ABSTRACT SCATTERING SYSTEMS: SOME SURPRISING CONNECTIONS

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In the late seventies, Mischa and I began a systematic study of algebraic scattering systems and the invariant forms acting on them.

In the late eighties we started working in multidimensional scattering—although many did not consider such approach as relevant.

In the late nineties our outlook was finally vindicated. Multidimensional abstract scattering systems appeared as counterparts of the input-output conservative linear systems in engineering.

It all started with the study of the Hilbert transform in terms of scattering...

The Hilbert transform operator

$$H : f \mapsto Hf = f * k$$

is given by convolution with the singular kernel

$$k(x) = p.v. \frac{1}{x}$$

for $x \in \mathbb{R}$.

The basic result of Marcel Riesz (1927) is

$$H \text{ is bounded on } L^p, \ 1 < p < \infty.$$

Objective: Similar boundedness properties in the "weighted" cases, for H and iterated $H = H_1 H_2 ... H_n$,

$$H : L^p(\mu) \to L^p(\nu), \ 1 < p < \infty$$

where μ, ν are general measures.

Scattering property of analytic projector P in L^2

Given $f \in L^2, f = f_1 + f_2$, where f_1 is analytic and f_2 is antianalytic.

Under this decomposition the Hilbert transform can be written as

$$Hf = -i\, f_1 + i\, f_2.$$

The analytic projector P, associated with the Hilbert tranform operator H, is defined as

$$Pf = P(f_1 + f_2) = f_1, \qquad P = \frac{1}{2}(I + iH).$$

The crucial observation is that P supports the SHIFT OPERATOR $S : f(x) \mapsto e^{ix} f(x)$.

Then, the RANGE of P is the set W_1 of analytic functions, and its KERNEL is the set W_2 of antianalytic functions:

$$S\, W_1 \subset W_1 \quad and \quad S^{-1}\, W_2 \subset W_2.$$

The scattering property of the 1-dimensional Hilbert transform provides the framework for a theory of invariant forms in scattering systems, leading to two-weight L^2-boundedness results for H.

The scattering properties are also essential for providing the two-weight L^2- boundedness of the product Hilbert transform in product spaces, where the analytic projectors supporting the n-dimensional shifts are at the basis of the lifting theorems in abstract scattering structures.

Notice that this fact, valid for the product Hilbert transforms, is not valid for the n-dimensional Calderón–Zygmund singular integrals which do not share the scattering property.

H IS BOUNDED ON L^2

In fact, H is an ISOMETRY on L^2:

$$\|Hf\|_2 = \|f\|_2$$

and this follows easily from the Plancherel Theorem for the Fourier transform.

The boundedness of H in L^2 can be obtained also through Cotlar's Lemma on Almost Orthogonality, which extends to Hilbert transforms in ergodic systems.

These are two different ways to deal with the boundedness of H in L^2. The same happens for $p \neq 2$.

TWO WAYS FOR $H : L^2$ TO $L^p, 1 < p < 2$

Checking that H is weakly bounded in L^1, and applying the Marcinkiewicz INTERPOLATION Theorem between $p = 1$ and $p = 2$, and then, from $p = 2$ to $p = \infty$ by duality.

By the "MAGIC IDENTITY":

$$(Hf)^2 = f^2 + 2\, H\,(f.Hf) \qquad (*)$$

valid for all "good" functions (f smooth with good decay at infinity).

Use first EXTRAPOLATION, since for $f \in L^2$, $(*)$ implies $Hf \in L^2$, then $f \in L^4$ implies $Hf \in L^4$, and $f \in L^{2^k}$ implies $Hf \in L^{2^k}, \forall\, k \geq 1$.

The boundedness of H in $L^{p'}, 1/p + 1/p' = 1, p = 2^k$, follows by duality, and interpolation gives the boundedness of H in $L^p, 1 < p < \infty$.

By polarization, the “Magic Identity” for an operator T becomes

$$T(f.Tg + Tf.g) = Tf.Tg - f.g.$$

Rubio de Francia, and Cotlar and Sadosky, used the Identity in dealing with the weighted Hilbert transform in Banach lattices. The same identity, and similar ones, were used extensively by Coifman-Meyer in harmonic analysis. Gohberg and Krein showed that the polarized identity holds for the Schatten class $\mathcal{S}^2$, and deduced the theorem of Krein and Macaev in a way similar to the passage from L^2 to L^p described above.

Gian-Carlo Rota used different “magic indentities” in his work in combinatorics, and his school encompased all particular cases in a general inequality. The indentities also hold in non-commutative situations, as in the non-commutative Hilbert transforms in von Neumann algebras.

H IS BOUNDED IN $L^2(\omega)$, $0 \leq \omega \in L^1$

HELSON-SZEGŐ (1960)

$\Longleftrightarrow$ $\omega = e^{u+Hv}$, u, v **real-valued bounded functions,** $\|v\|_\infty < \pi/2$

$\Longleftrightarrow$ log $\omega \in BMO$ **(with a special** BMO **norm)**

HUNT-MUCKENHOUPT-WHEEDEN (1973)

$\Longleftrightarrow$ $\omega \in A_2$:

$$\left(\frac{1}{|I|}\int_I \omega\right)\left(\frac{1}{|I|}\int_I \frac{1}{\omega}\right) \leq C\,, \quad \forall I \quad interval$$

Although both conditions are necessary and sufficient, the first is good to produce such weights, while the second is good at checking them.

DEFINITION OF u-BOUNDEDNESS

(Cotlar-Sadosky, 1981; Rubio de Francia, 1982)

An operator T acting in a Banach lattice X, $T : X \to L^0(\Omega)$ is u-bounded if $\forall f \in X$, $\|f\| \leq 1$, $\exists g \in X$, $g \geq |f|$, $\|g\| + \|Tg\| \leq C$.

u-boundedness of operators is considerably weaker than boundedness. For example,

T **is** u**-bounded in** L^∞ **iff** $T\,1 \in L^\infty$**.**

Now we can translate another equivalence for the Helson-Szegő theorem (p=2):

$\Longleftrightarrow \exists\, w \sim \omega$**, real-valued function, such that** $|H\,w(x)| \leq C\,w(x)$ $a.e.$

(here $w \sim \omega$ means $c\,\omega(t) \leq w(t) \leq C\,\omega(t), \forall\, t$)

$\Longleftrightarrow T_\omega : f \mapsto \omega^{-1}\,H(\omega\,f)$ **is** u**-bounded in** L^∞**.**

$\underline{H \text{ IS BOUNDED IN } L^p(\mathbb{T};\omega),\ 1 < p < \infty}$

HUNT-MUCKENHOUPT-WHEEDEN (1973)

$\Longleftrightarrow \omega \in A_p$:

$$\left(\frac{1}{|I|}\int_I \omega\right)\left(\frac{1}{|I|}\int_I \omega^{-1/(p-1)}\right)^{p-1} \leq C\,,\ \forall\, I\ interval$$

COTLAR-SADOSKY (1982)

$\Longleftrightarrow T_\omega$, **defined by**

$$T_\omega = \omega^{-2/p}\, H(\omega^{2/p}\, f), \qquad p \geq 2,$$

$$T_\omega = \omega^{2/p}\, H(\omega^{-2/p}\, f), \qquad p < 2,$$

is u**-bounded in** L^{p^*} **where** $1/p^* = |\,1 - 2/p\,|$.

H IS BOUNDED IN WEIGHTED $L^2(\mathbb{T}^2;\omega)$

The following are equivalent:

(1) The double Hilbert transform $H = H_1\,H_2$ **is bounded in** $L^2(\mathbb{T}^2;\omega)$

(2) $\omega = e^{u_1 + H_1\,v_1} = e^{u_2 + H_2\,v_2}$, u_1, u_2, v_1, v_2, **real-valued bounded functions,** $\|v_i\|_\infty < \pi/2$, $i = 1, 2$

(3) log $\omega \in bmo$ **(with a special** bmo **norm)**

(4) $\exists\, w_1, w_2$, $w_1 \sim \omega \sim w_2$, **such that**

$|H_1\,w_1(x)| \le C\,w_1(x),\ |H_2\,w_2(x)| \le C\,w_2(x)$ $a.e.$

(5) $T_1 : f \mapsto \omega^{-1}\,H_1(\omega\,f)$ **and** $T_2 : f \mapsto \omega^{-1}\,H_2(\omega\,f)$ **are simultaneously** u**-bounded in** $L^\infty(\mathbb{T}^2)$

(6) $\omega \in A_2^*$

H BDD. IN WEIGHTED $L^p(\mathbb{T}^2;\omega)$, $1 < p < \infty$

(Cotlar-Sadosky, 1990)

The following are equivalent:

(1) The double Hilbert transform $H = H_1\, H_2$ is bounded in $L^p(\mathbb{T}^2;\omega)$

(5) T_1 and T_2 are simultaneously u-bounded in L^{p^*}, where for $i = 1, 2$,

$$T_i : f \mapsto \omega^{-2/p}\, H_i(\omega^{2/p}\, f),\ if 2 \leq p <$$

and

$$T_i : f \mapsto \omega^{2/p}\, H_i(\omega^{-2/p}\, f),\ if 2 \leq p <$$

are simultaneously u-bounded in $L^{p^*}(\mathbb{T}^2)$.

(6) $\omega \in A_p^*$

LIFTING THEOREM FOR INVARIANT FORMS IN ALGEBRAIC SCATTERING STRUCTURES (Cotlar-Sadosky, 1979)

V **vector space,** σ **linear isomorphism in** V

W_+, W_-: **linear subspaces satisfying**

$$\sigma W_+ \subset W_+, \quad \sigma^{-1} W_- \subset W_-$$

$B_1, B_2 : V \times V \to \mathbb{C}$, **positive** σ**-invariant forms**

$\forall\, B_0 : W_+ \times W_- \to \mathbb{C} \ni \forall\, (x, y) \in W_+ \times W_-$

$$B_0(\sigma\, x, y) = B_0(x, \sigma^{-1}\, y),$$

$$|B_0(x, y)| \leq B_1(x, x)^{1/2}\, B_2(y, y)^{1/2}$$

$\exists\, B' : V \times V \to \mathbb{C} \ni \forall\, (x, y) \in V \times V$

$$B'(\sigma\, x, \sigma\, y) = B'(x,\, y),$$

$$|B'(x, y)| \leq B_1(x, x)^{1/2}\, B_2(y, y)^{1/2}$$

such that

$$B'(x, y) = B_0(x, y),\, \forall (x, y) \in W_+ \times W_-.$$

Let $V = \mathcal{P}$ be the set of trigonometric polynomials on $\mathbb{T}, \mathcal{P}_1, \mathcal{P}_2,$ the sets of analytic and antianalytic polynomials in $\mathcal{P}, \sigma = S$, the shift operator. The Herglotz-Bochner theorem translates to: B is positive and S-invariant in $\mathcal{P} \times \mathcal{P}$ iff $\exists\, \mu \geq 0$ such that

$$B(f,g) = \int f\, \overline{g}\, d\mu \qquad \forall f,\, g \in \mathcal{P}.$$

Since $\mathcal{P} = \mathcal{P}_1 \dot{+} \mathcal{P}_2$, the domain of B splits in four pieces $\mathcal{P}_i \times \mathcal{P}_j$ for $i, j = 1, 2$. A weaker concept of S-invariance is

$$B(Sf,\, Sg) = B(f,\, g)$$

for all $(f,\, g)$ in each quarter $\mathcal{P}_i \times \mathcal{P}_j, i, j = 1, 2$. Then the Lifting theorem asserts that B is positive in $\mathcal{P} \times \mathcal{P}$ and S-invariant in each quarter iff $\exists\, \mu = (\mu_{ij}) \geq 0 \ni \ \forall\, f_1, g_1 \in \mathcal{P}_1,\, f_2, g_2 \in \mathcal{P}_2$

$$B(f_1 + f_2,\, g_1 + g_2) = \sum_{i,j=1,2} \int f_i\, \overline{g}_j\, d\mu_{ij}.$$

Here $(\mu_{ij}) \geq 0$ means that the (complex) measures satisfy $\mu_{11} \geq 0, \mu_{22} \geq 0, \mu_{21} = \overline{\mu_{12}}$ and

$$|\mu_{12}(D)|^2 \leq \ \mu_{11}(D)\, \mu_{22}(D), \qquad \forall\, D \subset \mathbb{T}.$$

THE LIFTING THEOREM IN $\mathcal{P} \times \mathcal{P}$

If B is S-invariant in $\mathcal{P} \times \mathcal{P}$, then

$$B \geq 0 \Longleftrightarrow \mu \geq 0$$

but when B is S-invariant in each $\mathcal{P}_i \times \mathcal{P}_j$,

$$B \geq 0 \Longleftrightarrow \sum_{i,j} \int f_i \bar{f}_j \, d\mu_{ij} \geq 0$$

ONLY for $f_1 \in \mathcal{P}_1$, $f_2 \in \mathcal{P}_2$, which is far less than

$$\mu \geq 0 \Longleftrightarrow \sum_{i,j} \int f_i \bar{f}_j \, d\mu_{ij} \geq 0$$

for ALL $f_1, f_2 \in \mathcal{P}$.

Let $B_1 = B|\mathcal{P}_1 \times \mathcal{P}_1$, $B_2 = B|\mathcal{P}_2 \times \mathcal{P}_2$, $B_0 = B|\mathcal{P}_1 \times \mathcal{P}_2$. Then $B_1, B_2 \geq 0$, while B_0 is not positive but is bounded,

$$|B_0(f_1, f_2)| \leq B_1(f_1, f_1)^{1/2} \, B_2(f_2, f_2)^{1/2}.$$

Since $\mu_{11} \geq 0$, $\mu_{22} \geq 0$**,**

$$|B_0(f_1, f_2)| \leq \|f_1\|_{L^2(\mu_{11})} \|f_2\|_{L^2(\mu_{22})}.$$

By the definition of B_0 **it follows**

$$B_0(f_1, f_2) = \int f_1 \, \bar{f}_2 \, d\mu_{12}$$

only for $f_1 \in \mathcal{P}_1$, $f_2 \in \mathcal{P}_2$.

And since

$$|\mu_{12}(D)|^2 \leq \mu_{11}(D) \, \mu_{22}(D), \quad \forall \, D \subset \mathbb{T},$$

defining

$$B'(f_1, f_2) := \int f_1 \, \bar{f}_2 \, d\mu_{12}, \forall \, f_1, \, f_2 \in \mathcal{P}$$

it is $\|B'\| = \|B_0\|$**.**

H IS BOUNDED IN $L^2(\mathbb{T};\nu,\mu)$

Let the operator H be defined in $\mathcal{P}$ as

$$H(f_1+f_2)=-i\,f_1+i\,f_2.$$

The two-weight inequality for H is

$$\int_{\mathbb{T}}|Hf|^2d\mu\le\ M^2\int_{\mathbb{T}}|f|^2d\nu$$

or, equivalently,

$$(\star)=\sum_{i,j=1,2}\int_{\mathbb{T}}f_i\bar{f}_j\,d\rho_{ij}\ge 0,\qquad f_1\in\mathcal{P}_1,\ f_2\in\mathcal{P}_2$$

where

$$\rho_{11}=\rho_{22}=M^2\nu-\mu,\ \rho_{12}=\overline{\rho_{21}}=M^2\nu+\mu.$$

Defining $B(f,g)=B(f_1+f_2,\,g_1+g_2)=(\star)$, B is S-invariant in each quarter and non-negative. By the Lifting Theorem, $\exists(\mu_{ij})$, $i,j=1,2$, such that $\widehat{\rho}_{ii}(n)=\widehat{\mu}_{ii}(n),\forall\,n\in\mathbb{Z}$, while $\widehat{\rho}_{12}(n)=\widehat{\mu}_{12}(n)$ only for $n<0$. By the F. and M. Riesz theorem

$$\mu_{11}=\mu_{22}=M^2\nu-\mu,\ \mu_{12}=\overline{\mu_{21}}=M^2\nu+\mu-h,$$

$h\in H^1(\mathbb{T})$.

Then, the necessary and sufficient condition for boundedness in $L^2(\mathbb{T}; \nu, \mu)$, with norm M, is that for all $D \subset \mathbb{T}$ Borel sets,

$$|(M^2\nu + \mu)(D) - \int_D h\,dt\,| \leq (M^2\nu - \mu)(D)$$

where $h \in H^1(\mathbb{T})$.

In particular, μ is an absolutely continuous measure, $d\mu = w\,dt$ for some $0 \leq w \in L^1$.

In the case $\mu = \nu = \omega\,dt$ we return to the previous case:

The Hilbert transform H is a bounded operator in $L^2(\omega)$ with norm M

$\Longleftrightarrow$

$$|(M^2 + 1)\,\omega(t) - h(t)| \leq (M^2 - 1)\,\omega(t),\ a.e.\,\mathbb{T},$$

$h \in H^1(\mathbb{T})$, which is the source to all the equivalences we mentioned before.

LIFTING THEOREM FOR INVARIANT FORMS IN HILBERTIAN SCATTERING STRUCTURES

In 1974 Bill Helton described for the first time the relationship between discrete time systems theory, the theory of colligations, and the Lax-Phillips scattering theory.

The Lax-Phillips theory considers scattering systems defined on a Hilbert space $\mathcal{H}$, where the *outgoing* **and** *incoming* **spaces of the system are closed subspaces of $\mathcal{H}$, and the** *evolution* **is given by a one-parameter group of unitary operators.**

An explicit formula relates the scattering wave operators to the systems theory.

LAX-PHILLIPS SCATTERING SYSTEM (one evolution)

$$\mathfrak{S} = (\mathcal{K}; \mathcal{U}; \mathcal{W}, \mathcal{W}_*)$$

$\mathcal{K}$: **a Hilbert space, the** *ambient* **space**
$\mathcal{U}$: **unitary on** $\mathcal{K}$**, the** *evolution* **operator**
$\mathcal{W}, \mathcal{W}_* \subset \mathcal{K}$: *outgoing* **and** *incoming* **subspaces**

1. **Scattering** $\quad \mathcal{U}\,\mathcal{W} \subset \mathcal{W}$, $\mathcal{U}^*\,\mathcal{W}_* \subset \mathcal{W}_*$

2. $\bigcap_{n=0}^{\infty} \mathcal{U}^n \mathcal{W} = \{0\} = \bigcap_{n=0}^{\infty} \mathcal{U}^{*n} \mathcal{W}_*$

3. **Causality** $\quad \mathcal{W} \perp \mathcal{W}_*$

Orthogonal decomposition: $\mathcal{K} = \mathcal{W}_* \oplus \mathcal{V} \oplus \mathcal{W}$.
Then the *internal scattering subspace* **of** $\mathfrak{S}$

$$\mathcal{V} := \mathcal{K} \ominus [\mathcal{W} \oplus \mathcal{W}_*]$$

is semi-invariant. $\mathfrak{S}$ **is** *minimal* **if**

$$closure\,(\widetilde{\mathcal{W}} + \widetilde{\mathcal{W}}_*) = \mathcal{K}$$

for $\widetilde{\mathcal{W}} := \bigvee_{n \geq 0} \mathcal{U}^{*n} \mathcal{W}$ **and** $\widetilde{\mathcal{W}}_* := \bigvee_{n \geq 0} \mathcal{U}^{n} \mathcal{W}_*$.

Given two Hilbertian scattering systems $(\mathcal{K}_1;\mathcal{U}_1;\mathcal{W}_1,\mathcal{W}_{*1})$, $(\mathcal{K}_2;\mathcal{U}_2;\mathcal{W}_2,\mathcal{W}_{*2})$ **a form** $B:\mathcal{K}_1\times\mathcal{K}_2\to\mathbb{C}$ **is invariant if**

$$B(\mathcal{U}_1 f_1,\mathcal{U}_2 f_2)=B(f_1,f_2),\ \forall\, f_1\in\mathcal{K}_1,\ f_2\in\mathcal{K}_2.$$

A form $B_0:\mathcal{W}_1\times\mathcal{W}_{*2}\to\mathbb{C}$ **is also invariant,**

$$B_0(\mathcal{U}_1 f_1,f_2)=B_0(f_1,\mathcal{U}_2^{-1}f_2),\ \forall\, f_1\in\mathcal{W}_1,\ f_2\in\mathcal{W}_{*2}$$

since the subspaces $\mathcal{W}_i$, $\mathcal{W}_{*i}$, $i=1,2$ **are all invariant subspaces.**

LIFTING THEOREM FOR INVARIANT FORMS IN SCATTERING SYSTEMS (Cotlar-Sadosky, 1987)

Given two Hilbertian scattering systems as above, every invariant form $B_0:\mathcal{W}_1\to\mathcal{W}_{*2}$, $\|B_0\|\le 1$ **has an invariant lifting**

$$B':\mathcal{K}_1\times\mathcal{K}_2\to\mathbb{C},\ B'|\mathcal{W}_1\times\mathcal{W}_{*2}=B_0,\ \|B'\|=\|B_0\|.$$

Since the internal scattering subspaces $\mathcal{V}_1, \mathcal{V}_2$ are only semi-invariant, $B_0 : \mathcal{V}_1 \times \mathcal{V}_2 \to \mathbb{C}$ is only "essentially" invariant if

$$B_0(P_1 \mathcal{U}_1 f_1, f_2) = B_0(f_1, P_2 \mathcal{U}_2^{-1} f_2),\ \forall\, f_i \in \mathcal{V}_i,$$

where P_i is the orthogonal projection of $\mathcal{K}_i$ onto $\mathcal{V}_i$, $i = 1, 2$.

LIFTING THEOREM FOR ESSENTIALLY INVARIANT FORMS IN SCATTERING SYSTEMS
(Cotlar-Sadosky, 1993)

Given two Hilbertian scattering systems as before, for every essentially invariant form $B_0 : \mathcal{V}_1 \times \mathcal{V}_2 \to \mathbb{C}$, $\|B_0\| \leq 1$, there is an invariant lifting

$$B' : \mathcal{K}_1 \times \mathcal{K}_2 \to \mathbb{C},\ B'|\mathcal{V}_1 \times \mathcal{V}_2 = B_0,\ \|B'\| \leq 1.$$

Furthermore, $B' = 0$ on $\mathcal{W}_1 \times \mathcal{W}_{*2}$, $\mathcal{V}_1 \times \mathcal{W}_{*2}$, $\mathcal{W}_1 \times \mathcal{V}_2$.

A unitary operator $\mathcal{U} \in \mathcal{B}(\mathcal{K})$ is a strong dilation of a contraction operator $T \in \mathcal{B}(\mathcal{V})$ if $\mathcal{V}$ is a closed subspace of $\mathcal{K}$ and $T^n = P_{\mathcal{V}}\mathcal{U}^n|_{\mathcal{V}}$ is the compression of $\mathcal{U}^n$ to $\mathcal{V}$ for all $n \in \mathbb{N}$. A lemma of Sarason asserts that $\mathcal{U} \in \mathcal{B}(\mathcal{K})$ is a strong dilation of $T \in \mathcal{B}(\mathcal{V})$ iff $T = P_{\mathcal{V}}\mathcal{U}|_{\mathcal{V}}$ and $\mathcal{K} = \mathcal{W} \oplus \mathcal{V} \oplus \mathcal{W}_*$ where $\mathcal{U}\mathcal{W} \subset \mathcal{W}, \mathcal{U}^{-1}\mathcal{W}_* \subset \mathcal{W}_*$.

Sz.NAGY-FOIAS LIFTING THEOREM FOR UNITARY DILATIONS

Let, for $i = 1, 2, \mathcal{K}_i$ be Hilbert spaces, $\mathcal{V}_i \subset \mathcal{K}_i$ be closed subspaces of $\mathcal{K}_i$, and let $\mathcal{U}_i \in \mathcal{B}(\mathcal{K}_i)$ be unitary operators, while $T_i \in \mathcal{B}(\mathcal{V}_i)$ are contraction operators, such that $\mathcal{U}_i$ is a strong dilation of T_i.
If $X : \mathcal{V}_1 \to \mathcal{V}_2, \|X\| \leq 1$ is a contraction intertwining $T_1, T_2 :\ XT_1 = T_2X$, then there exists a contraction $Y : \mathcal{K}_1 \to \mathcal{K}_2, \|Y\| \leq 1$ such that Y intertwines $\mathcal{U}_1, \mathcal{U}_2$, and

$$X = P_{\mathcal{V}_2} Y|_{\mathcal{V}_1} \qquad (**)$$

where $P_{\mathcal{V}_2} : \mathcal{K}_2 \to \mathcal{V}_2$ is the orthoprojector.

PROOF. (Cotlar-Sadosky, 1993)

By Sarason's Lemma, for both $i = 1, 2$, $\mathcal{K}_i = \mathcal{W}_i \oplus \mathcal{V}_i \oplus \mathcal{W}_{*i}$ **and** $T_i = P_{\mathcal{V}_i} \mathcal{U}_i|_{\mathcal{V}_i}$**. Defining** $B_0 : \mathcal{V}_1 \times \mathcal{V}_2 \to \mathbb{C}$ **by**

$$B_0(f_1, f_2) = \langle X f_1, f_2 \rangle, \forall f_1 \in \mathcal{V}_1, f_2 \in \mathcal{V}_2$$

the intertwining condition for X **is equivalent to** B_0 **being essentially invariant with respect to** $\mathcal{U}_1$,$\mathcal{U}_2$**.**
If $B' : \mathcal{K}_1 \times \mathcal{K}_2 \to \mathbb{C}$ **is the lifting of** B_0**, define** $Y : \mathcal{K}_1 \to \mathcal{K}_2$ **by**

$$\langle Y f_1, f_2 \rangle := B'(f_1, f_2), \quad \forall\, 1 \in \mathcal{K}_1,\, f_2 \in \mathcal{K}_2.$$

Then $\|B'\| \leq 1$ **is equivalent to** $\|Y\| \leq 1$**, and its invariance is equivalent to** $Y \mathcal{U}_1 = \mathcal{U}_2 Y$**. Finally,** $B' = B_0$ **in** $\mathcal{V}_1 \times \mathcal{V}_2$ **is equivalent to** $(**)$**.**

MULTIDIMENSIONAL LAX-PHILLIPS SCATTERING SYSTEMS AND LINEAR ISO SYSTEMS IN ENGINEERING

The one-to-one correspondence between *conservative* **linear input/state/output (ISO) systems and** *Lax-Phillips* **scattering systems, where the** *transfer function* **of the linear system coincides with the** *scattering function* **of the scattering system has fundamental applications in H^∞ control theory.**

The *operator-valued* **transfer and scattering functions, analytic on the unit disk, and with H^∞ norm ≤ 1, are in the** *Schur class* $\mathcal{S} = \mathcal{S}(\mathbb{D})$.

Through the Cotlar-Sadosky *multi-evolution* **scattering systems, J. Ball, V. Vinnikov and C. Sadosky realized that we could deal also with** *multidimensional* **conservative ISO systems, and Schur-class functions in $\mathcal{S}(\mathbb{D}^n)$.**

For $n \geq 1$, to an n-dimensional conservative ISO system always corresponds an n-evolution orthogonal scattering system, with its scattering function equal to the transfer function of the ISO system.

Yet, when $n > 1$, an ISO system can be recovered from a scattering system only if there is a suitable *orthogonal decomposition* **of the internal scattering subspace.**

For $n = 1$ such a decomposition is trivial (and unique).

For $n = 2$ the orthogonal decompositions *always exist*; **in fact we construct explicitly two orthogonal decompositions which are extremal in a certain sense.**

For $n > 2$ the orthogonal decompositions do not necessarily exist (and when they exist, they need not be unique).

MAIN RESULT FOR $n = 2$:
(Ball-Sadosky-Vinnikov, 2005)

Every (minimal) 2-evolution scattering system admits a conservative ISO realization.

COROLLARY: Every Schur-class function on the bidisk admits a realization as the transfer function of a conservative 2D ISO system.

From this follows a new simple proof of

ANDÔ'S THEOREM ON LIFTINGS
OF COMMUTING CONTRACTIONS

–via the von Neumann inequality.

For $n = 2$**, a Schur-class function** $S \in \mathcal{S}(\mathbb{D}^2)$ **is an operator-valued analytic function on the unit bidisk, such that** $\|S\|_{H^\infty} \leq 1$.

If $S \in \mathcal{S}(\mathbb{D}^2)$ **is the** *transfer function* **of an** 2-**dimensional conservative discrete time ISO system, then the kernel** $I - S(z)S(w)^*$ **admits the**

AGLER DECOMPOSITION:

$I - S(z)S(w)^* =$

$$(1 - z_1 \overline{w_1})K_1(z, w) + (1 - z_2 \overline{w_2})K_2(z, w),$$

where $K_k(z, w), k = 1, 2,$ **are positive kernels on** $\mathbb{D}^2$**.**

By the above, any *transfer function* $S(z) = S(z_1, z_2)$ **satisfies the von Neumann inequality:**

$$\|S(T_1, T_2)\| \leq 1,$$

for all T_1, T_2**, commuting contractions on a Hilbert space.**

ALL FUNCTIONS $S \in \mathcal{S}(\mathbb{D}^2)$ SATISFY THE VON NEUMANN INEQUALITY

Sketch of our proof:

1. $S \in \mathcal{S}(\mathbb{D}^2)$

is equivalent to

2. S is the <u>scattering function</u> of a scattering system $\mathfrak{S}$ with two evolutions

implies (by our 2D theorem)

3. S is the <u>transfer function</u> of a bidimensional ISO system

implies (by the Agler decomposition)

4. S satisfies the von Neumann inequality

Now we can prove

5. Andô's Theorem:

ANY PAIR OF COMMUTING CONTRACTIONS HAS A JOINT UNITARY DILATION

If T_1, T_2 is a commuting pair of contraction operators on a Hilbert space $\mathcal{H}$, then there is a Hilbert space $\mathcal{K} \supset \mathcal{H}$, and a pair of commuting unitary operators $\mathcal{U}_1, \mathcal{U}_2$ on $\mathcal{K}$ such that, for any non-negative integers m, n,

$$T_1^m T_2^n = P_{\mathcal{H}} \, \mathcal{U}_1^m \mathcal{U}_2^n|_{\mathcal{H}}$$

Proof.

We view the commuting pair of contractions T_1 and T_2 as a representation π of the bidisk algebra $A(\mathbb{D}^2)$; the representation π is uniquely determined by

$$\pi(z_1) = T_1, \qquad \pi(z_2) = T_2.$$

We seek a $*$-representation ρ of the C^*-algebra $C(\mathbb{T}^2)$ with representation space $\mathcal{K} \supset \mathcal{H}$ which dilates π in the sense that

$$\pi(f) = P_{\mathcal{H}}\rho(f)|_{\mathcal{H}}, \; \forall f \in A(\mathbb{D}^2).$$

This holds if and only if π is completely contractive (Arveson, *Acta Mathematica*, 123, 1969).

The complete contractivity of π follows from the validity of the von Neumann inequality for every $S \in \mathcal{S}_2(\mathbb{C}, \mathbb{C})$, which can be realized as a transfer function. Since ρ is a $*$-representation, $\mathcal{U}_1 := \rho(z_1)$ and $\mathcal{U}_2 := \rho(z_2)$ are unitary operators on $\mathcal{K}$. The result now follows by restricting f to the monomials $f(z_1, z_2) = z_1^m z_2^n$.