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One of the first results of Mischa that came after Cotlar's Lemma:

Thm (Arocena-Cotlar-Sadosky)

T bdd $L^2(\sigma) \rightarrow L^2(\tilde{\nu})$

$$\left(\int_{\Pi} |Tf|^2 d\tilde{\nu} \leq M \int_{\Pi} |f|^2 d\sigma \right)$$

$$\Leftrightarrow \exists h \in H^1 \text{ s.t. } \begin{pmatrix} M\sigma - \tilde{\nu} & M\sigma + \tilde{\nu} - h d\tau \\ M\sigma + \tilde{\nu} - h d\tau & M\sigma - \tilde{\nu} \end{pmatrix} \geq 0$$

Many interesting corollaries. Among them:

Corollary T ^{Hilbert transform} bdd $L^2(\sigma) \rightarrow L^2(\tilde{\nu}) \Rightarrow \tilde{\nu}_s = 0$

On the arc Π we must require σ to be finite ($\sigma(\Pi) < \infty$)

On \mathbb{R} the condition should be replaced by $\int \frac{d\sigma(x)}{1+x^2} < \infty$

We introduce the operator

$$(*) \quad (Uf)(t) = f(t) - \alpha \int \frac{f(s) - f(t)}{s-t} d\mu(s)$$

viewed as an operator: $L^2(\mu) \rightarrow L^2(\nu)$

It comes from perturbation theory

$$A = A^* \text{ on } \mathcal{H}_0, A_\alpha := A + \alpha (\cdot, \varphi) \varphi$$

φ -cyclic, in the sense that $\text{span} \{A^n \varphi, n \geq 0\} = \mathcal{H}_0$

$$A \simeq M_S^0 \text{ on } L^2(\mu)$$

$$\mu = \mu_{\varphi}$$

$$M(z) = \int \frac{d\mu(x)}{x-z} = ((A-zI)^{-1} \varphi, \varphi)$$

$$\Leftrightarrow \int x^n d\mu(x) = (A^n \varphi, \varphi)$$

it is called Weil function

Let us assume $A = M_{\varphi}$ on $L^2(\mu)$

Then consider the operator A_{α} , which is given by $\varphi = 1$, so that

$$A_{\alpha} = M_{\varphi} + \alpha (\cdot, 1) 1 \sim M_t \text{ on } L^2(\nu), \nu = \nu_{\varphi}$$

Thm $U(M_{\varphi} + \alpha (\cdot, 1) 1) = M_t U$ with U as before (cf (*))

Proof $U M_{\varphi} - M_t U = -\alpha (\cdot, 1) 1_t, \alpha 1_s = 1_t$

$$\underbrace{M_t U M_{\varphi}} - M_t^2 U = -\alpha (\cdot, 1) t$$

|| known

$$U M_{\varphi} + \alpha (\cdot, 1) 1$$

$$U M_{\varphi}^2 - M_t^2 U = -\alpha [(\cdot, 1) t + (\cdot, s) 1]$$

$$(\#) U M_{\varphi}^n - M_t^n U = -\alpha [(\cdot, 1) t^{n-1} + (\cdot, s) t^{n-2} + \dots + (\cdot, s^{n-1}) 1_t]$$

$(\cdot, a) b$ is the integral op. with kernel $b(t) a(s)$

kernel given by $\sum_{k=0}^{n-1} s^k t^{n-1-k} = \frac{s^n - t^n}{s-t}$

We apply the operators to 1 and we get from (#) that

$$(U s^n)(t) - t^n = \int \frac{s^n - t^n}{s-t} d\mu(s)$$

and hence what we want. \square

Joint work with C. Liaw

Thm. Let $A = M_{\omega}$ on $L^2(\mu)$,

$$d\mu = \omega dt + \mu_S.$$

Let $\frac{1}{\omega} \in L^1_{loc}(\mathbb{I})$.
\ open interval

$$A_{\alpha} = A + \alpha (., 1) 1$$

$\nu = \nu_{\alpha}$ (spectral measure)

Then $\nu_S | \mathbb{I} = \emptyset$.

Thm. (Rigidity) Let V be invertible: $L^2(\mu) \rightarrow L^2(\nu)$

(given by *)

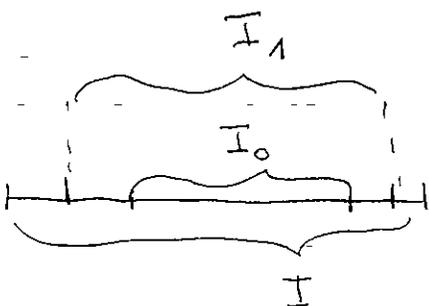
Then $\exists \varphi \in L^{\infty}$, $\frac{1}{\varphi} \in L^{\infty}$ s.t. $M_{\varphi} V$ -unitary -

If we know that $\text{supp } f, \text{supp } g$ are separated

then we can write down

$$(Uf, g) = \alpha \int \frac{\overline{g(t)} f(s)}{s-t} d\mu(s) d\nu(t)$$

$$\left| \int \frac{f(s) \overline{g(t)}}{s-t} d\mu(s) d\nu(t) \right| \leq C \|f\|_{L^2(\mu)} \|g\|_{L^2(\nu)}$$



take now ν_S instead of ν

then take $\nu_S^0 = \nu_S |_{I_0}$ & $d\omega$ instead of $d\mu$

ω can be replaced by $\omega_1 = \omega \chi_{I_1} + \chi_{\mathbb{R} \setminus I_1}$

$$\text{So } \left| \int \frac{f(s) \overline{g(t)}}{s-t} d\omega_1(t) d\nu_s^\circ(t) \right| \leq C \|f\|_{L^2(\omega_1)} \|g\|_{L^2(\nu_s^\circ)}$$

$$\text{Take } d\sigma = \frac{1}{\omega_1} ds, \quad \int \frac{d\sigma}{1+s^2} < \infty$$

Consider projection P_n onto polynomials of degree $\leq n$.

Then the estimate holds uniformly.

$$\sqrt{1_s} = 1_t$$

$$\sqrt{M_s + \alpha(\cdot, 1_s)} 1_t = M_t V$$

$$\underbrace{\sqrt{(M_s + \alpha(\cdot, 1_s)) 1_s}}_A = \underbrace{M_t}_B V \quad (\text{this says that } A \text{ \& } B \text{ are similar})$$

$$VA = BV$$

$$AV^* = V^*B$$

$$\sqrt{AV^*} = \sqrt{V^*B}$$

$$\Rightarrow BVV^* = VV^*B$$

$$\Rightarrow V = \underbrace{(V^*)^{-1/2}}_{M_{\sigma^{-1}}} u$$