• Fundamentals
  ○ **Graphs.** Be able to graph basic functions, such as polynomials (eg, \( f(x) = x^3 - x, x^2 + ax + b, x(x-1)^2(x+2)^3 \)), simple rational functions (eg, \( f(x) = 1/x, 1/x^2, x+1/x \)), trigonometric functions, inverse trig functions, logarithms, exponentials and their transformations.
  ○ **Algebra.** Be comfortable solving equations with quotients, powers, logarithms, exponents.
  ○ **Derivatives.**
    Find derivatives using product rule, quotient rule, chain rule, implicit differentiation. Including derivatives of \( x^p, e^x, a^x, \ln x, \log_a x, x^x, \sin x, \cos x, \tan x, \sec x, \cot x, \csc x, \sin^{-1} x, \cos^{-1} x, \tan^{-1} x, \sec^{-1} x. \)
    Use derivatives to find where functions are increasing, decreasing, concave up, concave down, find local max/min, sketch graphs.
  ○ **Fundamental Theorem of Calculus.** Understand the function \( \int_a^x f(t) \, dt \) and its derivative.
  ○ **Fundamental Theorem of Algebra.** What does it state?

• Special functions
  ○ Know the definition of hyperbolic functions, inverse trig functions, exponentials and logs of arbitrary base. Know their graphs, and be able to find their derivatives.
  ○ Find limits as \( x \to \pm\infty \) of rational functions, exponentials, logarithms, other (eg \( 1/\sqrt{x}, \frac{x^3-x^4}{\sqrt{1+2x^2}}, \frac{e^x-e^{-x}}{e^x+e^{-x}} \))
  ○ Use L’Hôpital’s rule when appropriate.

• Integration.
  ○ Find antiderivatives either directly, or using substitution (including trigonometric substitution), or integration by parts, or partial fractions. In some cases you may need a combination of methods.
  ○ Approximate integrals using the trapezoid rule, or the midpoint rule. Don’t memorize the rules. Understand where they come from.
  ○ Detect and evaluate improper integrals, using proper notation.
  ○ An improper integral \( \int_a^\infty f(x) \, dx, f \) continuous, converges if \( f(x) \) approaches zero **sufficiently fast!** How fast is fast enough?

• Differential Equations
  ○ Be able to check whether a given function solves a given differential equation (even if this function is given as an integral!).
  ○ Solve separable differential equations, \( dy/dx = f(x)g(y) \), possibly with an initial condition
  ○ What are the solutions of \( y’ = kx? \ y’ = ky? \ y’ = y^2? \ y’ = -y? \)
  ○ Plot direction fields and approximate solution curves. Given a direction field, read off steady states.
  ○ Know the main idea of Euler’s method.

• Series \( \sum_{n=n_0}^\infty a_n \)
  ○ Know the difference between a sequence and a series. What does it mean for a sequence to converge? What does it mean for a series to converge? What is the value of a series? what are partial sums?
A series converges if the summands approach zero **sufficiently fast**! How fast is fast enough?

- Understand the meaning of absolute convergence and conditional convergence.

- Be able to recognize and evaluate
  - geometric series
  - telescoping series
  - other series that happen to be known Taylor series for functions such as $f(x) = 1/(1 - x), e^x, \cos(x), \text{or} \sin(x)$ at particular values of $x$.

- Be able to determine whether a series converges absolutely, converges conditionally or diverges. Always state which test you are using (Divergence Test, Integral Test, Direct Comparison Test, Limit Comparison Test, Ratio Test).

- Be able to estimate alternating series by partial sums.

- Know examples of series that converge and series that diverge (for example, Harmonic, Alternating Harmonic, Geometric, and p-series).

- What is a power series in a variable $x$? What do you know about the values of $x$ where the power series converges? Where does a power series converge absolutely?

- When can you differentiate or integrate power series term-by-term? Does the interval of convergence change?

- **Taylor Series**

  - Know the definition of the Taylor series of a function $f(x)$ about a basepoint $x = a$
  
  
  $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$, and find Taylor series for any function using this formula.

  - Memorize the Taylor series about $x = 0$ for $e^x, \cos x, \sin x, 1/(1 - x)$ and their intervals of convergence.

  - Be able to find Taylor series from memorized series using substitution, integration, differentiation, and find their intervals of convergence.

  - What special properties do the Taylor polynomials $p_n(x)$ of a function $f(x)$ about a basepoint $x = a$ satisfy? What is $p_1(x)$?

  - Approximate a function by a Taylor polynomial and estimate the error made using Taylor’s Inequality, or by the Alternating Series Estimation test, when applicable.

  - Be able to apply the above 3 points to problems in physics and engineering in which there is a small parameter denoted by a variable not necessarily called $x$!

  - Complex numbers: Memorize and use Euler’s formula.

**PRACTICE PROBLEMS**

*Note: For complex variables, see homework.*

**Miscellaneous Series**

M1. Suppose you know that the series $f(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} \cos(k\pi x)$ converges. Approximate $f(1)$.

Since you know the series converges, what can you say about the approximation error $E_n = f(1) - \sum_{k=1}^{n} \frac{1}{k^2} \cos(k\pi)$ as $n \to \infty$?

M2. Show that $\cosh x \geq 1 + \frac{1}{2} x^2$ for all $x$.

M3. REVIEW Chapter 11, # 60 (application: force of gravity)
M4. Several important functions that arise in the mathematical sciences are given in terms of power series. An example is the Bessel function of order 0, given by

\[ J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n (n!)^2} \]

Find the interval of convergence of \( J_0 \).

**Fundamentals**

1. Memorize the graphs of \( \sin(x) \), \( \cos(x) \).

Using these graphs and the definitions of \( \tan(x) \), \( \sec(x) \), \( \sin^{-1}(x) \), \( \cos^{-1}(x) \), \( \tan^{-1}(x) \), obtain the graphs of these other trig and inverse trig functions.

Know the definitions \( \sinh(x) \), \( \cosh(x) \), \( \tanh(x) \), and use them to obtain their graphs.

Know how to graph exponentials and logarithms, \( \ln(x) \), \( e^x \), \( 2^x \), \( (1/2)^x \).

2. Find the derivatives of the following functions

(a) \( f(x) = x^{2^x} \)  
(b) \( f(x) = \frac{\tan^{-1}x}{x} \)  
(c) \( f(x) = \sinh(\ln x) \)

(d) \( f(x) = \int_1^\sqrt{x} \tan^{-1} s \, ds \)  
(e) \( f(x) = x \sin^{-1}(x^3) \)

**Integration**

3. Evaluate the following definite and indefinite integrals. Some of the definite ones may be improper, careful.

(a) \( \int \frac{1 + x - x^2}{x^2} \, dx \)  
(b) \( \int \frac{2^x}{5^x} \, dx \)  
(c) \( \int \frac{4 + x}{x^2 + 4} + \frac{3}{2 - 3x} \, dx \)

(d) \( \int_0^1 \sqrt{1 - x^2} \, dx \)  
(e) \( \int_2^\infty \frac{x^2}{x^2} \, dx \)  
(f) \( \int_0^{100} \frac{1}{\sqrt{x}} \, dx \)

(g) \( \int_1^{\infty} \frac{1}{x^2} \, dx \)  
(h) \( \int_0^1 \frac{dx}{\sqrt{1 - x^2}} \)  
(i) \( \int t \sin(2t) \, dt \)

(j) \( \int e^x \sin x \, dx \)  
(k) \( \int_5^\infty xe^{-x} \, dx \)  
(l) \( \int_0^2 9z^2 \ln z \, dz \)

(m) \( \int_0^{\pi/2} \sin^3 \theta \cos^2 \theta \, d\theta \)  
(n) \( \int \tan x \sech^2 x \, dx \)  
(o) \( \int \frac{\cos x}{1 + \sin^2 x} \, dx \)

(p) \( \int_0^2 \frac{1}{(x^2 + 4)^2} \, dx \)  
(q) \( \int_0^a x^2 \sqrt{a^2 - x^2} \, dx \)  
(r) \( \int_0^2 \frac{1}{t^4 \sqrt{t^2 - 1}} \, dt \)

(s) \( \int \frac{2 + x}{1 - x^2} \, dx \)  
(t) \( \int \frac{dv}{v^2 + 2v - 3} \)  
(u) \( \int_1^\infty \frac{\tan^{-1} x}{x^2} \, dx \)

**Differential Equations**
4. Verify that \( y(x) = e^{-x^2} \int_0^x e^{t^2} \, dt \), which is known as \textit{Dawson's integral}, is the solution of the initial value problem \( y'(x) = 1 - 2xy \), \( y(0) = 0 \).

5. Find an exact solution, given a guess of the form of the solution
   (a) §9.1, # 3. \hspace{1cm} (b) §9.1, # 4.

6. Find the set of all solutions \( y(t) \) to the differential equations
   (a) \( y' = t \) \hspace{1cm} (b) \( y' = y \)
   (c) Can you find some solutions to the differential equation \( y'' = -y' \) (that is, functions whose second derivative is the negative of themselves)? Once you found these, can you find solutions to \( y'' = -ky \), where \( k > 0 \)?


8. Consider the differential equation
   \[
   \frac{dy}{dx} = y^2
   \]
   with initial condition \( y(x_0) = y_0 \).
   (a) Plot the direction field for the differential equation as well as several solution curves.
       Describe the behaviour of the solutions as \( t \to \infty \), depending on the initial condition.
       Include steady states.
   (b) Find the solution \( y(x) \) if \( y(0) = 3 \). Highlight this solution in your plot in (a).
   (c) Find the solution \( y(x) \) if \( y(0) = 0 \). Highlight this solution in your plot in (a).

9. Consider the differential equation
   \[
   \frac{dI}{dt} = 15 - 3I
   \]
   with initial condition \( I(t_0) = I_0 \).
   (a) Plot the direction field for the differential equation as well as several solution curves.
       Describe the behaviour of the solutions as \( t \to \infty \), depending on the initial condition.
       Include steady states.
   (b) Find the solution \( I(t) \) if \( I(0) = 3 \). Highlight this solution in your plot in (a).
   (c) Find the solution \( I(t) \) if \( I(0) = 5 \). Highlight this solution in your plot in (a).

10. Consider the differential equation
    \[
    \frac{dP}{dt} = rP(1 - \frac{P}{M})
    \]
    with initial condition \( P(0) = P_0 \).
    (a) Plot the direction field for the differential equation as well as several solution curves.
       Describe the behaviour of the solutions as \( t \to \infty \), depending on the initial condition.
       Include steady states.
    (b) Find the solution \( P(t) \) if \( M < P_0 \). Highlight a sample solution in your plot in (a).
    (c) Find the solution \( P(t) \) if \( P_0 = M \). Highlight the solution in your plot in (a).

11. Find the solution \( y(x) \) to the following initial value problem
(a) \( \frac{dy}{dx} = \frac{e^x}{1 + 2y} \), \( y(0) = 2 \).

(b) \( \frac{dy}{dt} = t^2 y \), \( y(1) = -1 \).

(c) \( \frac{dy}{dx} = \frac{xy}{(y+1)(1-2x)} \), \( y(1) = 3 \). (implicit solution)

12. State Euler’s method to approximate the solution of the initial value problem \( \frac{dy}{dt} = f(t, y) \), \( y(0) = y_0 \) at \( t = 1 \) using steps of size \( \Delta t \). Explain the origin of the formula, including a picture.

13. §9.2, #20 (given direction field, graph Euler approximations)

14. §9.2, #25 (use \( h = 1, 0.5, 0.25 \) in (a) by hand, no need to program calculator/computer)

**Series**

12. Suppose \( \sum_{n=0}^{\infty} a_n = 3 \). Write a formula for the partial sum \( s_n \) of the series. What is \( \lim_{n \to \infty} a_n \)? What is \( \lim_{n \to \infty} s_n \)?

13. (a) What does it mean for a series to converge? What is the value of a series?

(b) A series \( \sum_{n=n_0}^{\infty} a_n \) converges if the summands \( a_n \) approach zero sufficiently fast! How fast is fast enough?

14. Using your own words, define what it means for a series \( \sum_{k=k_0}^{\infty} a_k \) to converge absolutely, and conditionally.

15. What is a p-series? Which p-series converge, which diverge?

What is a geometric series? Which geometric series converge, which diverge?

16. State a formula for
\( (a) \ S = \sum_{k=0}^{n} r^k \)\ (any value of \( r \), and any integer \( n \)). Can you derive this formula?
\( (b) \ S = \sum_{k=0}^{\infty} r^k, \ |r| < 1. \)
\( (c) \ The MacLaurin series for \( f(x) = 1/(1-x) \) and its radius of convergence.

17. State the integral test. Clearly state all conditions and conclusions.

18. Determine whether the following series converge absolutely, conditionally, or diverge. If it converges, can you find its value?

(a) \( \sum_{n=0}^{\infty} \frac{1+n^3}{1+2n^3} \) \( \quad \) (b) \( \sum_{n=10}^{\infty} (-1)^n \frac{1+n}{1+2n^3} \) \( \quad \) (c) \( \sum_{n=0}^{\infty} \frac{2(-1)^n3^{n+1}}{5^n} \)

(d) \( \sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+1} \) \( \quad \) (e) \( \sum_{n=0}^{\infty} (-1)^n \frac{3^n}{n!} \) \( \quad \) (f) \( \sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n!} \)

19. Consider the power series \( f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} \). Find the intervals of convergence of \( f \).

(a) Find the intervals of convergence of \( f \).
(b) Find series representations for \( f' \), \( f'' \), and their intervals of convergence.

**Taylor Series**

*Hint: to find McLaurin series, whenever possible, use the memorized series. That is, obtain new series by substitution, integration, differentiation, of known series.*

20. (a) What property defines the Taylor polynomial \( p_n \) of degree \( n \) of a function \( f(x) \) about a basepoint \( x = a \)?

(b) Write down a formula for \( p_n(x) \).

(c) If \( p_n \) is used to approximate a function \( f \) about \( x = a \), Write down an upper bound for the error \( |f(x) - p_n(x)| \).

(d) Find \( p_2(x) \) for \( f(x) = e^x \) about \( x = 0 \). Write down a formula for an upper bound for \( |f(x) - p_2(x)| \). Can you find an upper bound for \( |f(x) - p_2(x)| \) if \( |x| < .1 \)?

21. Write down a formula for the Taylor series of \( f \) about \( x = a \).

22. Here we address: How good is the approximation \( \sin \theta \approx \theta \) if \( \theta \) is small?

   (a) Find the Taylor series for \( f(\theta) = \sin \theta \) about \( \theta = 0 \). Find the linear approximation \( p_1(\theta) \).

   (b) State the Alternating Series Remainder Theorem.

   (c) Find an upper bound for \( |f(\theta) - p_1(\theta)| \) if \( |\theta| \leq 0.1 \) using the Alternating Series Remainder Theorem. Explain why you can use the ASRT.

   (d) Find an upper bound for \( |f(\theta) - p_1(\theta)| \) if \( |\theta| \leq 0.1 \) using Taylor’s Inequality. Which is a tighter bound, your result in (c) or in (d)?

23. Find the Maclaurin series for the following functions and state their radius and interval of convergence.

   (a) \( f(x) = \sin(x^3) \)

   (b) \( f(t) = \frac{1}{1 + 2t^2} \)

   (c) \( f(x) = xe^x \)

   (d) \( F(x) = \int_0^x \frac{\sin t}{t} \, dt \)

   (e) \( g(t) = \frac{1}{(1 + t)^2} \)

24. (a) Write down the linear approximation of a function \( f(x) \).

   (b) Find the linear approximation for \( f(x) = \sqrt{1 + x} \) about \( a = 0 \). Use it to approximate \( \sqrt{1.1} \) and \( \sqrt{0.9} \).

   (c) Find the linear approximation for \( f(x) = \sqrt{x} \) about \( a = 1 \).

25. Show that \( \pi - \pi^3/3! + \pi^5/5! - \pi^7/7! + \ldots \) converges to zero. How many terms must be computed to get within 0.01 of zero?

26. When a voltage \( V \) is applied to a series circuit consisting of a resistor \( R \) and an inductor \( L \), the current at time \( t \) is

   \[
   I = \left( \frac{V}{R} \right) \left( 1 - e^{-Rt/L} \right).
   \]

   Use Taylor series to deduce that \( I \approx Vt/L \) if \( R \) is small.