Let $C([0,1], \mathbb{R})$ be the following, infinite-dimensional vector space:

$$C([0,1], \mathbb{R}) = \{f : [0,1] \rightarrow \mathbb{R} \mid f \text{ is continuous} \}$$

For any $f$ and $g$ in $C([0,1], \mathbb{R})$, we define

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx$$

This is an inner product, a fact that is easy enough to prove except for the axiom that only the zero vector (zero function) satisfies $\langle f, f \rangle = 0$.

**Lemma 1.** Suppose $f : [0,1] \rightarrow \mathbb{R}$ is continuous. If $\langle f, f \rangle = 0$ then $f = 0$. (I.e. $\forall x \in [0,1], f(x) = 0$.)

**Proof.** We’ll prove the contrapositive.

Suppose $f$ is continuous and $f \neq 0$. This means at some $x_0$ in $[0,1]$, $f(x_0) \neq 0$. We can assume $0 < x_0 < 1$ since a continuous function cannot be zero on $(0,1)$ and nonzero at 0 or 1.

The square of $f$ is continuous and, and $f(x)^2 \geq 0$ for all $x$. Let $\delta = f(x_0)^2$ so that $\delta > 0$. The continuity of $f^2$

at $x_0$ tells us there exists some positive $\eta$ such that

$$x_0 - \eta < x < x_0 + \eta \Rightarrow f(x_0)^2 - \delta/2 < f(x)^2 < f(x_0)^2 + \delta/2$$

We don’t care about the upper bound on $f(x)$. Let $\mu = f(x_0)^2 - \delta/2$.

At this point, we’ve found $x_0$, $\eta$ and $\mu$ so that:

$$\eta > 0 \quad \mu > 0 \quad x_0 - \eta < x < x_0 + \eta \Rightarrow f(x_0)^2 > \mu$$

We can replace $\eta$ with a smaller positive value and the above equations hold, and we do so if needed to get

$$0 < x_0 - \eta \text{ and } x_0 + \eta < 1.$$
Using these facts, and the fact that $f(x)^2$ is always non-negative, we conclude

\[ \langle f, f \rangle = \]
\[ = \int_0^1 f(x)^2 \, dx \]
\[ = \int_0^{x_0 - \eta} f(x)^2 \, dx + \int_{x_0 - \eta}^{x_0 + \eta} f(x)^2 \, dx + \int_{x_0 + \eta}^1 f(x)^2 \, dx \]
\[ \geq \int_0^{x_0 - \eta} 0 \, dx + \int_{x_0 - \eta}^{x_0 + \eta} \mu \, dx + \int_{x_0 + \eta}^1 0 \, dx \]
\[ = 0 + 2\eta \mu + 0 \]
\[ > 0 \]