christ-Journé (1987): $T : L^2 \times L^2 \to L^1 \Longleftrightarrow K$ satisfies a bilinear WBP and the distributions $T^{*j}(1,1)$ are in BMO; here $T = T^{*0}$ and T^{*1}, T^{*2} are the transposes of T.

• Grafakos-Torres (2002): $T : L^p \times L^q \to L^r, 1/p + 1/q = 1/r < 2 \Longleftrightarrow$ $\sup_{\xi,\eta} \|T^{*j}(e^{ix\cdot\xi}, e^{ix\cdot\eta})\|_{BMO} \leq C$

Application: the Coifman-Meyer class

$$T_\sigma(f,g)(x) = \int_{\mathbb{R}^{2n}} \sigma(x,\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix\cdot(\xi+\eta)}\,d\xi d\eta,$$

where $\sigma \in BS^0_{1,0}$ (the Coifman-Meyer class), i.e.,

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \le C_{\alpha\beta\gamma}(1+|\xi|+|\eta|)^{-|\beta|-|\gamma|}$$

Then T_σ has a Calderón-Zygmund kernel

$$|\partial^\alpha K(x,y,z)| \le C_\alpha(|x-y|+|x-z|)^{-(2n+|\alpha|)}$$

and

$$T(e^{i\xi\cdot}, e^{i\eta\cdot})(x) = \sigma(x, \xi, \eta)e^{ix\cdot(\xi+\eta)}$$

which is (uniformly in all ξ, η) in L^∞.

The same observations apply to the transposes of T: $T^{*j}, j = 1, 2$, have Calderón-Zygmund kernels, and they behave well on the elementary objects $e^{i\xi\cdot}, e^{i\eta\cdot}$...this follows from a symbolic calculus for the transposes...

By the bilinear $T(1)$ theorem

$$T_\sigma : L^p \times L^q \to L^r, 1/p + 1/q = 1/r < 2$$

Coifman-Meyer essentially proved the same for $r > 1$ (and multipliers), but they used Littlewood-Payley theory...1978 result.

More general bilinear pseudodifferential operators

$$T_\sigma(f,g)(x) = \int_{\mathbb{R}^{2n}} \sigma(x,\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix\cdot(\xi+\eta)}\,d\xi d\eta$$

For $0 \le \delta \le \rho \le 1$ and $m \in \mathbb{R}$, we say that $\sigma \in BS^m_{\rho,\delta}$ if

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \le$$

$$C_{\alpha\beta\gamma}(1+|\xi|+|\eta|)^{m+\delta|\alpha|-\rho(|\beta|+|\gamma|)}$$

These are the bilinear analog of the classical Hörmander classes in the linear case.

Symbolic calculus for the transposes

• Interest: most L^p results depend on symmetric properties on the transposes!

If $\sigma(x, \xi, \eta) = \sigma(\xi, \eta)$ (multiplier), then the transposes of T_σ are easy to compute

$$T_\sigma^{*1} = T_{\sigma^{*1}} \text{ and } T_\sigma^{*2} = T_{\sigma^{*2}}$$

$$\sigma^{*1}(\xi, \eta) = \sigma(-\xi - \eta, \eta), \sigma^{*2}(\xi, \eta) = \sigma(\xi, -\xi - \eta)$$

Question: How about x-dependent symbols $\sigma(x, \xi, \eta)$?

The situation is more complicated...but the following calculus holds.

If $0 \leq \delta < \rho \leq 1$ *and* $\sigma \in BS^m_{\rho,\delta}$*, then* $\sigma^{*1}, \sigma^{*2} \in BS^m_{\rho,\delta}$ *and*

$$\sigma^{*1}(x,\xi,\eta) = \sum_\alpha \frac{i^\alpha}{\alpha!}\partial_x^\alpha \partial_\xi^\alpha \sigma(x,-\xi-\eta,\eta)$$

in the sense that for every $N > 0$

$$\sigma^{*1}(x,\xi,\eta) - \sum_{|\alpha|<N} \frac{i^\alpha}{\alpha!}\partial_x^\alpha \partial_\xi^\alpha \sigma(x,-\xi-\eta,\eta) \in BS^{m+(\delta-\rho)N}_{\rho,\delta}$$

and similarly for σ^{*2}.

- B.-Torres (2003): $\delta = 0, \rho = 1, m = 0$

- B.-Maldonado-Naibo-Okoudjou-Torres (2008): the general case

Question: What about $BS^m_{1,1}$?

Relevance of transposition calculus

The class $BS^0_{1,1}$ is the largest that produces operators with bilinear Calderón-Zygmund kernels. But...this class (or $BS^m_{1,1}$ in general) does not produce bounded operators on L^p spaces because it is not closed by transposition.

Nevertheless, these classes are bounded on Sobolev spaces of positive smoothness, and there is a pseudodifferential Leibniz rule.

If $\sigma \in BS^m_{1,1}$, $m \geq 0$, $s > 0$, then T_σ has a bounded extension from $L^p_{m+s} \times L^q_{m+s}$ into L^r_s. Moreover,

$$\|T_\sigma(f,g)\|_{L^r_s} \lesssim \|f\|_{L^p_{m+s}} \|g\|_{L^q} + \|f\|_{L^p} \|g\|_{L^q_{m+s}}$$

for all $1/p + 1/q = 1/r$, $1 < p, q, r < \infty$.

- B.-Torres (2003): $m = 0$; B.-Nahmod-Torres (2006): general case

- B. (2003): $m = 0$; boundedness on Lipschitz and Besov spaces

Bilinear paraproducts and pseudodifferential operators

All of the above results can essentially be recast in terms of bilinear paraproducts. For example,

$$T(f,g)(x) = \sum_Q \sigma_Q |Q|^{-1/2} \langle f, \phi_Q^1 \rangle \langle g, \phi_Q^2 \rangle \phi_Q^3(x),$$

where the sum runs over all dyadic cubes in $\mathbb{R}^n$, $\{\sigma_Q\} \in l^\infty$ and the functions ϕ_Q^i are families of *wavelets*,

$$\phi_Q^i(x) = |Q|^{-1/2} \phi^i(|Q|^{-1}(x - c_Q))$$

• B.-Maldonado-Nahmod-Torres (2007): paraproducts as bilinear Calderón-Zygmund operators

• Maldonado-Naibo (2008-09): nice connections to more general operators...Don't miss Diego's talk!!!

Linear vs bilinear pseudodifferential operators

The above results are analogous to the linear ones...

Question: Does everything hold the same in the bilinear setting?

NO! For example, Calderón-Vaillancourt's theorem fails!

In the linear case, $\sigma \in S^0_{0,0} \Rightarrow T_\sigma : L^2 \to L^2$, but

$$\sigma \in BS^0_{0,0} \not\Rightarrow T_\sigma : L^2 \times L^2 \to L^1$$

(or any $L^p \times L^q \to L^r, 1 \le p, q, r < \infty$)

$$\sigma \in BS^0_{0,0} \Leftrightarrow |\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x, \xi, \eta)| \le C_{\alpha\beta\gamma}$$

A substitute result

If $\sigma \in BS^0_{0,0}$ *and* $\partial^\gamma_\eta \sigma \in L^2_\xi L^1_\eta L^\infty_x$, $\partial^\beta_\xi \sigma \in L^2_\eta L^1_\xi L^\infty_x$, *then*

$$T_\sigma : L^2 \times L^2 \to L^1$$

- B.-Torres (2004): the proof uses a bilinear version of *Cotlar's lemma*

Let H *be a Hilbert space and* V *a normed space of functions closed under conjugation. If* $T_j : V \times H \to H, j \in \mathbf{Z}$, *is a sequence of bounded bilinear operators and* $\{a(j)\}_{j\in\mathbf{Z}}$ *is a sequence of positive real numbers such that*

$$\|T_i(f, T_j^{*2}(\bar{f}, g))\|_H + \|T_i^{*2}(\bar{f}, T_j(f, g))\|_H \leq a(i-j),$$

for all $f \in V, g \in H, \|f\|_V = \|g\|_H = 1$, *and for all* $i, j \in \mathbf{Z}$, *then*

$$\|\sum_{j=n}^{m} T_j\| \leq \sum_{i=-\infty}^{\infty} \sqrt{a(i)},\ \ n, m \in \mathbf{Z}, n \leq m.$$

...And a surprising connection

The *correct setting* for the study of this class is provided by the so-called *modulation spaces* $M^{p,q}$

$$f \in M^{p,q} \Leftrightarrow \|V_\phi f(x,\omega)\|_{L^p_x L^q_\omega} < \infty$$

Instructive statement: $f \in M^{p,q} \sim f \in L^p$ and $\widehat{f} \in L^q$.

Question: Why these spaces???

1. $BS^0_{0,0} \subseteq M^{\infty,1}(\mathbb{R}^{3n})$ and

2. Symbols in $M^{\infty,1}$ should yield operators that behave like pointwise multiplication both in time and frequency!

The following general result holds (and this applies to symbols that lie in a modulation space...hence can be quite rough!)

If $\sigma \in M^{\infty,1}(\mathbf{R}^{3n})$*, then* T_σ *extends to a bounded operator from* $M^{p_1,q_1} \times M^{p_2,q_2}$ *into* M^{p_0,q_0}*, where* $1/p_1 + 1/p_2 = 1/p_0$, $1/q_1 + 1/q_2 = 1 + 1/q_0$.

Consequently

If $\sigma \in BS^0_{0,0}$*, then* $T_\sigma : L^2 \times L^2 \to M^{1,\infty} \supseteq L^1$.

- B.-Gröchenig-Heil-Okoudjou (2005): time-frequency analysis proof

- B.-Okoudjou (2006): more general estimates on modulation spaces

What have we learned so far?

In terms of multipliers or symbols, the change made from linear to bilinear is, formally, to replace

$$|\xi| \quad \rightsquigarrow \quad |\xi| + |\eta|$$

Many linear results have bilinear (multilinear) counterparts...but not all...

A more dramatic change occurs if one replaces

$$|\xi| \quad \rightsquigarrow \quad |\xi - \eta| \,!$$

Composition of pseudodifferential operators

• Interest: boundedness on Sobolev spaces and (pseudodifferential) Leibnitz rules!

Let $J^m = (I - \Delta)^{m/2}$ and $\sigma \in BS^m_{1,0}$, $m \geq 0$. Then,

$$T_\sigma(f,g) = T_{\sigma_1}(J^m f, g) + T_{\sigma_2}(f, J^m g).$$

for some σ_1 and σ_2 in $BS^0_{1,0}$.

In particular, $T_\sigma : L^p_m \times L^q_m \to L^r$, $1/p + 1/q = 1/r$, $1 < p,q < \infty$.

However, if $\sigma \in BS^0_{1,0}$ and $a \in S^m_{1,0}$, then in general $L_a T_\sigma \notin \mathrm{Op}BS^m_{1,0}$

$L_a T_\sigma$ has a symbol that satisfies estimates in terms of $|\xi + \eta|$...

This provides another motivation to look at more general symbols!

The classes $BS^m_{\rho,\delta;\theta}$

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \leq$$

$$C_{\alpha\beta\gamma}(1 + |\eta - \xi \tan\theta|)^{m+\delta|\alpha|-\rho(|\beta|+|\gamma|)}$$

for $\theta \in (-\pi/2, \pi/2]$ (with the convention that $\theta = \pi/2$ corresponds to estimates in terms of $1 + |\xi|$ only)

In the one-dimensional case

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \leq$$

$$C_{\alpha\beta\gamma}(1 + \mathrm{dist}((\xi,\eta);\Gamma_\theta))^{m+\delta|\alpha|-\rho(|\beta|+|\gamma|)}$$

where Γ_θ is the line at angle θ with respect to the axis $\eta = 0$

Note that $\theta = -\pi/4, 0, \pi/2$ are the degenerate directions of the BHT.

Symbolic calculus

There exists a calculus for the composition with linear operators and for the transposes.

1. If $T_\sigma \in$ Op$BS^0_{1,0}$ and $L_a \in$ Op$S^m_{1,0}$, then $L_a T_\sigma \in$ Op$BS^m_{1,0;-\pi/4}$.

2. $\{BS^0_{1,0;\,\theta}\}_\theta$ is closed under transposition.

• B.-Nahmod-Torres (2006)

• Bernicot (2008): extension of the calculus to other classes of symbols

Note also that, in general, the classes $BS^m_{\rho,\delta}$ and $BS^m_{\rho,\delta;\,-\pi/4}$ are not comparable. The calculus for the transposes is *crucial* for boundedness on L^p spaces...later we will see a nice connection with the $T(1,1)$-theorem!!!

The symbols of the transposes can be computed explicitly, and it holds that

$$\sigma_\theta \rightsquigarrow \sigma_{\theta^{*1}}, \sigma_{\theta^{*2}}$$

where, for $\theta \neq 0, \pi/2, -\pi/4$,

$$\cot\theta + \cot\theta^{*1} = -1$$

$$\tan\theta + \tan\theta^{*2} = -1$$

In the degenerate directions

$$\{0, \pi/2, -\pi/4\}^{*1} = \{0, -\pi/4, \pi/2\}$$

$$\{0, \pi/2, -\pi/4\}^{*2} = \{-\pi/4, \pi/2, 0\}$$

Modulation invariant multiplier operators in 1-dimension

$$BHT(f,g)(x) = \int \mathrm{sign}(\xi-\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix\cdot(\xi+\eta)}d\xi d\eta$$

$$BHT : L^p \times L^q \to L^r, \ \ 1/p + 1/q = 1/r, \ \ r > 2/3$$

- Lacey-Thiele (1997-1999); Grafakos-Li (2004, uniform estimates)

$$T_m(f,g)(x) = \int m(\xi-\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix\cdot(\xi+\eta)}d\xi d\eta$$

$$|d^\alpha m(z)| \le C_\alpha |z|^{-\alpha}.$$

$$T_m : L^p \times L^q \to L^r, \ \ 1/p + 1/q = 1/r, \ \ r > 2/3$$

- Gilbert-Nahmod (2000); Muscalu-Tao-Thiele (2002, multilinear case)

Modulation invariant variable coefficient operators

Let now

$$T(f_1, f_2)(x) = \int_{\mathbb{R}^2} \sigma(x, \xi - \eta) \widehat{f}_1(\xi) \widehat{f}_2(\eta) e^{ix(\xi+\eta)} d\xi d\eta$$

where $\sigma(x, u) \in S^0_{1,0}$, i.e.,

$$|\partial_x^\alpha \partial_{\xi,\eta}^\beta \sigma(x, \xi - \eta)| \leq C_{\alpha\beta}(1 + |\xi - \eta|)^{-|\beta|}$$

These operators satisfy the *modulation invariance*

$$\langle T(f_1, f_2), f_3 \rangle = \langle T(e^{iz\cdot} f_1, e^{iz\cdot} f_2), e^{-i2z\cdot} f_3 \rangle$$

Question: What do we know about the kernel of such an operator?

Undoing the Fourier transforms,

$$= \int_{\mathbb{R}^2} k(x, x-y)\delta(z-2x+y) f_1(y) f_2(z)\, dydz$$

$$= \int_{\mathbb{R}} k(x,t) f_1(x-t) f_2(x+t)\, dt$$

(valid at least for functions with disjoint support)

$$k(x, x-y) = (\mathcal{F}_2^{-1}\sigma)(x, x-y)$$

Such a k is a (linear) Calderón-Zygmund kernel, but the Schwartz kernel of T is

$$K(x,y,z) = k(x, x-y)\delta(z-2x+y)$$

which is too singular to fall under the scope of the previous multilinear T(1)-Theorems.

Note that the BHT is obtained with $k(x,t) = 1/t$.

The linear to bilinear evolution of symbols

Linear Calderón-Zygmund theory

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Hilbert Transform

$$\mathrm{sign}(\xi)$$

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Hörmander-Mihlin Multipliers

$$|\partial^\alpha m(\xi)| \leq C_\alpha |\xi|^{-|\alpha|}$$

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Classical PDOs

$$|\partial_x^\beta \partial_\xi^\alpha \sigma(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{-|\alpha|}$$

Bilinear Calderón-Zygmund theory: $|\xi| \rightsquigarrow |\xi| + |\eta|$

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Bilinear Coifman-Meyer multipliers

$$|\partial_\xi^\alpha \sigma(\xi)| \leq C_\alpha (|\xi|)^{-|\alpha|}$$

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$$|\partial^{\alpha}_{\xi,\eta}\sigma(\xi,\eta)| \leq C_{\alpha}(|\xi| + |\eta|)^{-|\alpha|}$$

Bilinear Coifman-Meyer PDOs

$$|\partial^{\beta}_{x}\partial^{\alpha}_{\xi}\sigma(x,\xi)| \leq C_{\alpha\beta}(1 + |\xi|)^{-|\alpha|}$$

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$$|\partial^{\alpha}_{\xi,\eta}\sigma(\xi,\eta)| \le C_{\alpha}(|\xi| + |\eta|)^{-|\alpha|}$$

Bilinear Coifman-Meyer PDOs

$$|\partial^{\beta}_{x}\partial^{\alpha}_{\xi,\eta}\sigma(x,\xi,\eta)| \le C_{\alpha\beta}(1 + |\xi| + |\eta|)^{-|\alpha|}$$

Modulation invariant bilinear singular integrals: $\xi \rightsquigarrow \xi - \eta$

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Bilinear Hilbert Transform

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Bilinear modulation invariant multipliers

$$|\partial^\alpha m(\xi - \eta)| \leq C_\alpha |\xi - \eta|^{-\alpha}$$

Modulation invariant bilinear singular integrals: $\xi \rightsquigarrow \xi - \eta$

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Bilinear modulation invariant multipliers

$$|\partial^{\alpha} m(\xi - \eta)| \leq C_{\alpha} |\xi - \eta|^{-\alpha}$$

Bilinear modulation invariant PDOs

$$|\partial_x^{\beta} \partial_{\xi}^{\alpha} \sigma(x, \xi)| \leq C_{\alpha\beta} (1 + |\xi|)^{-|\alpha|}$$

Modulation invariant bilinear singular integrals: $\xi \rightsquigarrow \xi - \eta$

Bilinear Hilbert Transform

$$\mathrm{sign}(\xi - \eta)$$

Bilinear modulation invariant multipliers

$$|\partial^\alpha m(\xi - \eta)| \leq C_\alpha |\xi - \eta|^{-\alpha}$$

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Question: Why are these multipliers so much different than the Coifman-Meyer ones?

Resolving the singularities in the frequency plane

The Coifman-Meyer multipliers just blow up at the origin, i.e., they are singular only at a point in the $\xi\eta$ frequency plane.

- Littlewood-Paley theory

The latter symbols are singular along a line in the frequency $\xi\eta$-plane.

- Phase-space analysis (Whitney decomposition)

Trilinear forms

For symmetry purposes, we will look from now on at the trilinear form

$$\Lambda(f_1, f_2, f_3) = \langle T(f_1, f_2), f_3\rangle$$

$$= \langle T^{*1}(f_3, f_2), f_1\rangle = \langle T^{*2}(f_1, f_3), f_2\rangle$$

For the rest of this talk we will assume that all Calderón-Zygmund kernels considered satisfy

$$|\partial^\alpha k(x,t)| \le C|t|^{-|\alpha|-1} \quad t \neq 0, |\alpha| \le 1$$

Λ is said to be associated with a Calderón-Zygmund kernel k if

$$\Lambda(f_1, f_2, f_3) = \int_{\mathbb{R}^2} \prod_{j=1}^{3} f_j(x + \beta_j t) k(x,t)\, dxdt \qquad (1)$$

for some $\beta = (\beta_1, \beta_2, \beta_3)$ and all f_1, f_2, f_3 in $\mathcal{S}(\mathbf{R})$, $\cap_j \operatorname{supp} f_j = \emptyset$

Modulation symmetry

We assume that $\beta_1, \beta_2, \beta_3$ are different. Otherwise Λ reduces to a combination of a pointwise product and a bilinear form...This follows by a simple change of variables, and appropriately modifying the constants involved in the definition of a Calderón-Zygmund kernel.

We also assume β to be of unit length and perpendicular to $\alpha = (1, 1, 1)$. Let $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ be a unit vector perpendicular to α and β; note that $\gamma_j \neq 0$.

We impose the *modulation symmetry along the direction* γ:

$$\Lambda(f_1, f_2, f_3) = \Lambda(M_{\gamma_1 \xi} f_1, M_{\gamma_2 \xi} f_2, M_{\gamma_3 \xi} f_3)$$

for all $\xi \in \mathbb{R}$, where $M_\eta f(x) = e^{i\eta x} f(x)$.

From (1), the modulation symmetry holds for functions with disjoint support:

$$\Lambda(e^{i\gamma_1 \cdot} f_1, e^{i\gamma_2 \cdot} f_2, e^{i\gamma_3 \cdot} f_3)$$

$$= \int_{\mathbb{R}^2} \prod_{j=1}^{3} f_j(x + \beta_j t) e^{i(\alpha \cdot \gamma x + \beta \cdot \gamma t)} k(x,t)\, dx dt$$

$$= \int_{\mathbb{R}^2} \prod_{j=1}^{3} f_j(x + \beta_j t) k(x,t)\, dx\, dt$$

$$= \Lambda(f_1, f_2, f_3)$$

However...we want the *modulation symmetry to hold for all Schwartz functions, even when the representation formula (1) is not valid as an absolutely convergent integral!*

Modulation invariant $T(1,1)$-theorem

Assume Λ *is a trilinear form associated with a kernel* k *as in* (1) *and with modulation symmetry in the direction* γ. *Then,*

$$|\Lambda(f_1, f_2, f_3)| \lesssim \prod_{j=1}^{3} \|f_j\|_{L^{p_j}}$$

for all exponents $2 \leq p_1, p_2, p_3 \leq \infty$ *with*

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$$

if and only if *for all intervals* I, *all* L^2*-normalized bump functions* ϕ_I *and* ψ_I *supported in* I, *and all* f *in* $\mathcal{S}$ *we have the following restricted boundedness conditions*

$$|\Lambda(\phi_I, \psi_I, f)| \lesssim |I|^{-1/2}\|f\|_{L^2},$$

$$|\Lambda(\phi_I, f, \psi_I)| \lesssim |I|^{-1/2}\|f\|_{L^2},$$

$$|\Lambda(f, \phi_I, \psi_I)| \lesssim |I|^{-1/2}\|f\|_{L^2}.$$

Moreover, in such a case T satisfies

$$\|T(f_1, f_2)\|_{L^r} \lesssim \|f_1\|_{L^p}\|f_1\|_{L^q},$$

for $1/p + 1/q = 1/r, 1 < p, q \leq \infty, 2/3 < r < \infty$.

...This is the same range as for the BHT.

- B.-Demeter-Nahmod-Thiele-Torres-Villarroya (2008)

Reformulation of the $T(1,1)$-theorem

The Calderón-Zygmund trilinear form Λ with modulation symmetry in the direction γ is bounded if and only if

$$\begin{cases} T(1,1), T^{*1}(1,1), T^{*2}(1,1) \in BMO \\ \Lambda \in WBP \end{cases}$$

WBP:

$|\Lambda(\phi)| \lesssim |I|^{-1/2}$, for $\phi(x,y,z)$ any L^2-normalized adapted to $I \times I \times I$

L^2-normalized bump of order N adapted to interval I:

$$|\partial^\alpha \varphi(x)| \leq C|I|^{-1/2-\alpha}\left(1+\left|\frac{x-c(I)}{|I|}\right|^2\right)^{-N/2}, 0 \leq \alpha \leq N$$

The $T(1,1)$ theorem: sketch of proof

The boundedness of Λ immediately implies the restricted boundedness conditions. In fact, T, T^{*1}, T^{*2} : $L^4 \times L^4 \to L^2$ are enough to obtain these conditions.

Conversely, the restricted boundedness conditions are used to show that

$$\begin{cases} T(1,1), T^{*1}(1,1), T^{*2}(1,1) \in BMO \\ |\Lambda(\phi)| \lesssim |I|^{-1/2} \end{cases}$$

for $\phi(x,y,z)$ any L^2-normalized and adapted to $I \times I \times I$

These conditions imply the boundedness of the form Λ and so they are also necessary and sufficient.

The theorem is then reduced to the case

$$\begin{cases} T(1,1) = T^{*1}(1,1) = T^{*2}(1,1) = 0 \\ |\Lambda(\phi)| \lesssim |I|^{-1/2} \end{cases}$$

using some modulation invariant paraproducts. These conditions are used then to discretize the operator.

The proof of the theorem in the reduced case uses a phase-space analysis similar to the one used for the BHT... The difference: we use a Whitney decomposition in frequency in terms of *tubes* (rectangular boxes with square cross sections) not in terms of cubes as in the case of the BHT.

An application

Consider again the variable operator

$$T(f,g)(x) = \int_{\mathbb{R}^2} \sigma(x, \xi - \eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{ix(\xi+\eta)} d\xi d\eta$$

with $\sigma \in S^0_{1,0}$ The associated trilinear form is

$$\Lambda(f_1, f_2, f_3) = \langle T(f_1, f_2), f_3 \rangle$$

Note that this form has modulation symmetry in the direction $\gamma = (1,1,-2)/\sqrt{6}$ for all triples f_1, f_2, f_3, not just the ones with disjoint supports.

Consider

$$\phi_I = |I|^{-1/2}\phi_0((x - x_0)/|I|),$$

where ϕ_0 is adapted to and supported in the unit interval centered at the origin... so that ϕ_I is an L^2-normalized bump adapted to IThen $\widehat{\phi}_I$ is an L^2-normalized bump adapted to an interval of length $|I|^{-1}$, and

$$\|\widehat{\phi}_I\|_{L^1} = |I|^{-1/2}\|\widehat{\phi}_0\|_{L^1} \leq c|I|^{-1/2},$$

where c depends only on finitely many derivatives of ϕ_0.

Let then ϕ_I, ψ_I be two L^2-normalized bumps adapted to I and f be supported in CI. The estimate above applied to $\widehat{\phi}_I$ and $\widehat{\psi}_I$ provides the following information:

$$
\begin{aligned}
|\Lambda(\phi_I, \psi_I, f)| &\lesssim \|\widehat{\phi}_I\|_{L^1} \widehat{\psi}_I\|_{L^1} \|f\|_{L^1} \\
&\lesssim \|\widehat{\phi}_I\|_{L^1} \|\widehat{\psi}_I\|_{L^1} |I|^{1/2} \|f\|_{L^2} \\
&\lesssim |I|^{-1/2} \|f\|_{L^2}
\end{aligned}
$$

This gives the restricted boundedness condition for T!To obtain the other restricted boundedness conditions, write

$$
\Lambda(f, \phi_I, \psi_I) = \langle T^{*1}(\psi_I, \phi_I), f \rangle
$$

Using the symbolic calculus (B.-Nahmod-Torres, 2006) one gets that T^{*1} is a bilinear pseudodifferential operator

$$
T^{*1}(g, h)(x) = \int_{\mathbb{R}^2} \sigma_1(x, \xi, \eta) \widehat{g}(\xi) \widehat{h}(\eta) e^{ix \cdot (\xi + \eta)} d\xi d\eta
$$

σ_1 satisfies the estimates

$$|\partial_x^\mu \partial_{\xi,\eta}^\alpha \sigma_1(x,\xi,\eta)| \lesssim (1+|\xi+2\eta|)^{-|\alpha|}$$

The computations done with T can be repeated for T^{*1} and T^{*2}...Therefore, the modulation invariant $T(1,1)$ theorem implies that for our $T_\sigma \in \mathrm{Op}BS^0_{1,0;\pi/4}$, in which the symbol is assumed to have the form $\sigma(x,\xi-\eta)$ we have

$$T : L^p \times L^q \to L^r, 1/p + 1/q = 1/r < 3/2$$

The same argument will work for a $T_\sigma \in \mathrm{Op}BS^0_{1,0;\theta}$, with $\sigma(x,\xi - \eta \tan\theta)$

• Bernicot (2008): In general, $OpBS^0_{1,0;\theta} : L^p \times L^q \to L^r$; i.e., without dependence $\xi - \theta \tan\theta$ in the symbol...like in the work of Gilbert-Nahmod on multipliers generalizing the BHT!

A crucial difference

The operators we consider in the modulation invariant $T(1,1)$ theorem have kernels that satisfy minimal regularity requirements, therefore they cannot be expressed as smooth bilinear pseudodifferential operators in $OpBS^{0}_{1,0;\theta}$

On the other hand, the operators in $OpBS^{0}_{1,0;\theta}$ are not modulation invariant, so they do not have the kernel representation assumed by the $T(1,1)$ theorem. Nevertheless, the symbols are smooth, so it is not surprising that their boundedness can be achieved without appealing to a $T(1,1)$ theorem (this is the case also for many results on classical linear pseudodifferential operators!!!) Recall also that this boundedness was "predicted" to hold by the existence of a transposition symbolic calculus for these classes (B.-Nahmod- Torres, 2006)

Further application: “antisymmetric” forms

Consider $k(x,t)$ a Calderón-Zygmund kernel that satisfies the “antisymmetric” property

$$k(x+t,-t) = -k(x,t)$$

This unconventionally looking symmetry is due to the fact that we chose the kernel to be singular at $t=0$ and not at $x=t$, i.e., we work with the condition

$$|\partial^\alpha k(x,t)| \lesssim |t|^{-|\alpha|-1}$$

instead of the standard (and equivalent after a change of variable) condition

$$|\partial^\alpha k(x,t)| \lesssim |x-t|^{-|\alpha|-1}$$

We can always define

$$\Lambda(f_1, f_2, f_3) = p.v. \int f_1(x+t) f_2(x-t) f_3(x) k(x,t)\, dxdt$$

$$= \frac{1}{2} \int (f_1(x+t) f_2(x-t) f_3(x) - f_1(x) f_2(x+2t) f_3(x+t)) k(x,t)\, dxdt$$

since the integral is absolutely convergent.

It is straightforward to check that Λ is a well-defined modulation invariant form (even for functions without disjoint support) that satisfies the WBP!

Now compute

$$\Lambda(f_1, 1, f_3) = p.v. \int f_1(y) f_3(x) k_1(x,y)\, dx\, dy,$$

where $k_1(x,y) = k(x, y-x)$ is antisymmetric in the usual way, i.e.

$$\underline{k_1(y,x)} = k(y,x-y) =_{x-y:=-t} k(x+t,-t)$$

$$= -k(x,t) = -k(x,y-x) = \underline{-k_1(x,y)}$$

If the bilinear form with this kernel is bounded, we immediately get that

$$\Lambda(1,1,\cdot) = -\Lambda(\cdot,1,1) \in BMO$$

Similarly,

$$\Lambda(1,f_2,f_3) = p.v. \int f_2(y) f_3(x) k_2(x,y)\, dx\, dy,$$

where $k_2(x,y) = k(x,x-y)$, and the boundedness of the bilinear form with this kernel would give us also

$$\Lambda(1,\cdot,1) \in BMO$$

...thus $T(1,1) \Rightarrow$ boundedness of “antisymmetric” forms!

Why are "antisymmetric" forms relevant?

Bilinear Calderón commutators

$$BC^m(f,g,h) =$$

$$p.v. \int f(x+t)g(x-t)h(x)\frac{(A(x+t)-A(x))^m}{t^{m+1}}\,dtdx$$

with $\|A'\|_{L^\infty} < C$. Note that

$$k(x,t) = \frac{(A(x+t)-A(x))^m}{t^{m+1}}$$

satisfies

$$k(x+t,-t) = \frac{(A(x)-A(x+t))^m}{(-t)^{m+1}} = -k(x,t)!$$

The bilinear form associated to the kernel $k_1(x, y)$ is the usual Calderón commutator:

$$k_1(x, y) = k(x, y - x) = \frac{(A(y) - A(x))^m}{(y - x)^{m+1}}$$

A similar operator is obtained for the other kernel

$$k_2(x, y) = k(x, x - y) = \frac{(A(2x - y) - A(x))^m}{(x - y)^{m+1}}$$

Hence the previous scheme works and we get the boundedness of BC^m through the $T(1, 1)$ theorem!

Keeping track of the constants involved and expanding in the usual way in terms of commutators we also obtain the boundedness of the

Bilinear Cauchy integral

$$BC(f,g)(x) = p.v. \int \frac{f(x+t)g(x-t)}{t + i(A(x+t) - A(x))}\, dt$$

at least for $\|A'\|_{L^\infty} << 1$.

Thank you for attending this talk!!!