Math 163 — Review Exam 3: Solutions

Below are solutions to some problems in the review.

4. (a) \{a_n\} converges with \( \lim_{n \to \infty} a_n = 1/2 \). \( \sum_{n=1}^\infty a_n \) diverges since \( \lim_{n \to \infty} a_n \neq 0 \).
(b) \{a_n\} converges with \( \lim_{n \to \infty} a_n = 0 \). \( \sum_{n=1}^\infty a_n \) converges since geometric series with \( r = 9/10 < 1 \).
(c) \{a_n\} converges with \( \lim_{n \to \infty} a_n = 0 \). \( \sum_{n=1}^\infty a_n \) diverges since geometric series (we used Integral test to show this)
(d) \{a_n\} diverges since it oscillates between 3 and 1 as \( n \to \infty \). \( \sum_{n=1}^\infty a_n \) diverges since \( \lim_{n \to \infty} a_n \neq 0 \).
(e) \{a_n\} converges with \( \lim_{n \to \infty} a_n = 0 \). \( \sum_{n=1}^\infty a_n \) diverges since harmonic series (we used Integral test to show this)

5. (a) converges absolutely by Limit Comparison with \( \sum 1/n^2 \)
(b) converges absolutely by Direct Comparison with \( \sum 1/(1+n^2) \) which in turn converges by Limit Comparison with \( \sum 1/n^2 \)
(c) diverges by Limit Comparison with \( \sum 1/n \)
(d) diverges by Limit Comparison with \( \sum 1/\sqrt{n} \)
(e) converges absolutely by Integral Test, since \( f(x) = \frac{1}{x(\ln x)^2} \) (reason: denominator \( x(\ln x)^2 \) clearly increases as \( x \) increases) and
\[
\int_2^\infty \frac{dx}{x(\ln x)^2} = \lim_{t \to \infty} \int_2^t \frac{dx}{x(\ln x)^2} = \lim_{t \to \infty} \int_{\ln(2)}^{\ln(t)} \frac{du}{u^2} = \lim_{t \to \infty} \left[ -\frac{1}{\ln t} + \frac{1}{\ln 2} \right] = \frac{1}{\ln 2}
\]
which is finite
(f) diverges since \( \lim_{n \to \infty} \frac{(25/9)^n}{n^2} = \lim_{n \to \infty} \ln(25/9)^2(25/9)^n/2 = \infty \)
(g) converges absolutely by Ratio test
(h) converges conditionally. First show that it does not converge absolutely by Integral Test, then show it converges using Alternating Series test. Make sure to check all conditions in each case.

6. (a) Geometric series with \( r = 4/5 \), converges to 8
(b) Geometric series with \( r = 1/\sqrt{2} \) converges to \( \sqrt{2}/(\sqrt{2} - 1) \)
(c) Geometric series with \( r = -1/2 \) converges to \( -1/2 \)
(d) Diverges by Direct Comparison with \( \sum 1/\sqrt{n} \)
(e) Telescoping series, converges to \( -1 \) since partial sums are \( S_N = -1 + \frac{2}{N+1} \)
(f) Diverges since \( \lim_{n \to \infty} a_n = e \neq 0 \)
(g) Diverges since \( \lim_{n \to \infty} a_n = -\infty \neq 0 \)

(h) Geometric series with \( r = 1/e^2 \) converges to \( e^2/(e^2 - 1) \)

7. Ratio test for \( f \): \( \lim_{n \to \infty} |a_{n+1}/a_n| = |x| < 1 \) if \( x \in (-1,1) \) so radius of convergence is \( R = 1 \). Since series converges at \( x = \pm 1 \) the interval of convergence is \([-1,1] \).

The radius of convergence does not change under differentiation. So only have to check the endpoints \( x = \pm 1 \) for the derivative series \( \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} \). This series converges for \( x = -1 \) (alternating harmonic) and diverges for \( x = 1 \) (harmonic). So interval of convergence is \([-1,1] \).

The radius of convergence, again, does not change. So only have to check the endpoints \( x = \pm 1 \) for the second derivative series \( \sum_{n=1}^{\infty} \frac{(n-1)x^{n-2}}{n} \). This series diverges for \( x = \pm 1 \) since in either case \( \lim_{n \to \infty} a_n \neq 0 \).

10. For \( 1 + x > 0 \), \( \ln(1 + x) = \int \frac{dx}{1+x} = \int 1 - x^2 - x^3 + x^4 - \ldots dx = C + x - x^2/2 + x^3/3 - x^4/4 + x^5/5 - \ldots = x - x^2/2 + x^3/3 - x^4/4 + x^5/5 - \ldots \) since \( C = 0 \) which you obtain by substituting in \( x = 0 \). Therefore

\[
\ln(1 + x^2) = x^2 - x^4/2 + x^6/3 - x^8/4 + x^10/5 - \ldots
\]

and

\[
\ln(1.01) = 0.01 - 0.0001/2 + 0.000001/3 - 0.00000001/4 + \ldots = 0.01 - 0.0001/2 + 0.000001/3 + \text{error} = 0.00995033 + \text{error}
\]

where \( |\text{error}| \leq 0.00000001/4 = 0.25 \cdot 10^{-8} < 10^{-7} \) by the Alternating Series Estimation test, since the series is alternating and the terms \( |a_n| \) are decreasing and \( \to 0 \) as \( n \to \infty \).

11. \( \int_{0}^{0.2} \frac{1}{1+x^4} \, dx = \int_{0}^{0.2} 1 - x^4 + x^8 - x^{12} + \ldots \, dx = [x - \frac{x^5}{5} + \frac{x^9}{9} - \frac{x^{13}}{13} + \ldots ]_{0}^{0.2} \)

\[
= 0.2 - \frac{0.2^5}{5} + \frac{0.2^9}{9} - \frac{0.2^{13}}{13} + \ldots = 0.2 - \frac{0.2^5}{5} + \frac{0.2^9}{9} - \frac{0.2^{13}}{13} + \ldots
\]

\[
= 0.2 - \frac{0.2^5}{5} + \text{error} = 0.199936 + \text{error}
\]

where \( |\text{error}| \leq \frac{0.2^9}{9} < 10^{-7} \), by the Alternating Series Estimation Test.
12. Use alternating series estimation test, which applies since \( f(x) = \frac{1}{x^2} \geq 0 \) and decreasing, and thus

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{N} \frac{1}{n^2} + \text{error}
\]

where \( |\text{error}| \leq a_{N+1} = \frac{1}{(N+1)^2} \). Choosing \( N = 31 \) get

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = -0.82297 + \text{error}
\]

where \( |\text{error}| \leq a_{32} < 0.00098 < 0.001 \)

13. (a) This looks like a second derivative. Let’s see how we get there: Start with \( \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \),

\[
\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}, \quad \frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}.
\]

Therefore

\[
\frac{2x^2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^n, \quad |x| < 1
\]

Replace \( x \) by \( (x - 2)/3 \) and multiply by 5 to get

\[
\frac{5 \cdot 2(\frac{x-2}{3})^2}{(1 - (\frac{x-2}{3}))^3} = \sum_{n=2}^{\infty} 5n(n-1)(\frac{x-2}{3})^n, \quad |(x - 2)/3| < 1
\]

The left hand side simplifies to

\[
\frac{30(x-2)^2}{(5-x)^3}, \quad |x - 2| < 3.
\]

The radius of convergence is 3 (center \( c = 2 \)), so interval of convergence \((-1, 5)\). (By checking \( x = -1, 5 \), find that endpoints are not included.)

(b) Evaluating the expression in (a) at \( x = 3 \) (in the interval of convergence) get

\[
\sum_{n=2}^{\infty} \frac{5n(n-1)}{3^n} = \frac{15}{4}
\]