Department of Mathematics
Faculty of Mechanical Engineering
Slovak University of Technology in Bratislava

3rd International Conference
APLIMAT 2004

PLENARY LECTURE

ON THREE CLASSES OF REGULAR TOROIDS

SZILASSI Lajos (HU)

Abstract. As it is known, in a regular polyhedron every face has the same number of edges and every vertex has the same number of edges, as well. A polyhedron is called **topologically regular** if further conditions (e.g. on the angle of the faces or the edges) are not imposed. An ordinary polyhedron is called a **toroid** if it is topologically torus-like (i.e. it can be converted to a torus by continuous deformation), and its faces are simple polygons. A toroid is said to be **regular** if it is topologically regular. It is easy to see, that the regular toroids can be classified into three classes, according to the number of edges of a vertex and of a face. There are infinitelly many regular toroids in each classes, because the number of the faces and vertices can be arbitrarilly large. Hence, we study mainly those regular toroids, whose number of faces or vertices is minimal, or that ones, which have any other special properties. Among these polyhedra, we take special attention to the so called „Császár-polyhedron”, which has no diagonal, i.e. each pair of vertices are neighbouring, and its dual polyhedron (in topological sense) the so called „Szilassi-polyhedron”, whose each pair of faces are neighbouring. The first one was found by Ákos Császár in 1949, and the latter one was found by the writer of this paper, in 1977.

Keywords. regular, torus-like polyhedron

As it is known, in regular polyhedra, the same number of edges meet at each vertex, and each face has the same number of edges. A polyhedron is **topologically regular** if no further conditions (e.g. on the angles of the faces or the edges) are imposed. In this article we shall deal with such polyhedra, but we must first consider some concepts and simple facts concerning polyhedra.

We will assume that the edges of the polyhedra are to be straight line segments and the faces are to be planar.

A polyhedron is called **ordinary,** if any two points of the polyhedral body can be joined by a broken line not intersecting the surface of the polyhedron, and at each vertex, the faces containing that vertex form a cycle such that the adjacent members of the cycle are the adjacent faces (i.e. the faces with a common edge). In an ordinary polyhedron, each edge borders exactly two faces.

A polyhedron is **simple,** if it is ordinary, topologically sphere-like (i.e. it can be transformed to a sphere by continuous deformation), and its faces are simple polygons. (The simple polygon is topologically circle-like.) For example, convex polyhedra are simple, but not all simple polyhedra are convex. For simple polyhedra Euler's formula

$$V - E + F = 2$$

holds, where V, E and F are the numbers of vertices, edges and faces, respectively.

Application of this relationship readily shows that there are only five topologically regular, simple polyhedra, and each of them can be realized so that its faces and solid angles are regular and congruent, but it does not follow from Euler's formula.

An ordinary polyhedron is called a **toroid,** if it is topologically torus-like (i.e. it can be transformed to a torus by continuous deformation), and its faces are simple polygons. For toroids Euler's formula is modified to

$$V - E + F = 0 \,.$$

(We could have defined toroids more generally as ordinary, but not simple polyhedra, the surface of which is connected. This generalization will not be needed here.)

A toroid, in the stricter sense, is said to be **regular,** if the same number of edges meet at each vertex, and each face has the same number of edges. Studying toroids, we cannot expect all of the faces or solid angles to be regular and congruent, so the description *regular* here is clearly a topological property.

Let us consider some of the more interesting regular toroids. Several of the toroids mentioned here have been discovered by the author of this paper, including the one named after him, discovered in 1977 [7].

1. Assume each face of a regular toroid has a edges, and at each vertex meets exactly b edges. Both products $F * a$ and $V*b$ are equal to twice the number of edges, since every edge is incident with two faces and two vertices. Hence, from Euler's formula for toroids above,

$$\frac{2E}{a} + \frac{2E}{b} - E = 0 \ .$$

Since $E > 0$, this leads to the Diophantine equation

$$\frac{1}{a} + \frac{1}{b} - \frac{1}{2} = 0 \ .$$

This equation has only three integral solutions satisfying the conditions $a \geq 3$ and $b \geq 3$. Hence, we can distinguish three classes of regular toroids, according to the numbers of edges incident with each face and each vertex:

class $\mathbf{T}_1$: $a=3, b=6;$
class $\mathbf{T}_2$: $a=4, b=4;$
class $\mathbf{T}_3$: $a=6, b=3.$

As it is known, there are only three ways of tiling the plane with regular polygons; namely, with regular triangles, squares and regular hexagons. (Every edge must border exactly two faces; if this condition is omitted the tilings with triangles and squares are not unique.)

These three tilings correspond topologically to the three classes above. If a *sufficiently large* rectangle is taken from such a tiled plane, and the opposite edges are glued together, we obtain a map drawn on a torus which is topologically regular. (Two opposite edges of the rectangle are glued together first, to form a cylindrical tube, and then the remaining two, now circular edges are glued together, yielding a torus.) If the resulting regular map drawn on the torus has sufficiently many regions, there is no obstacle in principle to its realization with plane surfaces. We may say, therefore, that each of the three classes contains an infinite number of regular toroids. However, it is interesting to determine for each class the lowest number of faces or vertices required to construct a regular toroid in that class, possibly with the restricting condition that the faces or solid angles belong to as few congruence classes as possible.

2. With the use of sufficiently high number of triangles we can easily construct toroid belonging to class $\mathbf{T_1}$. (**Figure 1.**) It would be appropriate here to formulate the following problem; at least how many triangles are necessary to construct a toroid?

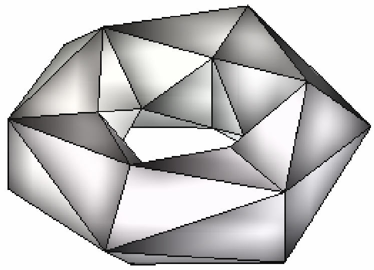


Figure 1.[1] *A toroid with 48 triangles in class* $\boldsymbol{T_1}$

At every vertex of a regular toroid in class $\mathbf{T_1}$ exactly six edges meet, so that such a toroid has at least seven vertices.

The Császár-polyhedron [1], [2], [6] (pp. 244-246) is such a toroid with only seven vertices. This polyhedron does indeed belong to class $\mathbf{T_1}$ for any two of its vertices are joined by an edge, and thus six edges meet at each vertex. The number of its vertices is the lowest possible not only in class $\mathbf{T_1}$, it can easily be seen that a toroid with less than seven vertices does not exist.

The model of the toroid, denoted by C_0 , which is constructed on the basis of the data published by professor Ákos Császár at Budapest University who is a member of the Hungarian Academy of Sciences [2], appears fairly crowded.

It has a dihedral angle which is greater than 352°. We have prepared a computer program to search for a less crowded version. Given the rectangular coordinates of the seven vertices, the program first checks whether the polyhedron defined by the seven points intersects itself. If it does not, the program then calculates the lengths of the edges, the interedge angles and the dihedral angles of the polyhedron. **Tables 2** and **3** show the data of five variants of the polyhedron. Variant C_1, can be obtained from C_0 by a slight modification of the coordinates of the vertices, whereas C_1 , C_2 , C_3 and C_4 is essentially different from each other.

Let us consider two models of Császár-polyhedron **essentially congruent** if:

- one of the polyhedra can be transferred into the other by gradually changing the coordinates of one of the polyhedra without causing self intersection in the surface;
- one of the polyhedra can be transferred into the other by reflection;
- one of the polyhedra can be transferred into the other by executing the above operations one after another;

otherwise the two Császár-polyhedra are **essentially different**.

[1] Most of the attached pictures were made with the software MAPLE. Then the pictures were made somewhat more plastic with the use of the software Corel PHOTO-PAINT.

Let us try to find a variant among the essentially congruent polyhedra which is not crowded too much and is aesthetically pleasing, and on the other hand, let us try to find two essentially different versions.

J. Bokowski and A. Eggert proved in 1986 that the Császár-polyhedron has only four essentially different versions.[1]

It is to be noted that in topological terms the various versions of Császár-polyhedron are isomorphic, there is only one way to draw the full graph with seven vertices on the torus. (**Figure 2**) The vertices of the polyhedra are marked accordingly. In this way the faces of the polyhedron can be identified with the same triplets of numbers.

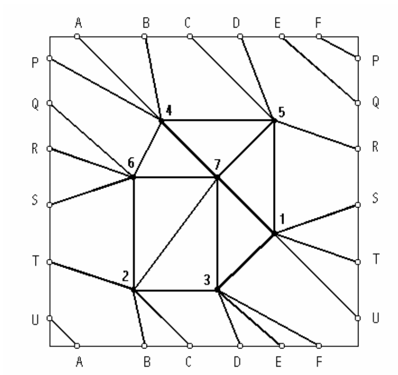


Figure 2. *The full graph with seven vertices which can be drawn on the torus*

(1-2-6)	(1- 4 -2)	(5-3-2)	(4-1-3)	(2-7-6)	(3-7-2)	(1-7-3)
(6-5-1)	(6- 3-5)	(2-4-5)	(3-6-4)	(5-7-1)	(4-7-5)	(6-7-4)

Table 1. *Faces of the Császár-polyhedron*

It may be observed that in all variants vertices *1* and *6*, 2 and 5 , 3 and 4, are reflected images of each other relative to the *z* axis of the coordinate system; accordingly, the pairs of faces written one under the other in the last two lines of **Table 1** are congruent. The solid angles corresponding to the foregoing vertex pairs are also congruent. Therefore, in all four variants the faces belong to seven congruence classes, and the solid angles to four congruence classes. For this reason the triangles shown one below another in **Table 1** are congruent.

The pictures of the four versions are shown below, together with the nets suitable for producing the models.

One variant is shown below for each of the four versions, which are (in our opinion) not crowded too much. **Table 2** includes the coordinates of the vertices, while **Table 3** shows the edge length values and the face angles belonging to the edges.

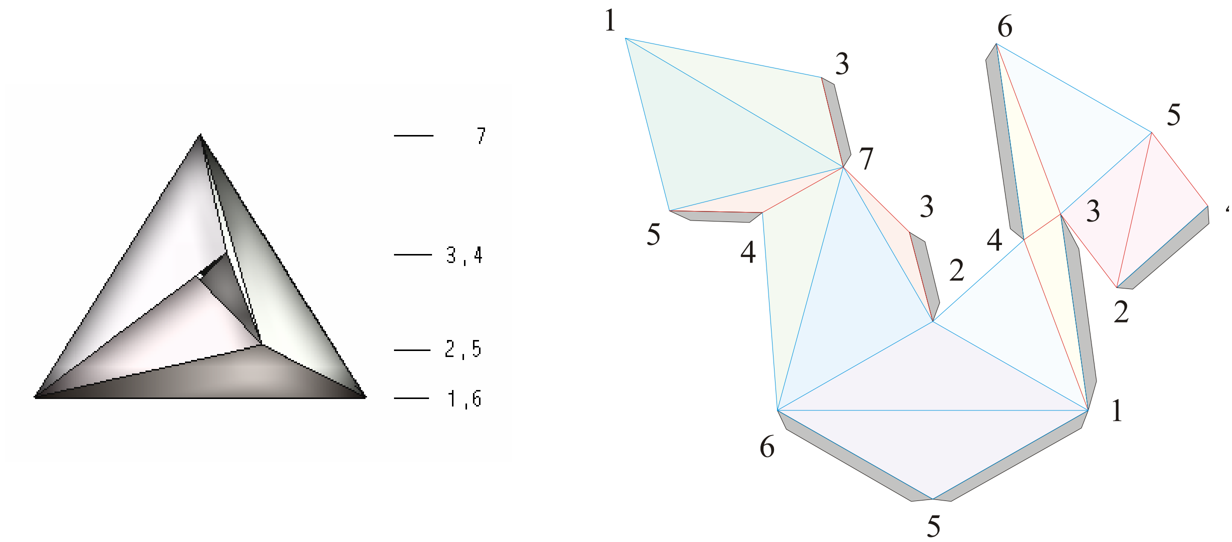


Figure 3. *Variant C_1 of the Császár-polyhedron*

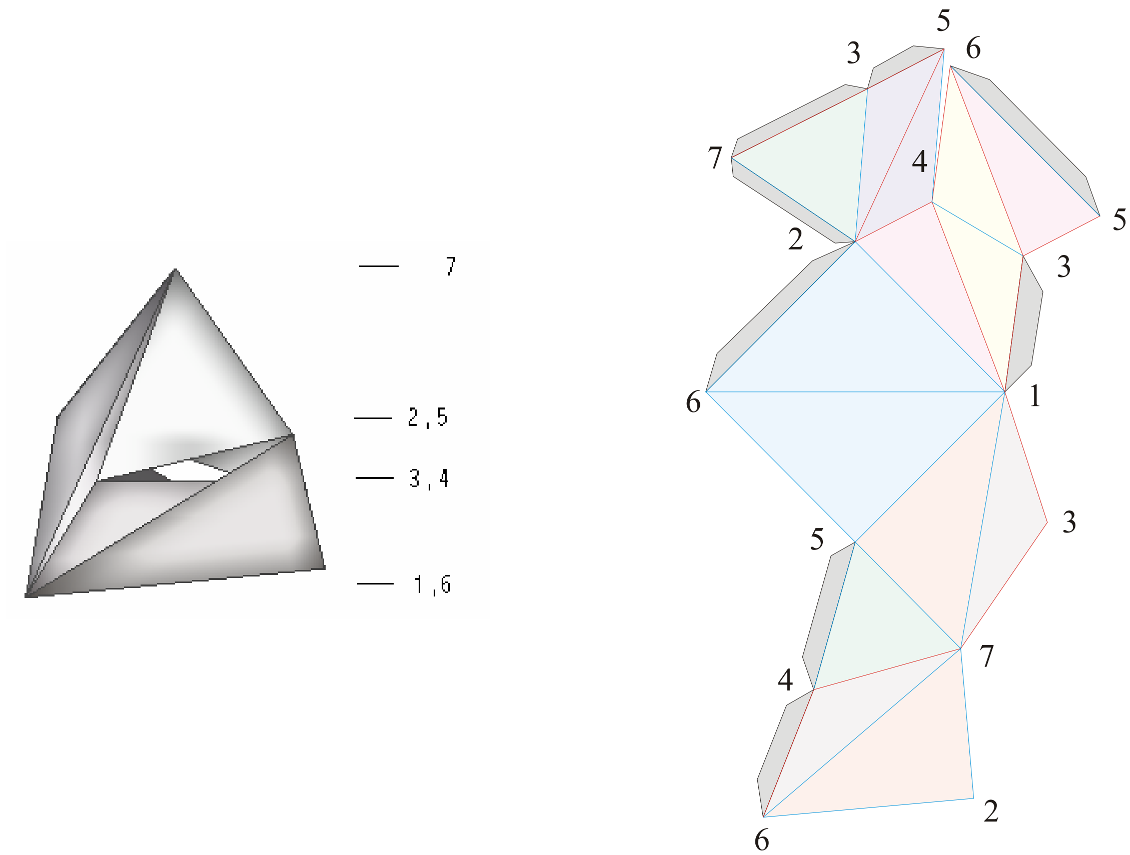


Figure 4. *Variant C_2 of the Császár-polyhedron*

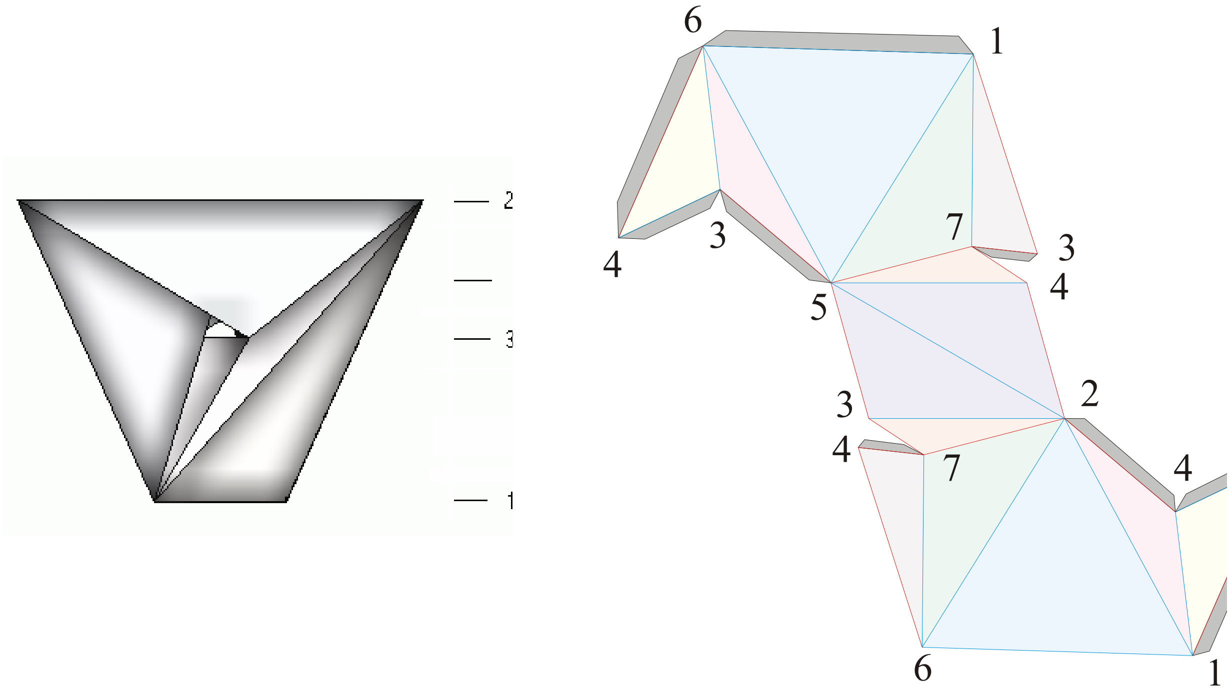


Figure 5. *Variant C_3 of the Császár-polyhedron*

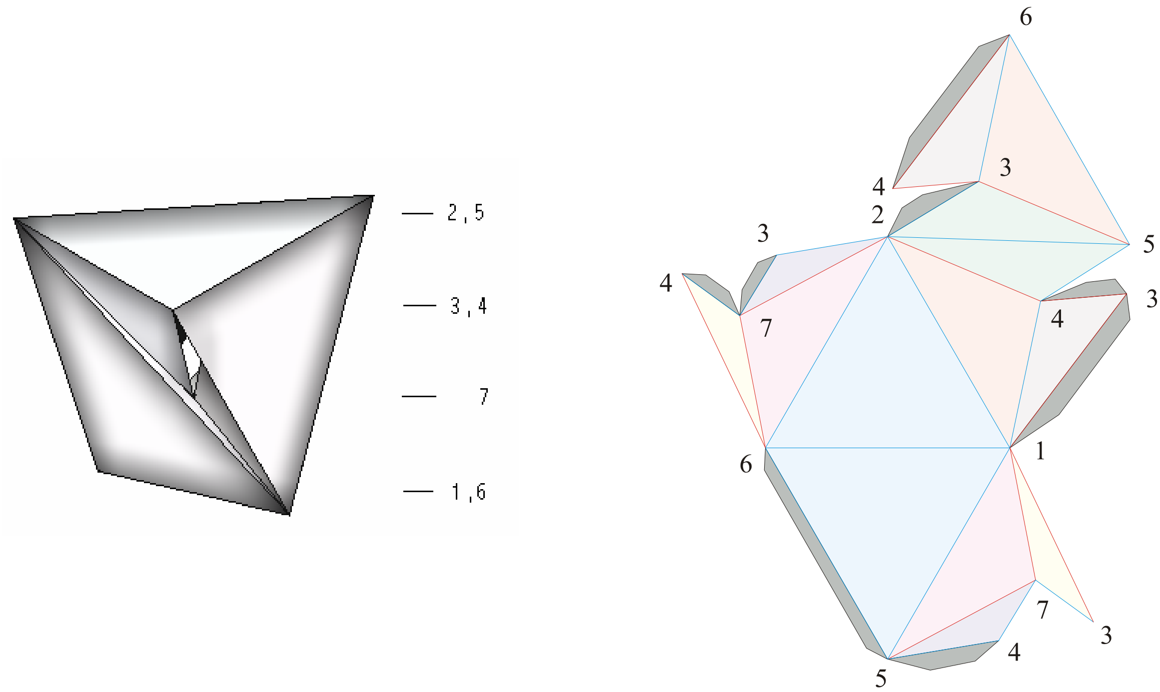


Figure 6. *Variant C_4 of the Császár-polyhedron*

	C_0			C_1			C_2			C_3			C_4		
Vertices	**x**	**y**	**z**	**x**	**y**	**z**	**x**	**y**	**z**	**x**	**y**	**z**	**x**	**y**	**z**
1.	3	-3	0	$4\sqrt{15}$	0	0	12	0	0	12	0	0	12	0	0
2.	3	3	1	0	8	4	0	$6\sqrt{2}$	$6\sqrt{2}$	0	12	$12\sqrt{2}$	0	12	$12\sqrt{2}$
3.	1	2	3	-1	2	11	3	-3	$6\sqrt{2}-3$	-4	-3	$\frac{13\sqrt{2}}{2}$	-3	3	$8\sqrt{2}$
4.	-1	-2	3	1	-2	11	-3	3	$6\sqrt{2}-3$	4	3	$\frac{13\sqrt{2}}{2}$	3	-3	$8\sqrt{2}$
5.	-3	-3	1	0	-8	4	0	$-6\sqrt{2}$	$6\sqrt{2}$	0	-12	$12\sqrt{2}$	0	-12	$12\sqrt{2}$
6.	-3	3	0	$-4\sqrt{15}$	0	0	-12	0	0	-12	0	0	-12	0	0
7.	0	0	15	0	0	20	0	0	$12\sqrt{2}$	0	0	$\frac{26\sqrt{2}}{3}$	0	0	$4\sqrt{2}$

Table 2. *Coordinates for the four variants of Császár-polyhedron*

	C_0		C_1		C_2		C_3		C_4	
Edges	**d**	**f**	**d**	**f**	**d**	**f**	**d**	**f**	**d**	**f**
(1-6)	8.5	153°	31.0	127°	24	90°	24	71°	24	71°
(2-5)	8.5	321°	16.0	344°	16.9	270°	24	54°	24	56°
(3-4)	4.5	253°	4.5	257°	8.5	114°	10	76°	8.5	286°
(2-4)=(5-3)	6.7	78°	12.3	69°	6.9	296°	12.6	204°	16.3	191°
(2-3)=(5-4)	3.0	216°	9.3	209°	12.2	35°	17.4	42°	11.0	103°
(3-7)=(4-7)	12.2	269°	9.3	279°	12.2	291°	5.9	244°	7.1	22°
(2-7)=(5-7)	14.6	18°	17.9	36°	12	61°	12.9	340°	16.5	307°
(1-5)=(6-2)	6.1	87°	17.9	90°	16.9	90°	24	53°	24	22°
(1-2)=(6-5)	6.1	44°	17.9	67°	16.9	15°	24	51°	24	66°
(1-4)=(6-3)	5.1	352°	18.3	343°	16.2	237°	12.6	157°	14.8	39°
(1-3)=(6-4)	6.2	58°	19.9	57°	11.0	279°	18.7	339°	19.0	272°
(1-7)=(6-7)	15.3	76°	25.3	57°	20.8	24°	17.1	74°	13.3	272°

Table 3. *Data for the four variants of the Császár-polyhedron*
d: *Edge length*
f: *Face angles belonging to edges*

3. Class $\mathbf{T_2}$ of regular toroids consists of those torus-like ordinary polyhedra in which four edges meet at each vertex and the faces are quadrilaterals. (**Figure 7.**) This type of regular toroids is the easiest to construct.

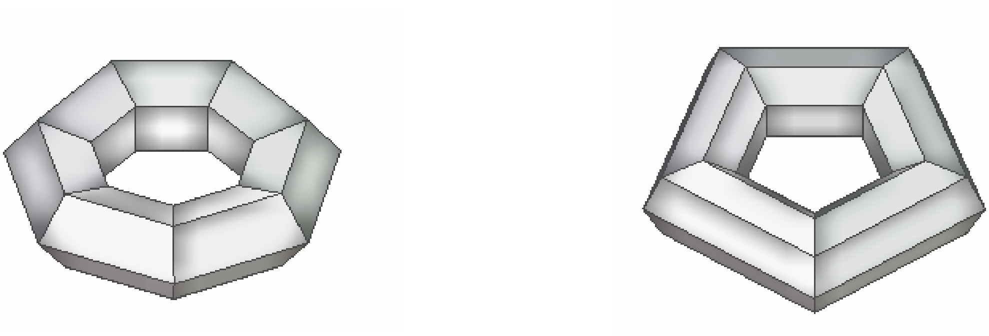


Figure 7. *Two toroids with 35 faces in class* $\boldsymbol{T_2}$

Let us take an (e.g. regular) *p*-sided polygon, and rotate it by *(k/q)* 2π where q is an integer $\gamma \geq 3$, and $k = l, 2, ..., q$, about a straight line *t* which lies in the plane of the polygon, but does not intersect it. The resulting toroid, which consists of $p \, . \, q$ trapezia (or rectangles), is regular and belongs to class $\boldsymbol{T_2}$. As an example, for $p = q = 3$ the toroid in **Figure 8** is obtained. This is the member with the lowest number of faces in class $\boldsymbol{T_2}$ since every toroid of type $\boldsymbol{T_2}$ has at least nine vertices (and nine faces). (Each vertex is incident with four faces which altogether have a total of nine vertices; for ordinary polyhedra any two of these nine vertices must be distinct.)

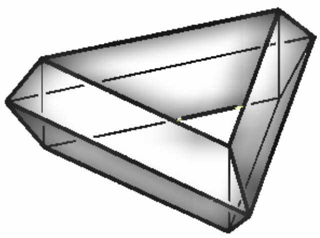


Figure 8. *A toroid with minimal faces in class* $\boldsymbol{T_2}$

In the case $p = 3, q = 4$ (or $p = 4, q = 3$) the above procedure yields a regular toroid in $\boldsymbol{T_2}$**,** with *12* faces and *12* vertices. However, it is not known whether there exist a regular toroid in T_2 with *10* or *11* faces (vertices), though a graph having *10* or *11* vertices (and regions) and which belongs to class $\boldsymbol{T_2}$ **,** can be drawn on torus. (**Figure 9.**) If so it would have to be obtained by a different method, since this one requires that F be a product of two integers each ≥ 3.

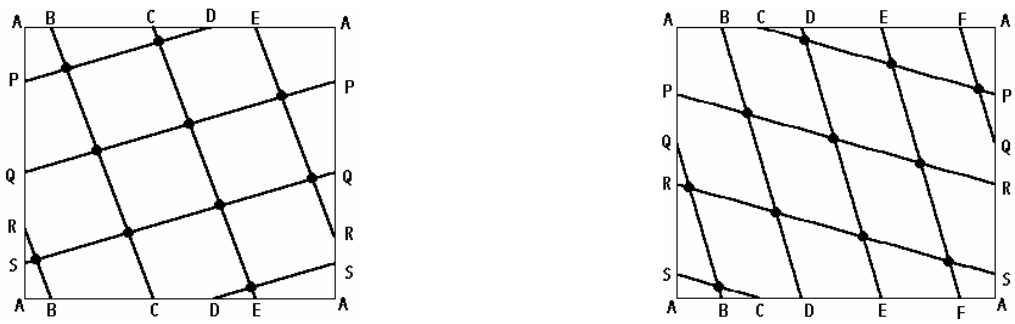


Figure 9. *A graph belonging to class* $\boldsymbol{T_2}$ *drawn on a torus. Vertices 10 and 11have the same number of regions.*

4. If constructing toroids belonging to class $\boldsymbol{T_3}$**,** care should be taken to make sure that the six points defining one region (face) are in the same plane. This requirement can be easily met, if the toroid has "sufficiently high number" of faces. (**Figure 10.**)

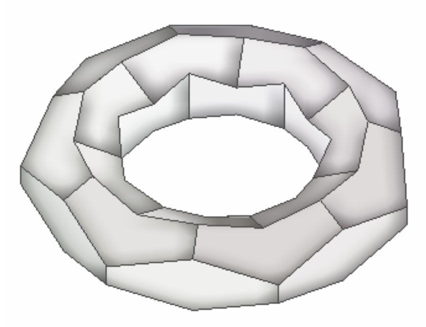
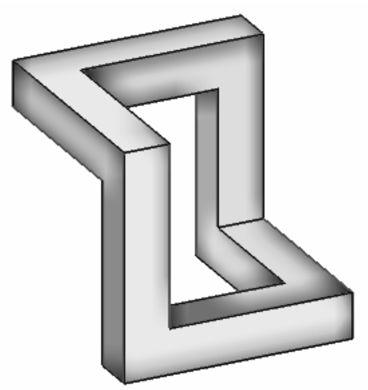


Figure 10. *A toroid with 42 faces in class* $\boldsymbol{T_3}$

It can be seen that any toroid of Class $\boldsymbol{T_3}$ has a concave face, and even all the faces could be concave. Such toroid is shown in **Figure 11**, which consists of 12 L-shaped hexagons, i.e. two pairs of 6 congruent hexagons. Yet another unique feature is that any face is perpendicular to the adjacent faces, and the angle of the meeting faces and the angle of the polygons have only two values in terms of congruency.

A somewhat more complicated toroid is shown below, which also belongs to Class $\boldsymbol{T_3}$, and consists of 24 **L**-shaped hexagons. In terms of congruency, its faces are divided into four groups (**Figure 12**).



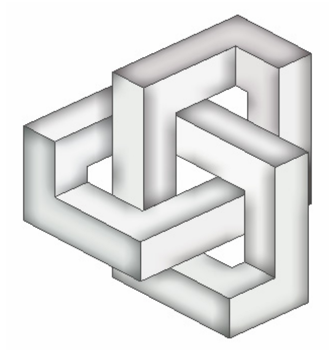
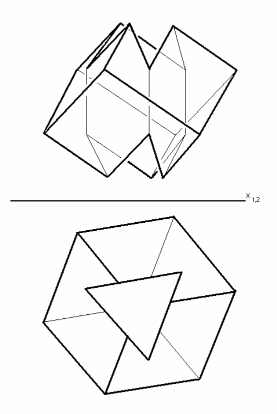


Figure 11. 12. *Toroids of class* $\boldsymbol{T_3}$ *consisting of* **L**-*shaped hexagons*

We shall prove that there exists in class $\boldsymbol{T_3}$ a regular toroid with nine faces such that its faces and solid angles belong to two, respectively three, congruence classes.

Let us project a cube perpendicularly to a plane π perpendicular to one of its internal diagonals selected in advance. The resulting projection is a regular hexagon, therefore the projections of any two skew facial diagonals parallel to π, that are incident with adjacent faces of the cube, trisect each other. This means that the line segments joining the corresponding points of trisection of the facial diagonals in question are parallel to the selected internal diagonal.



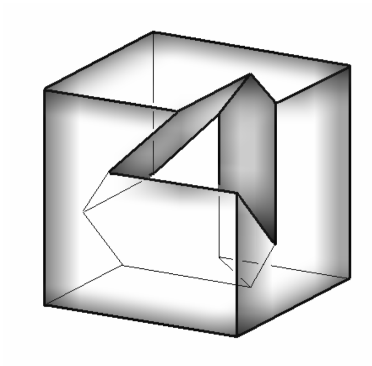


Figure 13. , 14. *A toroid with nine faces which are of two kinds concerning their congruency*

Utilizing this we can pierce the cube with a triangular prism, whose edges are parallel to the diagonal of the cube and pass through the points of trisection of the pairs of skew facial diagonals of the cube (**Figure 13**).

The *external* part of the resulting toroid surface consists of the six mutually congruent concave hexagons remaining from the faces of the cube, while its *internal* part is formed by the three, mutually congruent convex hexagons arising during the penetration **(Figure 14**).

Let us now establish the dual of this polyhedron, which would form a regular octahedron from the originally assumed cube (**Figure 15**). This can be accomplished with a spatial polarity referring to a sphere, where the centre of the sphere coincides with that of the cube. Because the original shape is centrally symmetric, the dual is also centrally symmetric. In this way we obtain a toroid of Class T_1, which has two kinds of solid angles and three kinds of polygon angles in terms of congruency. The six triangles forming the "inner part" of the toroid are regular, because these triangles were obtained from the regular corners of the cube.

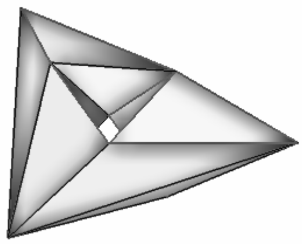


Figure 15. *A toroid belonging to Class T_1 having two kinds of solid angles in terms of congruency*

5. We have just created a regular toroid consisting of nine faces only (**Figure 14**). One might wonder; is it possible to create a (regular) toroid from even less faces?

As we have seen, the most important property of the Császár-polyhedron is that any two vertices are joined by an edge. There is a very close relationship, so-called duality, between this polyhedron and the polyhedron with the lowest number of faces in class T_3**,** the main characteristic of the latter being that any two faces have a common edge (**Figure 16**).) This relationship is partly topological, however, if we create a new polyhedron by means of projective transformation from a given polyhedron, e.g. with the use of polarity referring to a sphere (as we did above), then the new polyhedron will be metric as well.

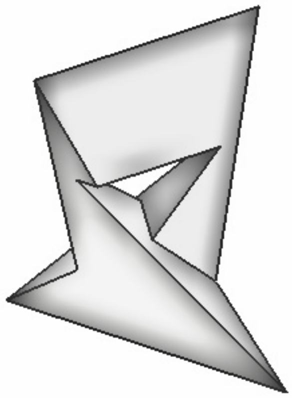


Figure 16. *Toroid with seven faces topologically isomorphic with Heawood's seven colors toroidal map*[2]

The data of the faces and the network needed to fit them together are given in **Figure 17**, based on the drawings of Stewart [5] (pp. 248-249).

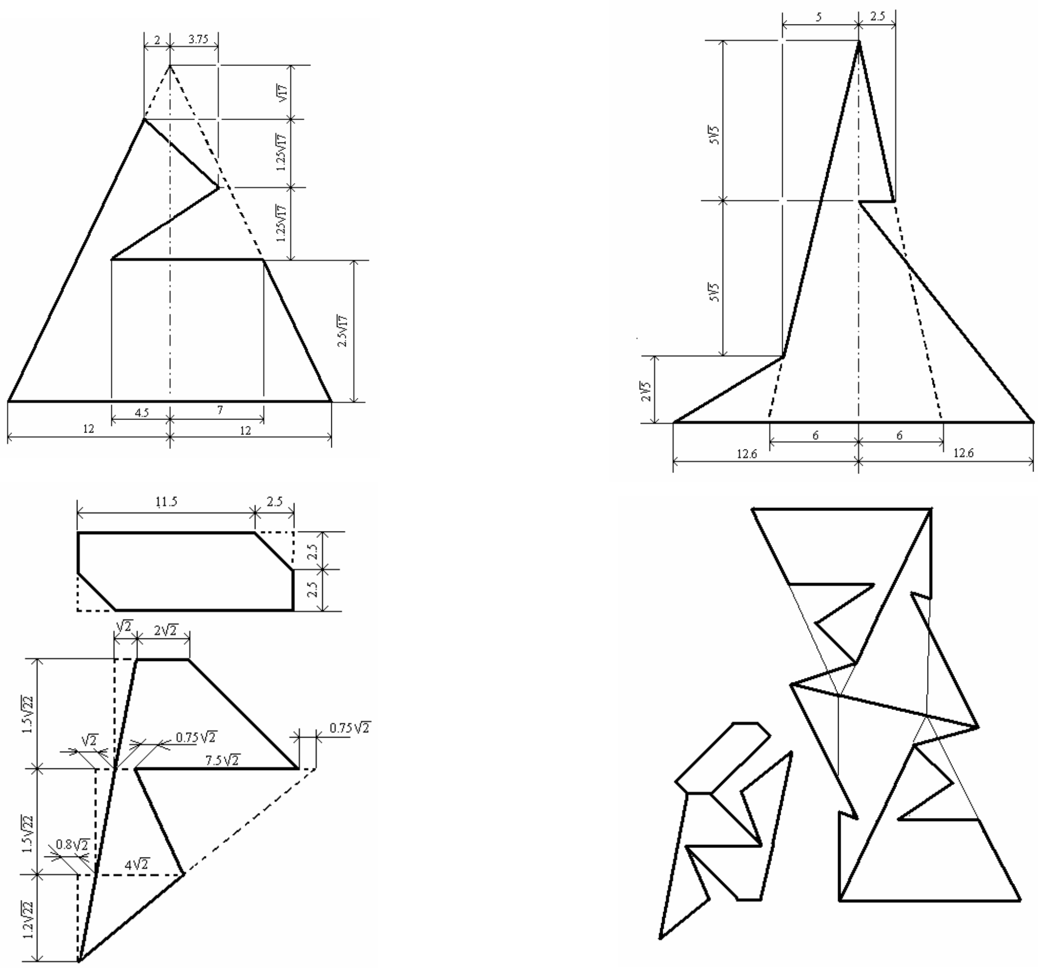


Figure 17. *Data for making seven faced toroid*

[2] It was Martin Gardner who used the term "Szilassi polyhedron" first to identify this polyhedron [3]

The author of this paper constructed this seven faced polyhedron in 1977 after producing the dual of the Császár-polyhedron using the spherical polarity of a sphere. In this way, however, the structure obtained consisted of self-intersecting polygons. A computer assisted analysis had to be used to find the undesirable intersections, as a result of which the data could be modified to obtain the above (in **Figure 17.**) polyhedron bordered by simple polygons.

Once one model of the structure is known, it is easy to develop a straightforward method to construct the polyhedron. This will be described briefly below.

Consider a tetrahedron, which has an axis of symmetry. Assume, that this axis is aligned with the *z* axis of the coordinate system. The structure will be established in a way that this axial symmetry will be maintained. (**Figure 18, 19.**)

One pierces the tetrahedron with a triangular prism, the edges of which are parallel with the *(xy)* plane of the coordinate system, and let one of its edge be aligned with two opposite edges of the tetrahedron, and let the other two edges be aligned with the axis of symmetry of two faces.

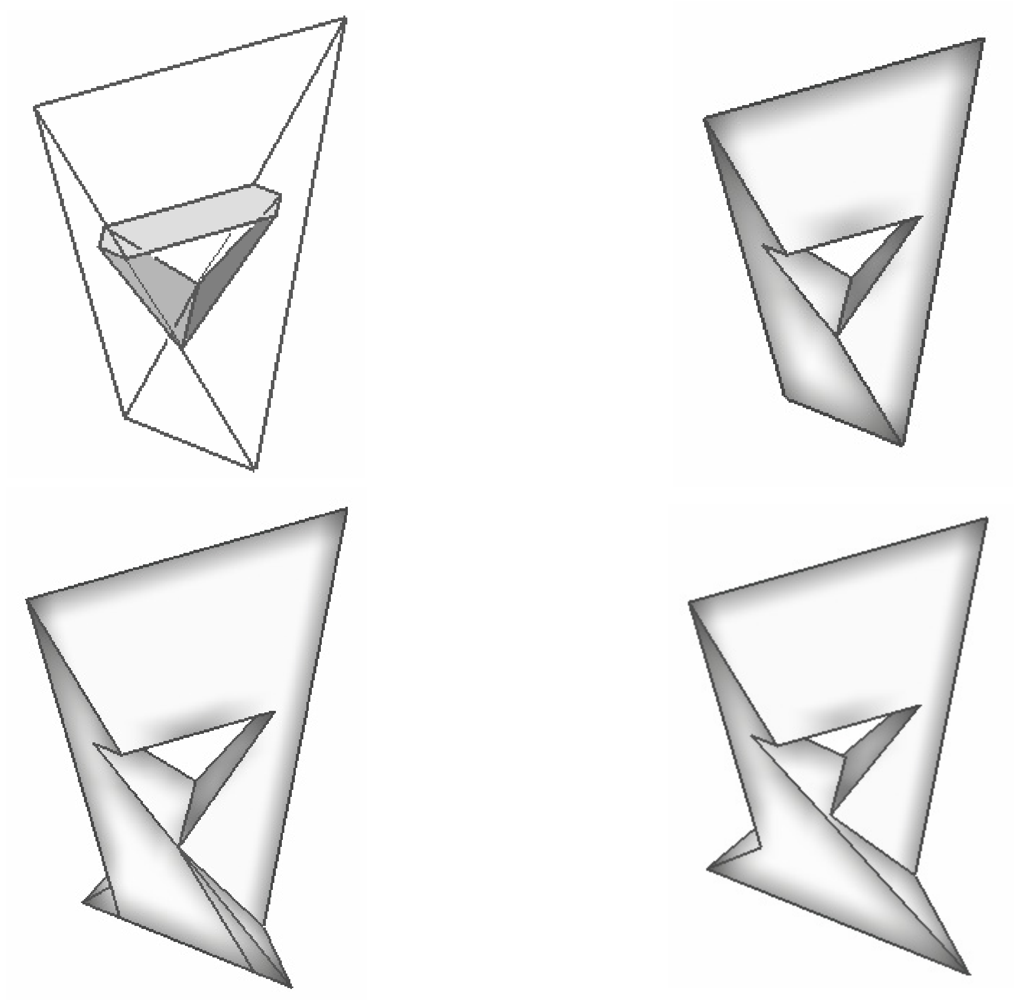


Figure 18., 19., 20., 21. *Deduction of the seven faced toroid*

Two quadrangles of the toroid thus obtained are not yet adjacent to two-two faces of the tetrahedron. For this reason we may complement the structure with two smaller tetrahedrons, the face planes of which coincide with the planes of the faces already obtained. In the polyhedron thus produced any two faces are adjacent, however, two faces are not simple hexagons; one of the vertices of each of these hexagons is on the opposite edge (**Figure 20**).

We can eliminate this undesirable coincidence by shifting the edge of the piercing prism (i.e. the edge which intersected two edges of the tetrahedron) closer to the opposite face (**Figure 21**). Now we obtained the ordinary polyhedron we have been looking for.

Starting from the same tetrahedron now we can produce more or less airy (harmonic) variants of the obtained toroid.

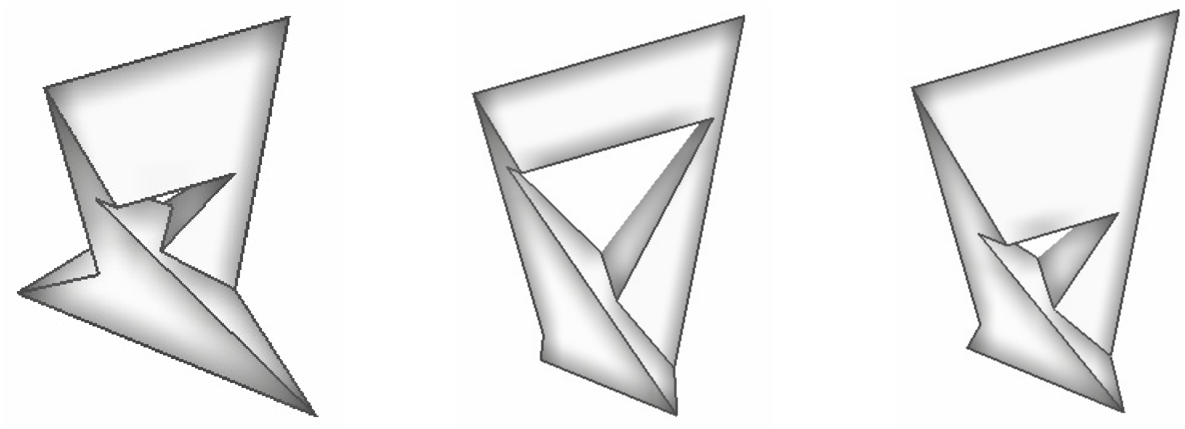


Figure 22.. , 23.. , 24.. *Several variants of the seven faced toroid*

These seven faced toroids are not regarded as essentially different, even if they can be transformed into each other only by reflecting in a plane. The author does not know whether this polyhedron has an <u>essentially different</u> version (similarly to the Császár-polyhedron). Any variant of this polyhedron, too, is symmetrical about the *z* axis of the coordinate system. Thus, its faces belong to four congruence classes, and its vertices into seven congruence classes.

It can be seen that a toroid cannot be constructed with fewer than seven facial planes, hence this polyhedron has the lowest number of faces in any toroid, not only in class $\boldsymbol{T_3}$ **.**

6. Various problems arose in the last two centuries in connection with the colouring of maps, e.g. the four-colour problem. In 1890 it was proved by Heawood that seven colours are sufficient for colouring any map drawn on a torus. At the same time he also showed that seven colours are necessary too, by drawing on the torus a map consisting of seven regions, any two of which were adjacent, so that in order to get the required colouring each region had to be given a different colour. By the polyhedron described here Heawood's seven-colour map can be constructed from seven simple planar hexagons, that is Heawood's toroidal map topologically isomorphic to this toroid.

Heawood's seven-colour map can also be realized with a toroid each face of which is a regular polygon. (In this case, of course, the regions no longer consist of a single face.) The network of this construction (due to Stewart [6], p. 199), together with the separate *interior* and *exterior* halves of the toroid, may be seen in **Figure 25**. The numbers in the polygons denote the colours. Examination of **Figure 25** or of a model prepared on this basis confirms that every colour is in fact adjacent to all of the other colors.

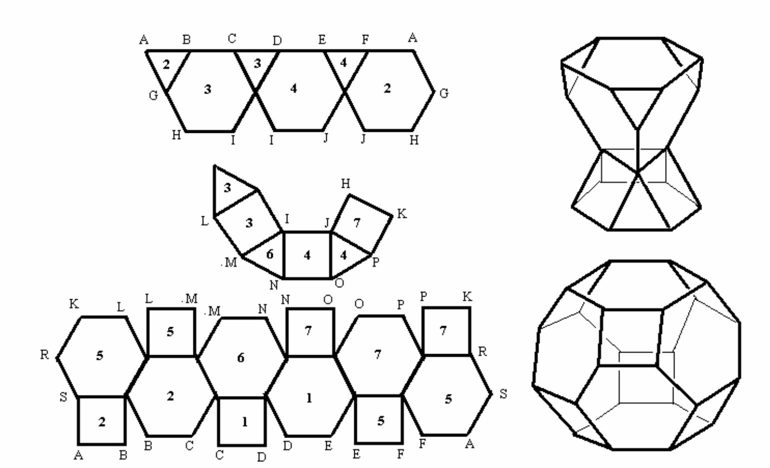


Figure 25. *Realization of Heawood's map with a toroid having regular polygons*

A similarly interesting construction is the toroid in which the regions are not only adjacent, but also congruent. Each such region consists of four triangles which are congruent in pairs. For its construction, let us consider the regular heptagon $\boldsymbol{A_1, A_2, A_3, \dots A_7}$.

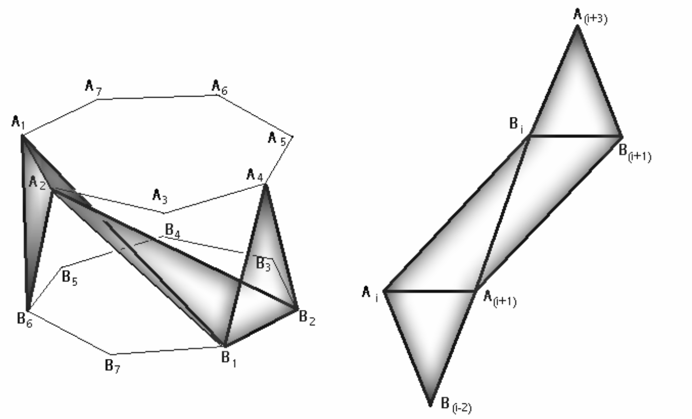


Figure 26. *Realization of Heawood's map with toroid constructed of seven regions any of which is congruent and neighbouring with each other. Every region consists of four triangles.*

We rotate it by the angle $\frac{5}{2}\frac{2\pi}{7}$ around its centre, and then shift it in the direction perpendicular to its plane. In this way we obtain the regular heptagon B_1, B_2, B_3, ...B_7. For each i = 1, 2, ..., 7, the figure consisting of the triangles $A_iA_{i+1}B_{i-2}\triangle$, $A_iB_iA_{i+1}\triangle$, $B_iB_{i+1}A_{i+1}\triangle$ and $B_iA_{i+3}B_{i+1}\triangle$ is coloured one colour and is considered as one region (**Figure 26**). (If some index does not lie between 1 and 7, 7 is either added to it or subtracted from it so as to yield a number between I and 7, i.e. indices are taken (modulo 7)+1.)

If the i-th region is rotated by the angle $\frac{2\pi}{7}$ around the axis joining the centres of the two regular heptagons, we obtain the $(i + 1)$-th region. These regions are therefore indeed congruent and altogether form a toroid. Examining the indices of the edges bordering the regions one can see that each of them is indeed adjacent to all of the others. For example, the neighbour of the i-th region along the edge $\overline{A_iB_i}$ is the *(i* + 1)-th region, and its neighbour along the edge $\overline{A_iB_{i-2}}$ is the (*i*-3)-th region.

For the construction of the polyhedron, we may arbitrarily fix the distance of the planes of the two regular heptagons or, for example, the sides of the isosceles triangle $A_iA_{i+1}B_{i-2}\triangle$. From these, the other data may be calculated. **Table 4** provides the data of three variants of the toroid, differing in their edge lengths.

Edges i=1,2,…7	V_1		V_2		V_3	
	d	**F**	**d**	**F**	**d**	**f**
$\overline{A_iA_{i+1}}$	6	64° 1'	6	51°45'	6	43°21'
$\overline{A_iB_{i-2}}$	6	150°13'	8	152°13'	10	153° 1'
$\overline{A_iB_i}$	13.48	51°12'	14.48	65°11'	13.48	74°33'
$\overline{A_{i+1}B_i}$	10.04	332°13'	11.35	325°13'	10.04	320°43'

Table 4. *The toroid with seven congruent, pairwishe adjacent regions*
d*: Edge length*
f*: Face angles belonging to edges*

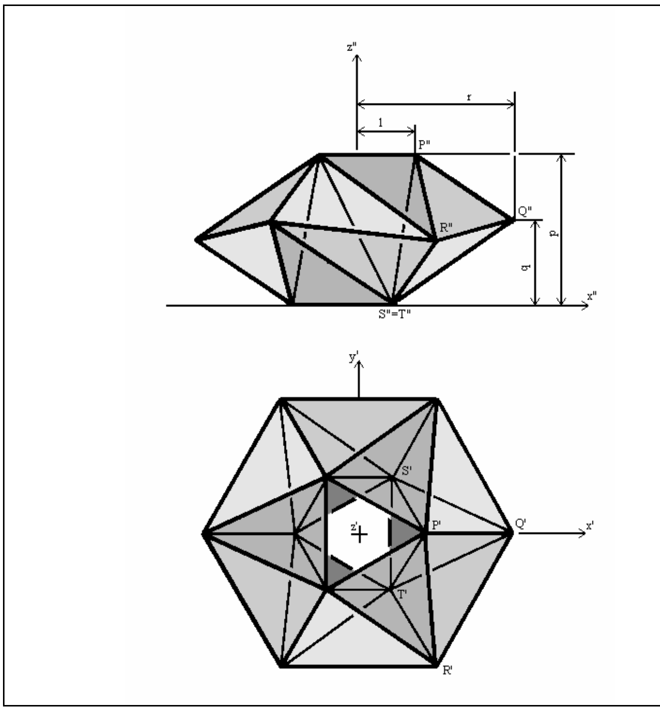
This polyhedron is a regular toroid in class $\mathbf{T_1}$ its faces belong to two congruence classes and its solid angles to one congruence class, i.e. they are congruent. It has 7·4 = 28 faces.

A toroid of the same kind (i.e. with congruent solid angles and with two types of faces) in class $\mathbf{T_2}$ having fewer faces and vertices can also be obtained if we set out this construction from a regular hexagon instead of a heptagon. This has *12* vertices and only *24* faces. A regular toroid with an even smaller number of faces cannot be obtained in this way, because, for instance, for a regular pentagon all of the edges $\boldsymbol{A_iB_i}$**,** meet in one point, the centre of symmetry of the figure.

7. What is the smallest number of faces needed to construct a toroid consisting purely of congruent faces? Stewart [6] (p. 250) constructed a regular toroid in class $\boldsymbol{T_1}$ consisting of 36 congruent isosceles triangles.

We shall show that there exists a toroid with a total of 24 faces that are congruent isosceles triangles.

For the construction we take a regular triangle, rotate it by *60°* around its centre, and then shift it by a distance *p* (to be determined later) in the direction perpendicular to its plane. The resulting six points define six isosceles triangles with base *a* and side *b.* These form the *interior* part of the toroid. The *exterior* part of the toroid surface will be constructed from isosceles triangles congruent with the previous ones. For this, six new vertices must be taken so that their distance from each other is *b* and from the previous vertices *a* or *b,* respectively. The six vertices are situated so that three of them are at a distance g from the plane of the lower regular triangle we started with, while the three between them are at a distance q from the plane of the upper regular triangle. **Figure 27** presents a two-imageplane picture of the construction. The perpendicular projection of the outer six points to the first image plane is a regular hexagon. Let r be the radius of its circumscribed circle, while the radius of the circumscribed circle of the regular triangle we started with is the unity.



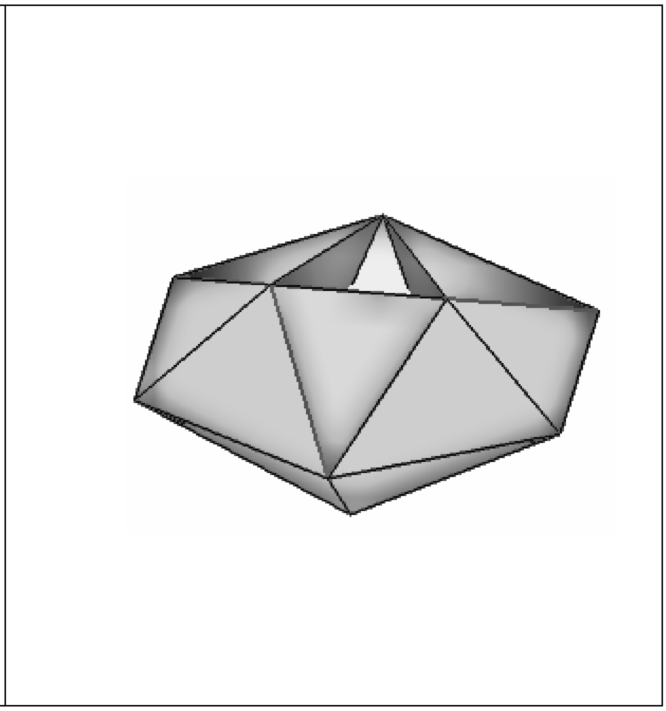


Figure 27., 28. *A toroid with 24 faces consisting of only congruent triangles*

With these variables the rectangular coordinates of the points *P, Q, R, S, T* in **Figure 27.** can be expressed as follows:

$$P(1,0,p)\ ,\ Q(r,0,q)\ ,\ R\left(\frac{r}{2},\frac{-r\sqrt{3}}{2},p-q\right)\ ,\ S\left(\frac{1}{2},\frac{\sqrt{3}}{2},0\right)\ ,\ T\left(\frac{1}{2},\frac{-\sqrt{3}}{0},0\right)\ .$$

Our task is to select the data so that the conditions $\overline{ST}=\overline{PQ}=a$ and $\overline{SP}=\overline{PR}=\overline{QR}=b$ be satisfied. Five equations can be set up from the coordinates of the points:

$$a^2=3\ ,$$

$$a^2=(r-1)^2+(q-p)^2\ ,$$

$$b^2=1+p^2\ ,$$

$$b^2=\left(\frac{r}{2}-1\right)^2+\frac{3}{4}r^2+q^2\ ,$$

$$b^2=r^2+(p-2q)^2\ .$$

The positive roots of this system of equations are:

$$p=\frac{3+\sqrt{2}}{2}\ ,\ q=\frac{1+\sqrt{2}}{2}\ ,\ r=1+\sqrt{2}\ ,\ a=\sqrt{3}\ ,b=\frac{\sqrt{15+6\sqrt{2}}}{2}\ .$$

From this, the single parameter determining the toroid is the ratio

$$b:a=\frac{\sqrt{5+2\sqrt{2}}}{2}=1.39896...\approx 1.4\ .$$

It must be noted that this toroid is not regular, since five edges meet at six of its vertices, and seven edges meet at the other six. The corresponding six solid angles of either type are congruent.

We are not aware of the existence of a toroid with less than 24 congruent faces.(**Figure 28.)**

Finally, it should be mentioned that polyhedra in **Figures 3. - 6. , 12. , 14., 15., 16.** . and **26**, all of the toroids mentioned above have a dextro and a laevo variant; although these are congruent, they can be transformed into each other only by reflection in a plane.

References

[1] BOKOWSKI, J. - EGGERT, A: *All realizations of Möbius' torus with 7 vertice,* Topologie Strucurale N° 17 (1991) Montreal, Canada (p.:59.- 78.).

[2] CSÁSZÁR, Á.: *A polyhedron without diagonals,* Acta Sci. Math. Universitatis Szegediensis 13,1949-50, pp.140-142.

[3] GARDNER M.: *The Minimal Art ,* Scientific American 1978/11.

[4] GARDNER M.: *Fractal Music, Hypercards, and More Mathematical Recreation from Scientific American Magazine*, New York, W.H. Freeman, 1992., pp. 118-120.

[5] STEWART, B. M.: *Adventures Among the Toroids,* Okemos, Michigan, Revised second edition, 1980.

[6] SZILASSI, L.: *Regular Toroids,* Structural Topology # 13. 1986., Montreal, Canada pp. 69-80.

[7] SZILASSI L.: *Egy poliéder, melynek bármely két lapja szomszédos,* Juhász Gyula Tanárképző Főiskola Tudományos Közleményei II. Szeged, 1977., pp. 130-139. (In Hungarian.)

Contact address

Lajos Szilassi , University of Szeged, Faculty of Juhász Gyula Teachers' Training College, Department of Mathematics, H 6725 Szeged, Boldogasszony sgt. 6. Hungary

e-mail: szilassi@jgytf.u-szeged.hu