TOPICS  (Lectures 13-23)

0. Methods of Integration
   Partial fractions.
   You need to be able to use all methods of integration, throughout course.

1. Numerical Integration
   Derive the trapezoid rule.
   Apply each of the rules below to approximate an integral using \( n \) equally spaced subintervals.
   - Rectangle rule, using right or left endpoints.
   - Trapezoid rule.
   - Midpoint rule.
   - Simpson’s rule.
   Given a formula for the error, can you find the value of \( n \) for which the error is guaranteed to be less than a prescribed tolerance?
   Given a formula for the error, approximately by how much does the error decrease if \( n \) is doubled, for each method above?

2. Improper Integrals
   Recognize improper integrals, with either
   - Unbounded integrands
   - Unbounded domains
   Rewrite improper integrals as a limit or a sum of 2 limits, as appropriate
   Evaluate limits, using proper notation throughout

3. Differential Equations \( y' = f(t, y), \ y'' = f(t, y, y') \)
   Find the family of solutions to a differential equation (general solution)
   - Using antidifferentiation where appropriate \( (y' = f(t), \ y'' = f(t)) \)
   - Using separation of variables for \( y' = g(t)h(y) \)
   Find the particular solution to an initial value problem
   What are autonomous differential equations?
   Find steady solutions for autonomous systems
   Sketch direction fields and integral curves for simple examples
   Eulers method
   - Sketch solution to Eulers method in a direction field
   - Given an initial value problem \( y' = f(t, y), y(t_0) = y_0 \), approximate the solution \( y(t) \) at some time \( t > t_0 \) using Euler’s method, with one or two steps.
PRACTICE PROBLEMS

0. Evaluate the following indefinite integrals.  
(a) \[ \int \frac{a}{x^2 - bx} \, dx \]
(b) \[ \int \frac{x^3 - 2x^2 - 4}{x^3 - 2x^2} \, dx \]

1. Let \( T_n \) be the approximation of the integral \( I = \int_a^b f(x) \, dx \) by the trapezoid rule, using \( n + 1 \) uniformly spaced points \( x_j = a + j \Delta x, \, j = 0, \ldots, n \), where \( \Delta x = (b-a)/n \).

(a) Write down a formula for \( T_n \).

(b) Find the approximation \( T_4 \) of the integral \( I = \int_0^1 e^x \, dx \).

(c) Sketch a graph of \( f(x) = e^x \) and the area given by the approximation \( T_4 \) (b). Using solely your sketch, determine whether \( T_4 \) is an over- or an underestimate. Explain.

(d) For any function \( f \) with a continuous second derivative, the error of the trapezoid approximation can be shown to be

\[ |I - T_n| = \frac{(b-a)^3}{12n^2} |f''(c)| \]

for some \( c \in [a, b] \). Use this formula to find an upper bound for the error in your approximation in (b). Compare this upper bound with the actual error.

(e) What do you expect would the error be if you used \( n = 8 \) instead of \( n = 4 \)?

(f) What should the value of \( n \) be so that the approximation error \( |I - T_n| \) of the integral in (b) is guaranteed to be less than \( 10^{-4} \)?

2. For which values of \( p \) does \( \int_a^\infty \frac{dx}{x^p}, \, a > 0 \) converge? For which does it diverge?

3. Determine if the following integrals converge or diverge. If they converge, compute their value. Use correct notation throughout.

(a) \[ \int_{-\infty}^{\infty} \cos(\pi t) \, dt \]
(b) \[ \int_0^1 \frac{x - 1}{\sqrt{x}} \, dx \]
(c) \[ \int_0^3 \ln(x) \, dx \]
(d) \[ \int_{-\infty}^{\infty} \frac{x^2}{9 + x^6} \, dx \]
(e) \[ \int_3^6 \frac{dx}{\sqrt{x-3}} \]
(f) \[ \int_0^\infty e^{-as} \, ds, \, a > 0 \, \text{a constant} \]
(g) \[ \int_0^1 \frac{2}{1 - s^2} \, ds \]
(h) \[ \int_{-\infty}^{\infty} ye^{-3y} \, dy \]
(i) \[ \int_3^5 \frac{dx}{4x^2 - 9} \]

4. §7.8, # 62 (average speed of molecules in ideal gas) Hint: begin by setting \( a = M/(2RT) \), thus simplifying the integrand.

5. Review p 556, # 77 (evaluate \( \int_0^\infty f'(x) \, dx \) if \( \lim_{x \to \infty} f(x) = 0 \))

6. Review p 556, # 80 (evaluating potential given by improper integral) The following figure may help

![Diagram of charge distribution](image-url)
7. Given the linear, homogeneous differential equation \( y'' + 2y' + 2y = 0 \) where \( y = y(t) \). Show that \( y_1(t) = e^{rt} \sin t \) and \( y_2(t) = e^{rt} \cos t \) both solve the equation for a particular value of \( r \). Find that value.

8. Sketch the direction fields and several solution curves for (a) \( y' = y \), (b) \( y' = x \), (c) \( y' = y^2 \), where \( y = y(x) \). What are the constant solutions?

9. §9.3, 11, 16

10. HW Consider the initial value problem

\[
\frac{dP}{dt} = P(2 - P), \quad P(0) = 1/2
\]

(a) Plot the direction field for this differential equation. Include several solution curves. Clearly indicate the one solution that solves the initial value problem.

(b) Find the solution \( P(t) \) of the initial value problem.

(c) Find the limits \( \lim_{t \to \infty} P(t) \) and \( \lim_{t \to -\infty} P(t) \). Do these limits agree with the graph of \( P(t) \) that you plotted in (a)?

(d) Approximate \( P(1) \) using Euler’s method with \( \Delta t = 1/2 \). Compare the approximation with your exact result in (b).


12. Psychologists interested in learning theory study learning curves. A learning curve is the graph of a function \( P(t) \), the performance of someone learning a skill as a function of the training time \( t \). The performance is always nonnegative, \( P \geq 0 \). A model for learning is given by

\[
\frac{dP}{dt} = k(M - P)
\]

where \( M \) is the maximum level of performance of which the learner is capable.

(a) Draw a direction field and several solution curves for this differential equation.

(b) At what level of performance is the rate of improvement \( dP/dt \) the largest?

(c) Find the performance \( P(t) \) if the initial performance is \( P(0) = M \).

(d) Find the performance \( P(t) \) if the initial performance is \( P(0) = P_0 < M \).

(e) What does the number \( k \) measure?