Convergence of the Solutions of the Compressible to the Solutions of the Incompressible Navier–Stokes Equations *

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We study the slightly compressible Navier–Stokes equations. We first consider the Cauchy problem, periodic in space. Under appropriate assumptions on the initial data, the solution of the compressible equations consists—to first order—of a solution of the incompressible equations plus a function which is highly oscillatory in time. We show that the highly oscillatory part (the sound waves) can be described by wave equations, at least locally in time. We also show that the bounded derivative principle is valid; i.e., the highly oscillatory part can be suppressed by initialization. Besides the Cauchy problem, we also consider an initial-boundary value problem. At the inflow boundary, the viscous term in the Navier–Stokes equations is important. We consider the case where the compressible pressure is prescribed at inflow. In general, one obtains a boundary layer in the pressure; in the velocities a boundary layer is not present to first approximation.

1. INTRODUCTION

We consider the compressible Navier–Stokes equations in the following simplified form

\[ u_t + (u \cdot \nabla)u + \nabla p = \nu \Delta u + F, \]  
\[ \varepsilon^2 \{ p_t + (u \cdot \nabla) p \} + \nabla \cdot u = g. \]  

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Here \( v > 0, \varepsilon > 0 \). We restrict the discussion to two space dimensions and use the notations

\[
\mathbf{u} = (u(x, y, t), v(x, y, t)), \quad p = p(x, y, t)
\]

for the velocity field and the pressure. Then \( \nabla \cdot \mathbf{u} = u_x + u_y \) and \( \nabla \times \mathbf{u} = v_x - u_y \) denote the dilatation and the vorticity, respectively. The inhomogeneous terms \( F = F(x, y, t) \) and \( g = g(x, y, t) \) are assumed to be \( C^\infty \)-smooth for simplicity. We want to discuss the limiting behaviour of the solutions of (1.1) as \( \varepsilon \to 0 \), under appropriate initial and boundary conditions. The limiting equations

\[
\begin{align*}
U_t + (\mathbf{U} \cdot \nabla)\mathbf{U} + \nabla P &= \nu \Delta \mathbf{U} + \mathbf{F}, \\
\nabla \cdot \mathbf{U} &= g
\end{align*}
\] (1.2a, 1.2b)

describe incompressible flow if \( g = 0 \). We allow an inhomogeneous term \( g \neq 0 \), since inhomogeneous equations like \( \nabla \cdot \mathbf{U} = g \) have to be solved below to derive an asymptotic expansion. We refer to equations like (1.2) as incompressible problems also if \( g \neq 0 \).

In Section 2 we shall discuss the Cauchy problem where all functions are assumed to be \( 1 \)-periodic in \( x \) and \( y \). We use the notation

\[
(f, g) = \int_0^1 \int_0^1 f(x, y) g(x, y) \, dx \, dy, \quad \|f\| = (f, f)^{1/2}
\]

to denote the \( L_2 \)-scalar product and norm. Clearly, (1.2b) is only solvable if

\[
(1, g(\cdot, t))) = \int_0^1 \int_0^1 g(x, y, t) \, dx \, dy = 0, \quad t \geq 0.
\] (1.3)

Henceforth we assume in our discussion of the Cauchy problem that (1.3) is satisfied. For Eqs. (1.1a), (1.1b) we give initial conditions

\[
\mathbf{u} = \mathbf{u}_0(x, y), \quad p = p_0(x, y) \quad \text{at} \ t = 0. \tag{1.1c}
\]

For (1.2a), (1.2b) we can only prescribe the velocity

\[
\mathbf{U} = \mathbf{U}_0(x, y) \quad \text{at} \ t = 0, \tag{1.2c}
\]

where

\[
\nabla \cdot \mathbf{U}_0 = g(\cdot, 0)
\]

is required for consistency. At each time \( t \) the incompressible pressure \( P \) is determined—up to a constant \( \tilde{P}(t) \)—by an elliptic equation: taking the
divergence of (1.2a) and using (1.2b), one obtains
\[
\Delta P + U_x^2 + 2U_y V_x + V_y^2 + U g_x + V g_y = H,
\]
\[
H = -g_t + \nu \Delta g + \nabla \cdot F. \quad (1.4)
\]
If an initial velocity \( u_0(x, y) \) for (1.1) is given, we construct \( U_0(x, y) \) such that
\[
\forall \cdot U_0 = g(\cdot, 0), \quad \nabla \times U_0 = \nabla \times u_0,
\]
\[
(1, U_0) = (1, u_0), \quad (1, V_0) = (1, v_0). \quad (1.5)
\]
The incompressible problem (1.2a)-(1.2c) has a solution \( U, P \) in \( 0 \leq t < \infty \); it is unique up to a time-dependent function \( \overline{P}(t) \) which can be added to \( P \). We fix the constant such that
\[
0 = (1, p_0 - P(\cdot, 0) - \overline{P}(0)),
\]
\[
0 = (1, P_t + \overline{P}_t + (U \cdot \nabla)\overline{P}), \quad t \geq 0. \quad (1.6)
\]
Then we prove in Section 2,

**Theorem 1.1.** Assume the initial data satisfy
\[
\nabla \cdot u_0 = g(\cdot, 0) + O(\varepsilon), \quad p_0 = P(\cdot, 0) + \overline{P}(0) + O(1).
\]
For any \( T > 0 \) and \( 0 < \varepsilon \leq \varepsilon_0(T) \), the compressible problem (1.1a)-(1.1c) has a unique solution in \( 0 \leq t \leq T \). It can be written in the form
\[
\begin{align*}
\mathbf{u} &= U + \mathbf{u}_1 + O(\varepsilon^2), \\
p &= P + \overline{P}(t) + p_1 + O(\varepsilon),
\end{align*}
\]
where \( \mathbf{u}_1, p_1 \) are highly oscillatory in time. The functions \( \mathbf{u}_1, \varepsilon p_1 \) and their space derivatives can be estimated by the initial data
\[
\mathbf{u}_0 - U_0, \quad \varepsilon(p_0 - P(\cdot, 0) - \overline{P}(0))
\]
and their space derivatives.

As we shall make more precise in Theorem 2.7, the highly oscillatory functions \( \mathbf{u}_1, p_1 \) are—to first order and locally in time—determined by solutions of
\[
\nabla \times \mathbf{u}_t = 0, \quad (\nabla \cdot \mathbf{u})_t = \frac{1}{\varepsilon^2} \Delta (\nabla \cdot \mathbf{u}), \quad \overline{p}_{tt} = \frac{1}{\varepsilon^2} \Delta \overline{p}.
\]
Thus \( \mathbf{u}_1, p_1 \) represent the sound waves which oscillate on the fast time scale \( t/\varepsilon \); to first order they do not create vorticity. The \( O(\varepsilon^2) \) and \( O(\varepsilon) \)
terms in the decomposition (1.7) contain the result of the interaction between the fast and the slow time scale. Under our assumptions, these interaction terms are of smaller order than both the fast and the slow part of the solution.

In numerical calculations one is usually not interested in the highly oscillatory part of the solution. Then the effect of compressibility is contained in the $O(\varepsilon^2)$ and $O(\varepsilon)$ terms in (1.7). These are of interest if $\varepsilon$ is not too small. To suppress the highly oscillatory part, one chooses initial data such that a couple of time derivatives of the solution are bounded independently of $\varepsilon$ at $t = 0$. The bounded derivative principle stated next justifies this initialization.

**Theorem 1.2.** If the initial data for (1.1) are chosen such that two time derivatives of the solution are bounded independently of $\varepsilon$ at $t = 0$, then

$$u = U + \varepsilon^2 U_1 + u_1 + O(\varepsilon^4),$$

$$p = P + \bar{P}(t) + \varepsilon^2(P_1 + \bar{P}_1(t)) + p_1 + O(\varepsilon^3).$$

Here $U_1, P_1$ are solutions of linearized incompressible equations, and $u_1 = O(\varepsilon^2), p_1 = O(\varepsilon)$ are highly oscillatory in time. The highly oscillatory part is suppressed further if more than two time derivatives stay bounded at $t = 0$.

In Section 3 we consider an initial-boundary value problem with prescribed inflow velocity at $x = 0$ and outflow velocity at $x = 1$. For the compressible equations an extra boundary condition for $p$ at inflow is needed. We consider the simple choice to prescribe $p(0, y, t) = p_b(y, t)$ at inflow and show that this leads, in general, to a boundary layer in the pressure. In the velocities a boundary layer is not present to first approximation. We can derive an asymptotic expansion and obtain to leading order

$$u = U + \varepsilon^2 u_{1b} + O(\varepsilon^2),$$

$$p = P + \bar{P}(t) + p_{1b} + O(\varepsilon).$$

Here $p_{1b}$ is the boundary layer function

$$p_{1b}(x, y, t) = \left(p_b(y, t) - P(0, y, t) - \bar{P}(t)\right)e^{-ax^2/\nu^2},$$

$$\alpha(y, t) = \frac{1 - \varepsilon^2 U^2(0, y, t)}{\nu U(0, y, t)}.$$

The limit process from compressible to incompressible flow has been considered earlier, e.g., by Ebin [1] and by Klainerman and Majda [2]. Both papers consider the Euler equations ($\nu > 0$), but for the Cauchy problem the cases $\nu > 0$ and $\nu = 0$ are very similar.
The asymptotic expansion derived by Klainerman and Majda differs from our expansion, however. Their correction terms to the incompressible solution contain both slowly varying and highly oscillatory components. We first expand the slow part of the solution. This allows us to isolate the highly oscillatory part $u_1$, $p_1$ and to relate it to the solutions of wave equations.

For the initial-boundary value problem the assumption $\nu > 0$ is important because the case $\nu = 0$ would require different boundary conditions. Our discussion of the boundary layer at inflow seems to be new.

2. THE CAUCHY PROBLEM

All functions in this section are assumed to be 1-periodic in $x$ and $y$, and $C^\infty$-smooth for simplicity. We first show an asymptotic expansion of the slow part of the solution $u, p$ of (1.1). Then we consider the highly oscillatory part in a time interval $0 \leq t \leq T$, $T = O(1)$. We show that this part is essentially described by wave equations in each subinterval of length $O(\sqrt{\epsilon})$, under suitable assumptions on the initial data. In Section 2.4 we prove validity of the bounded derivative principle.

2.1. Expansion of the Slow Part

Suppose the $C^\infty$-functions $u = u^\epsilon$, $p = p^\epsilon$ solve (1.1a)-(1.1c) in $0 \leq t \leq T$ for $0 < \epsilon \leq \epsilon_0(T)$. We assume (1.3) and (1.5), and denote the solution of (1.2a)-(1.2c) and (1.6) by $U, P + \bar{P}(t)$. Defining new variables $u', p'$ by

$$u = U + u', \quad p = P + \bar{P}(t) + p',$$

we obtain

$$u' + (U \cdot \nabla)u' + (u' \cdot \nabla)U + (u' \cdot \nabla)u' + \nabla p' = \nu \Delta u',
\epsilon^2 \{ p' + (U \cdot \nabla)p' + (u' \cdot \nabla)P + (u' \cdot \nabla)p' \} + \nabla \cdot u' = \epsilon^2 g_1$$

with

$$g_1 = -\{ P_t + \bar{P}_t + (U \cdot \nabla)P \}.$$

We recall that $(1, g_1(\cdot, t)) = 0$ by (1.6). The initial conditions for $u', p'$ read

$$u' = u_0 - U_0, \quad p' = p_0 - P(\cdot, 0) - \bar{P}(0) \quad \text{at } t = 0.$$
We first determine the slow part of \( u', p' \). To this end, we write

\[
    u' = \varepsilon^2 U_1 + u'', \quad p' = \varepsilon^2 (P_1 + \bar{P}_1(t)) + p'',
\]

where we define \( U_1, P_1 \) as the solution of the linearized incompressible problem

\[
    U_{1t} + (\mathbf{U} \cdot \nabla) U_1 + (U_1 \cdot \nabla) U + \nabla P_1 = \nu \Delta U_1,
\]

\[
    \nabla \cdot U_1 = g_1, \quad (1, P_1(\cdot, t)) = 0,
\]

\[
    U_1 = U_{1,0} \quad \text{at} \; t = 0.
\]

Here the initial data \( U_{1,0} \) are defined as the solution of

\[
    \nabla \cdot U_{1,0} = g_1(\cdot, 0), \quad \nabla \times U_{1,0} = 0, \quad (1, U_{1,0}) = (1, V_{1,0}) = 0.
\]

Then \( u'', p'' \) satisfy

\[
    u'' + (\mathbf{U}'' \cdot \nabla) u'' + (\mathbf{u}'' \cdot \nabla) U'' + \nabla p'' = \nu \Delta u'' + \varepsilon^4 F_2,
\]

\[
    \varepsilon^2 \{ P'' + (\mathbf{U}'' \cdot \nabla) P'' + (\mathbf{u}'' \cdot \nabla) P'' + (\mathbf{p}'' \cdot \nabla) P'' \} + \nabla \cdot u'' = \varepsilon^4 g_2
\]

with

\[
    U^{(1)} = U + \varepsilon^2 U_1, \quad P^{(1)} = P + \varepsilon^2 P_1,
\]

\[
    F_2 = - (U_1 \cdot \nabla) U_1,
\]

\[
    g_2 = - \{ P_{1t} + \bar{P}_{1t} + (\mathbf{U} \cdot \nabla) P_1 + (U_1 \cdot \nabla) P \} - \varepsilon^2 (U_1 \cdot \nabla) P_1.
\]

We choose \( \bar{P}_1(t) \) such that \( (1, g_2(\cdot, t)) = 0 \). The equations for \( u'', p'' \) have the same structure as the equations for \( u', p' \), but the inhomogeneous terms have been reduced to \( O(\varepsilon^4) \). Clearly, the process can be continued, and one obtains

**Lemma 2.1.** Let \( U_j, P_j + \bar{P}_j(t), j = 1, \ldots, l \), be defined recursively as the solutions of the linear incompressible problems

\[
    U_{jt} + (\mathbf{U}^{(j-1)} \cdot \nabla) U_j + (U_j \cdot \nabla) U^{(j-1)} + \nabla P_j = \nu \Delta U_j + \varepsilon^{2j-4} F_j,
\]

\[
    \nabla \cdot U_j = g_j, \quad (1, P_j(\cdot, t)) = 0,
\]

\[
    U_j = U_{j,0} \quad \text{at} \; t = 0.
\]
Here we define

\[ U^{(j-1)} = U + \sum_{i=1}^{j-1} \varepsilon^{2i} U_i, \quad P^{(j-1)} = P + \sum_{i=1}^{j-1} \varepsilon^{2i} P_i, \]

\[ F_1 = 0, \quad F_j = -(U_{j-1} \cdot \nabla) U_{j-1}, \quad 2 \leq j \leq l, \]

\[ g_j = -\left( P_{j-1,t} + \overline{P}_{j-1,t} + (U^{(j-1)} \cdot \nabla) P_{j-1} + (U_{j-1} \cdot \nabla) P^{(j-1)} \right) \]

\[ - \varepsilon^{2j-2}(U_{j-1} \cdot \nabla) P_{j-1}. \]

The initial data \( U_{j,0} \) are determined by

\[ \nabla \cdot U_{j,0} = g_j(\cdot, 0), \quad \nabla \times U_{j,0} = 0, \quad (1, U_{j,0}) = (1, V_{j,0}) = 0, \]

and \( \overline{P}_{j-1}(t) \) is chosen such that

\[ (1, g_j(t)) = 0, \quad 0 \leq t \leq T, \]

and \( \overline{P}_{j-1}(0) = 0 \). All derivatives of the functions \( U_j, P_j + \overline{P}_j(t) \) are bounded independently of \( \varepsilon \). The error terms \( \tilde{u}, \tilde{p} \) defined by

\[ u = U + \varepsilon^2 U_1 + \cdots + \varepsilon^{2j} U_j + \tilde{u} = U^{(j)} + \tilde{u}, \]

\[ p = P + \overline{P}(t) + \varepsilon^2 \{ P_1 + \overline{P}_1(t) \} + \cdots + \varepsilon^{2j} \{ P_j + \overline{P}_j(t) \} + \tilde{p} - P^{(j)} + \tilde{p} \]

satisfy the equations

\[ \tilde{u}_t + (U^{(j)} \cdot \nabla) \tilde{u} + (\tilde{u} \cdot \nabla) U^{(j)} + (U^{(j)} \cdot \nabla) \tilde{u} + \nabla \tilde{p} = \nu \Delta \tilde{u} + \varepsilon^{4j} F_{l+1}, \]

\[ \varepsilon^2 \{ \tilde{p}_t + (U^{(j)} \cdot \nabla) \tilde{p} + (\tilde{u} \cdot \nabla) P^{(j)} + (\tilde{u} \cdot \nabla) \tilde{p} \} + \nabla \cdot \tilde{u} = \varepsilon^{2j+2} g_{l+1}, \]

\[ \tilde{u} = u_0 - U^{(j)}(\cdot, 0), \quad \tilde{p} = p_0 - P^{(j)}(\cdot, 0) \quad \text{at } t = 0. \]

\textbf{Proof}: The formal expansion follows easily if one writes out the quadratic terms as in

\[ ((U + u') \cdot \nabla)(U + u') = (U \cdot \nabla)U + (U \cdot \nabla)u' \]

\[ + (u' \cdot \nabla)U + (u' \cdot \nabla)u', \]

and similarly for \( ((U + u') \cdot \nabla)(P + p') \). \( L_2 \) -bounds for the functions follow by standard energy estimates; bounds for derivatives follow from the differentiated equations.

The process described above allows one to reduce the inhomogeneous terms in the differential equations to arbitrarily high order in \( \varepsilon \). Clearly, the solution \( \tilde{u}, \tilde{p} \) of the above error equation is only small, however, if the initial data are small.
2.2. Linearization about the Slow Part of the Solution and the Estimate of the Remainder

In our notation we ignore the dependence on \( l \) in Lemma 2.1 and set

\[
\tilde{U} := U^{(l)} = U + \varepsilon^2 U_1 + \cdots + \varepsilon^{2l} U_l,
\]

\[
\tilde{P} := P^{(l)} = P + \bar{P}(t) + \varepsilon^2 (P_1 + \bar{P}_1(t)) + \cdots + \varepsilon^{2l} (P_l + \bar{P}_l(t)).
\]

(For the discussion below, any choice \( l \geq 1 \) would be sufficient.) In the error equations of Lemma 2.1 we first neglect the nonlinear terms and the forcing terms. Then we obtain the linearized compressible equations

\[
\begin{align*}
\dot{u}_1 + (\tilde{U} \cdot \nabla) u_1 + (u_1 \cdot \nabla) \tilde{U} + \nabla p_1 &= \nu \Delta u_1, \\
\varepsilon^2 \{p_{1t} + (\tilde{U} \cdot \nabla) p_1 + (u_1 \cdot \nabla) \tilde{P}\} + \nabla \cdot u_1 &= 0,
\end{align*}
\]

(2.2)

\[
\begin{align*}
u_1 &= u_0 - \tilde{U}(\cdot, 0) =: \nu_{0,0}, \\
p_1 &= p_0 - \tilde{P}(\cdot, 0) =: p_{1,0} \quad \text{at } t = 0.
\end{align*}
\]

We recall that all derivatives of the coefficients \( \tilde{U}, \tilde{P} \) are bounded independently of \( \varepsilon \). To symmetrize the underlying hyperbolic system, we use the variable \( q = \varepsilon p_1 \) and show

**Lemma 2.2.** Suppose \( u_1, q \) solve

\[
\begin{align*}
u_1 + (\tilde{U} \cdot \nabla) u_1 + (u_1 \cdot \nabla) \tilde{U} + \frac{1}{\varepsilon} \nabla q &= \nu \Delta u_1, \\
q_t + (\tilde{U} \cdot \nabla) q + \varepsilon(u_1 \cdot \nabla) \tilde{P} + \frac{1}{\varepsilon} \nabla \cdot u_1 &= 0
\end{align*}
\]

in \( 0 \leq t \leq T \) and satisfy initial conditions

\[
u_1 = u_{1,0}, \quad q = q_0 \quad \text{at } t = 0.
\]

For any \( k = 0, 1, \ldots \) it holds that

\[
\max_{0 \leq t \leq T} \{ \|u_1(\cdot, t)\|_{H^k} + \|q(\cdot, t)\|_{H^k} \} \leq C_k \{ \|u_{1,0}\|_{H^k} + \|q_0\|_{H^k} \}
\]

with \( C_k \) independent of \( \varepsilon \). Here

\[
\|v\|_{H^k}^2 = \sum_{r+s \leq k} \left| \frac{\partial^{r+s}}{\partial x^r \partial y^s} v \right|^2
\]

denotes the usual Sobolev norm.

**Proof.** For \( k = 0 \) the result follows through integration by parts, if one observes that the contributions of the \((1/\varepsilon)\)-terms cancel. To obtain the
estimate for derivatives, one differentiates the equation; it is important to
note that the large parts have constant coefficients.

Using the previous lemmas, we obtain that \( p_1 = (1/\varepsilon)q \) remains
bounded if the initial data satisfy

\[
u_0 - \hat{U}(\cdot, 0) = O(\varepsilon), \quad p_0 - \hat{P}(\cdot, 0) = O(1),
\]
or, equivalently,

\[
\nabla \cdot u_0 = g(\cdot, 0) + O(\varepsilon), \quad p_0 - P(\cdot, 0) - \hat{P}(0) = O(1). \quad (2.3)
\]

Henceforth we assume (2.3).

**Estimate of the remainder.** Let us write

\[
u = \hat{U} + u_1 + u', \quad p = \hat{P} + p_1 + p',
\]

where \( \hat{U} = U^{(l)} \), \( \hat{P} = P^{(l)} \) is the slow part of the solution constructed in
Lemma 2.1, and \( u_1, p_1 \) is the solution of the linearized system (2.2). For
the remainder terms \( u', p' \) we obtain

\[
u' + \left( (\hat{U} + u_1) \cdot \nabla \right) u' + (u' \cdot \nabla)(\hat{U} + u_1) + (u' \cdot \nabla)u' + \nabla p'
\]
\[
= \nu \Delta u' + \varepsilon^{2l} F_{l+1} - (u_1 \cdot \nabla) u_1,
\]
\[
\varepsilon^2 \left\{ p' + \left( (\hat{U} + u_1) \cdot \nabla \right) p' + (u' \cdot \nabla)(\hat{P} + p_1) + (u' \cdot \nabla)p' \right\} + \nabla \cdot u'
\]
\[
= \varepsilon^{2l+2} g_{l+1} - \varepsilon^2 (u_1 \cdot \nabla) p_1
\]

with homogeneous initial conditions

\[
u' = 0, \quad p' = 0 \quad \text{at} \ t = 0.
\]

We set \( \varepsilon p' = q' \) and divide the equation for \( p' \) by \( \varepsilon \) to obtain

\[
u' + \cdots + \frac{1}{\varepsilon} \nabla q' = \nu \Delta u' + O(\varepsilon^2),
\]
\[
q' + \cdots + \frac{1}{\varepsilon} \nabla \cdot u' = O(\varepsilon^2).
\]

(We need \( l \geq 1 \) so that the forcing is of size \( O(\varepsilon^2) \) in the last equation.)

By assumption (2.3) and Lemma 2.2 all space derivatives of the variable
coefficients of the above system are bounded independently of \( \varepsilon \). Also, all
space derivatives of the forcing are \( O(\varepsilon^2) \). Then standard arguments (see,
e.g., [5]) show that the Cauchy problem for \( u', q' \) has a unique \( C^\infty \)-solution
in \( 0 \leq t \leq T \) if \( 0 < \varepsilon \leq \varepsilon_0(T) \). Also,

\[
\|u'(\cdot, t)\| + \|q'(\cdot, t)\| = O(\varepsilon^2).
\]
Using the simple back transformation \( p' = (1/\varepsilon)q' \), we have proved
Theorem 1.1 with the exception of the statement that \( u_1, p_1 \) are highly
oscillatory in time.

2.3. Behaviour of the Highly Oscillatory Part

In this section we discuss the behaviour of the solution \( u_1, p_1 \) of the
linearized problem (2.2) and prove Theorem 2.7 formulated below. We
recall that the initial data \( u_{1,0}, p_{1,0} \) are constructed so that \( \nabla \times u_{1,0} = 0 \)
and that their spatial averages are zero. The coefficients \( \tilde{u}, \tilde{p} \) are uni-
formly smooth; i.e., all derivatives—including time derivatives—are
bounded independently of \( \varepsilon \). We make the change of variables

\[
\tau = t/\varepsilon, \quad \tilde{u}(x, y, \tau) = u_1(x, y, \varepsilon \tau),
\]
\[
\tilde{q}(x, y, \tau) = \varepsilon p_1(x, y, \varepsilon \tau).
\]

If we drop the \( \tilde{\varepsilon} \) sign in our notation, then (2.2) becomes

\[
\begin{align*}
\partial_t u + \varepsilon \{(U \cdot \nabla)u + (u \cdot \nabla)U\} + \nabla q &= \varepsilon \nu \Delta u, \\
\partial_t q + \varepsilon (U \cdot \nabla)q + \varepsilon^2 (u \cdot \nabla)p + \nabla \cdot u &= 0, (2.4)
\end{align*}
\]

with \( q_{1,0} = \varepsilon p_{1,0} \). In Section 2.2 we had assumed \( u_{1,0} = O(\varepsilon), q_{1,0} = O(\varepsilon) \)
(see (2.3)), which was important to estimate the remainder terms. In this
section the problem is linear, and consequently the size of the initial data
is unimportant. We assume, for simplicity, that the initial data \( u_{1,0}, q_{1,0} \)
and all their space derivatives are \( O(1) \). Then we obtain as in Lemma 2.2
that

\[
u = O(1), \quad q = O(1) \quad \text{in} \quad 0 \leq \tau \leq T/\varepsilon. \quad (2.5)
\]

(Here we use the notation \( g = O(1) \) if for each \( k = 0, 1, \ldots \) we have that

\[
\max_{0 \leq \tau \leq T/\varepsilon} \|g(\cdot, \tau)\|_{H^k} \leq C_k
\]

with \( C_k \) independent of \( \varepsilon \).)

If we neglect the terms multiplied by \( \varepsilon \) in (2.4), then we obtain the
constant coefficient symmetric hyperbolic system

\[
\begin{align*}
\partial_t \tilde{u} + \nabla \tilde{q} &= 0, \\
\partial_t \tilde{q} + \nabla \cdot \tilde{u} &= 0. \quad (2.6)
\end{align*}
\]

Let us first discuss (2.6). We define the vorticity and dilatation of \( \tilde{u} \) by

\[
\xi = \nabla \times \tilde{u}, \quad \tilde{s} = \nabla \cdot \tilde{u}
\]
and summarize some elementary results in

**Lemma 2.3.** (i) Suppose $\tilde{u}, \tilde{q}$ solve (2.6). Then

$$\xi_\tau = 0, \quad \tilde{s}_\tau = \Delta \tilde{s}, \quad \tilde{q}_\tau = \Delta \tilde{q},$$

and the spatial averages stay constant,

$$(1, \tilde{u}_\tau) - (1, \tilde{v}_\tau) - (1, \tilde{q}_\tau) = 0.$$

(ii) Conversely, if $\xi, \tilde{s}, \tilde{q}$ and the spatial averages $(1, u), (1, v)$ are known, then we can obtain $\tilde{u}$ by solving the inhomogeneous Cauchy–Riemann system

$$\tilde{\bar{u}}_x + \tilde{\bar{v}}_y = \tilde{s}, \quad \tilde{\bar{v}}_x - \tilde{\bar{u}}_y = \tilde{\xi}.$$  

(2.7)

Estimates for the solution $\tilde{u}$ of (2.7) can easily be obtained by Fourier expansion in $x, y$.

It will be important below to have estimates for time-integrals of the solutions of (2.6). To this end, let $\phi(x, y, \tau)$ denote a solution of the wave equation,

$$\phi_\tau = \Delta \phi, \quad \phi = \phi_0, \quad \phi_\tau = \phi_{\tau, 0} \quad \text{at } \tau = 0,$$

where $(1, \phi_0) = (1, \phi_{\tau, 0}) = 0$.

Then we obtain by Fourier expansion

$$\phi(x, y, \tau) = \sum_{k \neq 0} \hat{\phi}(k, \tau) e^{2\pi i (k_1 x + k_2 y)}, \quad k = (k_1, k_2) \in \mathbb{Z}^2$$

with

$$\hat{\phi}(k, \tau) = \hat{\phi}_0(k) \cos 2\pi k \tau + \frac{1}{2\pi k} \hat{\phi}_{\tau, 0}(k) \sin 2\pi k \tau, \quad k^2 = k_1^2 + k_2^2.$$

Since the integrals

$$\int_0^r \cos(2\pi k \sigma) \, d\sigma, \quad \int_0^r \sin(2\pi k \sigma) \, d\sigma, \quad k \neq 0,$$

are bounded uniformly in $\tau$ and $k \neq 0$, we can estimate integrals

$$\int_0^r \phi(x, y, \sigma) \, d\sigma$$
in terms of the initial data, with constants independent of $\tau$. Using Lemma 2.3, one obtains

**Lemma 2.4.** Let $(\tilde{u}, \tilde{q})$ solve (2.6) with initial data

\[ \tilde{u} = \tilde{u}_0 = O(1), \quad \tilde{q} = \tilde{q}_0 = O(1) \quad \text{at } \tau = 0. \]

Let $\phi = \phi(x, y, \tau)$ denote any of the functions $\tilde{u}, \tilde{v}, \tilde{q}$, or any space derivative of these functions.

(i) If $\bar{\xi}_0 = 0$, $(1, \tilde{u}_0) = (1, \tilde{v}_0) = (1, \tilde{q}_0) = 0$ for the initial data then

\[ \int_0^\tau \phi(x, y, \sigma) \, d\sigma = O(1); \tag{2.8} \]

i.e., we have a bound independent of $x, y, \tau$.

(ii) If $\bar{\xi}_0 > 0$, $(1, \tilde{u}_0), (1, \tilde{v}_0), (1, \tilde{q}_0)$ are $O(\eta)$, then one obtains an extra term $O(\eta \tau)$ on the right-hand side of (2.8).

Next we formulate a result in which we compare the solution $w = (u, q)$ of (2.4) with the solution $\bar{w} = (\bar{u}, \bar{q})$ of (2.6) to the same initial data

\[ w = \bar{w} = w_0 = O(1) \quad \text{at } \tau = 0. \]

We recall that the vorticity $\xi_0$ and the spatial averages are zero at $\tau = 0$.

**Lemma 2.5.** In the interval $0 \leq \tau \leq T/\varepsilon$ it holds that

\[ w - \bar{w} = O(\varepsilon \tau). \]

For the vorticity and the spatial averages we have that

\[ \xi, (1, u), (1, v)(1, q) \text{ are } O(\varepsilon + \varepsilon^2 \tau^2), \quad 0 \leq \tau \leq T/\varepsilon. \]

In particular,

\[ u - \bar{u} \text{ and } q - \bar{q} \text{ are } O(\sqrt{\varepsilon}) \quad \text{in } 0 \leq \tau \leq 1/\sqrt{\varepsilon}, \]

\[ \xi, (1, u), (1, v), (1, q) \text{ are } O(\varepsilon) \quad \text{in } 0 \leq \tau \leq 1/\sqrt{\varepsilon}. \]

**Proof.** (1) We can write (2.4) and (2.6) in the form

\[ w_\tau + Sw = \varepsilon Mw, \quad \bar{w}_\tau + S\bar{w} = 0, \]

where $M$ is a differential operator whose coefficients depend smoothly on $t = \varepsilon \tau$. By (2.5) we have

\[ (w - \bar{w})_\tau + S(w - \bar{w}) = O(\varepsilon). \]
Then Lemma 2.3 and Duhamel's principle yield

\[ w - \bar{w} = O(\varepsilon \tau). \]

(2) To estimate the vorticity \( \xi \), we write the first equation (2.4) in the form

\[ u_\tau + \nabla q = \varepsilon M_1 u \]

and obtain

\[ \xi = \varepsilon \nabla \times M_1 (u - \bar{u}) + \varepsilon \nabla \times M_1 \bar{u} \]

\[ = O(\varepsilon^2 \tau) + \varepsilon \nabla \times M_1 \bar{u}. \]

Integration in \( \tau \) gives us

\[ \xi(x, y, \tau) = O(\varepsilon^2 \tau^2) + \varepsilon \int_0^\tau (\nabla \times M_1 \bar{u})(x, y, \sigma) \, d\sigma. \]

To estimate the integral, we recall (2.8). If \( \psi = \psi(t), 0 \leq t \leq T \), is in \( C^1[0,T] \) and

\[ \int_0^T \phi(\sigma) \, d\sigma = O(1), \]

then

\[ \int_0^\tau \psi(\varepsilon \sigma) \phi(\sigma) \, d\sigma = \psi(\varepsilon \tau) \int_0^\tau \phi(\sigma) \, d\sigma - \varepsilon \int_0^\tau \psi'(\varepsilon \sigma) \left( \int_0^\sigma \phi(\sigma_1) \, d\sigma_1 \right) \, d\sigma \]

\[ = O(1 + \varepsilon \tau). \]

Therefore,

\[ \xi(x, y, \tau) = O(\varepsilon^2 \tau^2 + \varepsilon + \varepsilon^2 \tau^2) = O(\varepsilon + \varepsilon^2 \tau^2). \]

Estimates for derivatives of \( \xi \) and for the spatial averages can be shown in a similar way.

In \( \tau \)-intervals of length \( O(1/\varepsilon) \) the estimates of the previous lemma only yield \( O(1) \) bounds. However, if we subdivide the interval \( 0 \leq \tau \leq T/\varepsilon \) into \( O(1/\sqrt{\varepsilon}) \) intervals of length \( 1/\sqrt{\varepsilon} \), then \( w = (u, q) \) is always \( O(\sqrt{\varepsilon}) \)-close to a solution \( \bar{w} = (\bar{u}, \bar{q}) \) of (2.6) in each subinterval.

**Lemma 2.6.** Divide the interval \( 0 \leq \tau \leq T/\varepsilon \) into subintervals

\[ I_j = [\tau_j, \tau_{j+1}], \quad \tau_j = j \frac{1}{\sqrt{\varepsilon}}, j = 0, 1, \ldots, O\left(\frac{1}{\sqrt{\varepsilon}}\right), \]
and let $\bar{u}_j, \bar{q}_j$ denote the solution of (2.6) in $I_j$ to initial data

$$u_j = \bar{u}, \quad q_j = \bar{q} \quad \text{at } \tau = \tau_j.$$ 

Then

$$u - \bar{u}_j, \quad q - \bar{q}_j \quad \text{are } O(\sqrt{\varepsilon}) \quad \text{in } I_j$$

and

$$\xi - \bar{\xi}_j, \quad (1, u - \bar{u}_j), \quad \text{etc. are } O(\varepsilon) \quad \text{in } I_j.$$ 

These estimates are uniform in $j$,

$$j = 0, 1, \ldots, O(1/\sqrt{\varepsilon}).$$

**Proof.** The $O(\sqrt{\varepsilon})$-estimate of $w - w_j$ follows from the global bound (2.5); see the proof of the previous lemma.

To prove $\xi - \bar{\xi}_j = O(\varepsilon)$ in $I_j$, we use a recursive argument in $j$. Assume for some $j$ we have an estimate

$$\xi, (1, u), (1, v), (1, q) \quad \text{are } O(j\varepsilon) \quad \text{at } \tau = \tau_j. \quad (2.9)$$

Then Lemma 2.4(ii) yields

$$\int_{\tau_j}^{\tau} \phi(x, y, \sigma) \, d\sigma = O(1) + O(j\varepsilon(\tau - \tau_j)) = O(1), \quad \tau \in I_j,$$

where $\phi$ is $u$, $v$, or $q$, or a space derivative of these functions. Then we can argue as in the second part of the proof of the previous lemma and obtain

$$\xi(x, y, \tau) - \xi(x, y, \tau_j) = O\left(\varepsilon + \varepsilon^2(\tau - \tau_j)^2\right) = O(\varepsilon), \quad \tau \in I_j.$$

For the spatial averages one proceeds in the same way. This shows (2.9) at $\tau = \tau_{j+1}$, and the lemma is proved since $\bar{\xi}_j(x, y, \tau) = \xi(x, y, \tau), \tau \in I_j$.

The proof of the previous lemma shows the global bounds

$$\xi, (1, u), (1, v), (1, q) \quad \text{are } O(\sqrt{\varepsilon}), \quad 0 \leq \tau \leq T/\varepsilon.$$ 

Thus far we have assumed the scaling $w_0 = (u_{1,0}, q_{1,0}) = O(1)$ for the initial data. If we now use the assumption (2.3), we gain a power of $\varepsilon$. In terms of the original variables $u_1 = u_1(x, y, t), \quad p_1 = p_1(x, y, t)$ we have proved the following result.

**Theorem 2.7.** Suppose the initial data satisfy (2.3), and let $u_1, p_1$ denote the solution of the linearized compressible problem (2.2). Then, in the
global interval $0 \leq t \leq T$,

$$\xi_1 = \nabla \times u_1, (1, u_1), (1, v_1), (1, \varepsilon p_1) \text{ are } O(\varepsilon^{3/2}).$$

Locally, for any fixed $0 \leq t_0 \leq T$, let $\bar{u} = \bar{u}(t_0), \bar{p} = \bar{p}(t_0)$ denote the solution of

$$\bar{u}_t + \nabla \bar{p} = 0, \quad \varepsilon^2 \bar{p}_t + \nabla \cdot \bar{u} = 0, \quad (2.10)$$

$$\bar{u} = u, \quad \bar{p} = \bar{p} \quad \text{at } t = t_0.$$

In the interval $t_0 \leq t \leq t_0 + \sqrt{\varepsilon}$ it holds that

$$\xi_1 - \xi, (1, u_1 - \bar{u}), (1, v_1 - \bar{u}), (1, \varepsilon p_1 - \varepsilon \bar{p}) \text{ are } O(\varepsilon^2).$$

Roughly speaking, except for a small vorticity and small spatial averages, the solution $u_1, p_1$ is highly oscillatory in time, but slowly varying in space, since the solutions of (2.10) have this property.

Remark. The estimates of the previous theorem are not sharp. To explain this, we consider a system of ordinary differential equations

$$\frac{dw}{dt} + \frac{1}{\varepsilon} \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} w + M(t)w = 0,$$

$$w = \begin{bmatrix} w^I \\ w^{II} \end{bmatrix}, \quad w^I(0) = w^{II}(0) = O(1), \quad w^{II}(0) = 0.$$

Here we assume $A^* = -A$, $\det A \neq 0$, and a smooth matrix function $M = M(t)$. One can show that the slow part $w^{II}$ remains $O(\varepsilon)$ in $0 \leq t \leq T$; see Kreiss [3]. This suggests that $\xi_1 = O(\varepsilon^2)$ under the assumptions of Theorem 2.7, which can indeed be proved.

2.4 Validity of the Bounded Derivative Principle

In numerical calculations one is often not interested in the highly oscillatory part of the solution. Then one chooses the initial data such that a couple of time derivatives of the solution stay bounded independently of $\varepsilon$ at $t = 0$. We show here that this initialization indeed suppresses the oscillations in an $O(1)$ time interval, i.e., the "bounded derivative principle" is valid. For a discussion of the bounded derivative principle see [3, 4]. In the case considered here, we assume that all data $u_0, p_0, F_0, g$ are $O(1)$, at least. Then one time derivative of the solution $u, p$ of (1.1) is bounded independently of $\varepsilon$ if and only if

$$\nabla \cdot u_0 = g(\cdot, 0) + O(\varepsilon^2). \quad (2.11)$$
This is also equivalent to \( u_0 - U_0 = O(\varepsilon^2) \). Under the assumption (2.11) we have, in general,
\[
    p_0 - P^{(l)}(\cdot, 0) = O(1) \quad \text{at } t = 0,
\]
where \( P^{(l)} \) denotes the slow part of the pressure constructed in Section 2.1. Therefore, \( u_1 = O(\varepsilon) \), \( p_1 = O(1) \), and the highly oscillatory part is not suppressed.

Two time derivatives of the solution \( u, p \) of (1.1) are bounded independently of \( \varepsilon \) at \( t = 0 \) if and only if (2.11) holds and
\[
    (\nabla \cdot u)_t(\cdot, 0) - g_t(\cdot, 0) + O(\varepsilon^2).
\] (2.12)

Using (1.1a) and (1.4) we obtain that (2.12) is equivalent to
\[
    \Delta p_0 = \Delta P(\cdot, 0) + O(\varepsilon^2).
\]
The latter condition is also equivalent to
\[
    p_0 - P(\cdot, 0) - \bar{P}(0) = O(\varepsilon^2). \quad (2.13)
\]
If this is assumed, then the initial data \( u_{1,0}, p_{1,0} \) in (2.2) are \( O(\varepsilon^2) \), and consequently
\[
    (u_1, \varepsilon p_1) = O(\varepsilon^2) \quad \text{in } 0 \leq t \leq T.
\]

Now we use the remainder estimate (see Section 2.2), where we assume an expansion with \( l \geq 2 \) of the slow part \( \bar{U} = \bar{U}^{(l)}, \bar{P} = \bar{P}^{(l)} \). Then the remainder functions \( u', q' = \varepsilon p' \) satisfy equations
\[
    u_t' + \cdots + \frac{1}{\varepsilon} q_t' = \nu \Delta u' + O(\varepsilon^4),
\]
\[
    q_t' + \cdots + \frac{1}{\varepsilon} \nabla \cdot u' = O(\varepsilon^4).
\]
Consequently, \( (u', \varepsilon p') = O(\varepsilon^4) \), and we have proved Theorem 1.2.

3. An Initial-Boundary Value Problem

In this section we consider the compressible equations (1.1a), (1.1b) in the domain \( 0 \leq x, y \leq 1 \) with an inflow boundary condition at \( x = 0 \) and an outflow boundary condition at \( x = 1 \). We assume that all data and the solutions are \( 1 \)-periodic in \( y \) and \( \mathcal{C}^\infty \)-smooth. The velocities are prescribed
\[
    u(x, y, t) = G(x, y, t) \quad \text{at } x = 0, x = 1, \quad (3.1a)
\]
where
\[ G(x, y, t) = \begin{bmatrix} G^{(1)}(x, y, t) \\ G^{(2)}(x, y, t) \end{bmatrix}, \quad x = 0, x = 1, \]
is given with
\[ G^{(1)}(x, y, t) > 0 \quad \text{at } x = 0, x = 1. \]

At the inflow boundary \( x = 0 \) the pressure is an ingoing characteristic variable for (1.1b), and an additional boundary condition is needed. We consider here the simple choice to prescribe the pressure
\[ p(0, y, t) = p^b(y, t) \quad (3.1b) \]
at inflow. We assume initial conditions
\[ u = u_0, \quad p = p_0 \quad \text{at } t = 0. \quad (3.2) \]

There are two kinds of difficulties, namely sound waves generated at \( t = 0 \) interacting with the boundary and—even for smooth flow—the occurrence of a boundary layer at the inflow boundary. (In addition, there are compatibility problems between boundary and initial data.) In this paper we restrict ourselves to a study of the boundary layer at inflow and put strong assumptions on the data so that no difficulties arise from the starting conditions at \( t = 0 \). To this end, consider the incompressible problem (1.2a), (1.2b) supplemented by
\[ U(x, y, t) = G(x, y, t) \quad \text{at } x = 0, x = 1, \quad (3.3a) \]
\[ U = u_0 \quad \text{at } t = 0 \quad (3.3b) \]
and the (artificially imposed) side-condition
\[ (1, P(\cdot, t)) = 0, \quad t \geq 0. \quad (3.3c) \]
The incompressible problem only has a solution if \( \nabla \cdot u_0 = g(\cdot, 0) \) and
\[ \int_0^1 \{G^{(1)}(1, y, t) - G^{(1)}(1, y, t)\} dy = (1, g(\cdot, t)), \quad t \geq 0, \quad (3.4) \]
(see (1.2b) and (3.3a)) hold.
Henceforth we make the (reasonable) assumption that the problem (1.2a), (1.2b), (3.3a)–(3.3c) has a unique smooth solution \( U, P \) for \( t \geq 0 \). In addition, to simplify our argument below, we assume
\[ U(x, y, t) \geq y > 0 \]
for \(0 \leq x, y \leq 1, t \geq 0\). Then, using a still undetermined function \(\overline{P}(t)\), we introduce new variables \(u', p'\) by

\[
\begin{align*}
\quad u &= U + u', \\
\quad p &= P + \overline{P}(t) + p'.
\end{align*}
\]

For \(u', p'\) we obtain Eqs. (2.1). The initial and boundary conditions read

\[
\begin{align*}
\quad u' &= 0, \\
\quad p' &= p_0 - P(\cdot, 0) - \overline{P}(0) =: p_0' \quad \text{at } t = 0, \\
\quad u' &= 0 \quad \text{at } x = 0, x = 1, \\
\quad p' &= p^b(y, t) - P(0, y, t) - \overline{P}(t) =: p_1^b(y, t) \quad \text{at } x = 0. \quad (3.5)
\end{align*}
\]

To avoid all difficulties arising from the start-up, we now make the strong assumption that

\[
\begin{align*}
\quad p'_0 &= 0, \\
\quad p^b_1 &= 0, \\
\quad g_1 &= 0 \quad \text{in } 0 \leq t \leq \delta
\end{align*}
\]

for some \(\delta > 0\). (This assumption can be weakened. We only need to assume that a finite number of \(t\)-derivatives of the solution \(u', p'\) vanish at \(t = 0\); the required number of \(t\)-derivatives depends on the number of terms in the asymptotic expansion derived below. Also, instead of just subtracting the incompressible solution, the initialization process of Section 2 can be employed to derive less restrictive sufficient conditions on the data at \(t = 0\).) We fix \(T > 0\); then the considerations in Section 3.2 will prove that the compressible problem (1.1a), (1.1b), (3.1a)-(3.1c) has a unique \(C^\infty\)-solution \(u = u^e, p = p^e\) in \(0 \leq t \leq T\) for \(0 < \epsilon \leq \epsilon_0(T)\).

It will be proved that—except for a boundary layer at \(x = 0\)—the compressible solution \(u, p\) is close to the incompressible solution \(U, P + \overline{P}(t)\) if \(\overline{P}(t)\) is suitably chosen.

### 3.1. Reduction of Inhomogeneous Terms and Asymptotic Expansion

We expect that the boundary condition for \(p'\) at \(x = 0\) generates a boundary layer on the scale \(x/\epsilon^2\). Therefore, we expect that (2.1) can—to first approximation—be replaced by

\[
\begin{align*}
\quad Uu_{1x} + p_{1x} &= \nu u_{1xx}, \quad (3.6a) \\
\quad Uv_{1x} + p_{1y} &= \nu v_{1xx}, \quad (3.6b) \\
\quad \epsilon^2 Up_{1x} + u_{1x} &= 0 \quad (3.6c)
\end{align*}
\]

with

\[
U - U(0, y, t) = G^{(1)}(0, y, t).
\]
We shall determine a boundary layer solution, i.e., a solution which decays like
\[ e^{-\alpha x/e^2}, \quad \alpha \geq \alpha_0 > 0, \]
for \(0 \leq x \leq 1\) and which satisfies the boundary condition for \(p'\) at \(x = 0\). Using (3.6c) to eliminate \(u_{1x}, u_{1xx}\) from (3.6a), we obtain
\[
(1 - \epsilon^2 U^2) p_{1x} + \nu \epsilon^2 U p_{1xx} = 0.
\]
The desired boundary layer solution is
\[
p_1(x, y, t) = p^b_1(y, t) e^{-\alpha x / \epsilon^2},
\]
\[
\alpha(y, t) = \frac{1 - \epsilon^2 U^2}{\nu U}, \quad U = G^{(1)}(0, y, t).
\]
Then (3.6) is satisfied if we choose the exponentially decaying functions
\[
u_1(x, y, t) = -\epsilon^2 U p^b_1 e^{-\alpha x / \epsilon^2},
\]
\[
u_1(x, y, t) = \epsilon^4 \beta e^{-\alpha x / \epsilon^2}
\]
with
\[
\beta(x, y, t, \epsilon) = \frac{p^b_{1y} - p^b_1 \alpha y x / \epsilon^2}{\nu \alpha^2 + \epsilon^2 \alpha U}.
\]
(Note that \(\nu_1 = O(\epsilon^4)\) near \(x = 0\).)

The boundary layer function \(u_1\) satisfies the boundary condition \(u' = 0\) at \(x = 0\) only up to order \(O(\epsilon^2)\). To prove existence of a solution below, we shall need that the inhomogeneous terms can be reduced to higher order. To this end, we define new variables \(u'', p''\) by
\[
u' = u_1 + u'', \quad p' = p_1 + p''
\]
and use the abbreviations (different from Section 2)
\[
U^{(1)} = U + u_1, \quad P^{(1)} = P + p_1.
\]
Then we obtain from (2.1) and (3.5)
\[
u'' + (U^{(1)} \cdot \nabla) u'' + (u'' \cdot \nabla) U^{(1)} + (u'' \cdot \nabla) U^{(1)} + \nabla p'' = \nu \Delta u'' + \epsilon^2 H_1,
\]
\[
\epsilon^2 \left\{ p'' + (U^{(1)} \cdot \nabla) p'' + (u'' \cdot \nabla) P^{(1)} + (u'' \cdot \nabla) P^{(1)} \right\} + \nabla \cdot u''
\]
\[
= \epsilon^2 g_1 + \epsilon^2 h_1,
\]
\[
u'' = 0, p'' = 0 \quad \text{at } t = 0
\]
\[
u'' = \epsilon^2 G_1 \quad \text{at } x = 0, x = 1,
\]
\[
p'' = 0 \quad \text{at } x = 0.
\]
Here

\[ e^2 H_1 - \left\{ u_{1t} + (U \cdot \nabla) u_1 + (u_1 \cdot \nabla) U \\
+ (u_1 \cdot \nabla) u_1 + \nabla p_1 \right\} + \nu \Delta u_1, \]

\[ g_1 = -\left\{ P_t + \bar{P}_t + (U \cdot \nabla) P \right\}, \]

\[ h_1 = -\left\{ p_{1t} + (U \cdot \nabla) p_1 + (u_1 \cdot \nabla) P \\
+ (u_1 \cdot \nabla) p_1 \right\} - \varepsilon^{-2} \nabla \cdot u_1, \]

\[ G_1(x, y, t) = -\varepsilon^{-2} u_1(x, y, t) = O(1). \]

The components of \( H_1 \) consist of boundary layer functions of the form

\[ O\left(1 + \frac{x}{\varepsilon^2}\right) e^{-ax/\varepsilon^2}, \]

because (by (3.6a))

\[ \varepsilon^{-2}\{ -Uu_{1x} - p_{1x} + \nu u_{1xx} \} = \varepsilon^{-2}\{ U(0, y, t) - U(x, y, t) \} u_{1x} \]

\[ - O\left(\frac{x}{\varepsilon^2}\right) e^{-ax/\varepsilon^2}. \]

Similarly,

\[ h_1 = O\left(1 + \frac{x}{\varepsilon^2}\right) e^{-ax/\varepsilon^2} \]

is a boundary layer function.

**Determination of the outer part of \( u^o, p^o, \) and adjustment of \( \bar{P}(t) \).** We write

\[ u^o = \varepsilon^2 U_1 + u^o, \quad p^o = \varepsilon^2\{ P_1 + \bar{P}_1(t) \} + p^o, \quad (3.9) \]

where we define \( U_1, P_1 \) to be the solution of the linearized incompressible problem

\[ U_{1t} + (U \cdot \nabla) U_1 + (U_1 \cdot \nabla) U + \nabla P_1 = \nu \Delta U_1, \]

\[ \nabla \cdot U_1 = g_1, \quad (1, P_1(\cdot, t)) = 0, \]

\[ U_1 = 0 \quad \text{at } t = 0, \]

\[ U_1(x, y, t) = G_1(x, y, t) \quad \text{at } x = 0, x = 1. \]

Solvability requires—similar to (3.4)—that the data satisfy

\[ \int_0^1 \{ G_1^{(1)}(1, y, t) - G_1^{(1)}(0, y, t) \} \, dy = (1, g_1(\cdot, t)), \quad t \geq 0. \quad (3.10) \]
This consistency condition leads to a first-order differential equation for $\bar{P}(t)$: we recall (see (3.8))

$$G^{(1)}(x, y, t) = G^{(1)}(0, y, t)\{p^b(y, t) - P(0, y, t) - \bar{P}(t)\}e^{-ax/\varepsilon^2},$$

and

$$g_1 = -\{P_t + \bar{P}_t + (U \cdot \nabla)P\}.$$

Thus we obtain from (3.10) a linear equation

$$\bar{P}_t(t) + a(t)\bar{P}(t) = b(t)$$

with smooth coefficients $a(t), b(t)$. The value $\bar{P}(0)$ is determined by our assumption $p_0 = 0$ and thus $\bar{P}(t)$ is fixed. The adjustable constant $\bar{P}_t(t)$ is fixed by a similar consistency condition when the next smooth terms $U_2, P_2 + \bar{P}_2(t)$ are determined.

**Determination of the next boundary layer term.** For the functions $u''', p'''$ one obtains equations

$$u''' + (U^{(2)} \cdot \nabla)u''' + (u''' \cdot \nabla)U^{(2)} + (u''' \cdot \nabla)u''' + \nabla p''' = \nu A u''' + \varepsilon^2 H_2 + \varepsilon^4 H_2^s,$n

$$\varepsilon^2\{p''' + (U^{(2)} \cdot \nabla)P''' + (u''' \cdot \nabla)P^{(2)} + (u''' \cdot \nabla)P'''\} + \nabla \cdot u''' = \varepsilon^2 h_2^s + \varepsilon^4 h_2^s,$n

$$u'' = 0, p''' = 0 \quad \text{at } t = 0,$n

$$u''' = 0 \quad \text{at } x = 0, x = 1,$n

$$p''' = -\varepsilon^2\{P_1 + P_1(t)\} \quad \text{at } x = 0.$$n

Here $H_2, h_2$ are of boundary layer type, whereas $H_2^s, h_2^s$ are smooth functions; i.e., all derivatives are bounded independently of $\varepsilon$.

Now we determine a boundary layer solution of inhomogeneous equations (3.6), namely

$$Uu_{2x} + p_{2x} = \nu u_{2xx} + H_2^{(1)},$$n

$$Uv_{2x} + p_{2x} = \nu v_{2xx} + H_2^{(2)},$$n

$$\varepsilon^2 U p_{2x} + u_{2x} = h_2,$n

$$p_2 = -P_1 - \bar{P}_1(t) \quad \text{at } x < 0.$$n

(As before, $U = U(0, y, t) = G^{(1)}(0, y, t)$.) The solution components $u_2, v_2, p_2$ are $O(\varepsilon^2), O(\varepsilon^4), O(1)$, respectively. The corrections $\varepsilon^2 u_2, \varepsilon^2 p_2$ are added to $u_1, p_1$ to improve the approximation in the boundary layer.
This process can be continued. By solving linearized incompressible problems with smooth data and boundary layer equations with boundary layer type data, we can reduce the forcing functions and inhomogeneous boundary data to any order in $\varepsilon$. One obtains an asymptotic expansion

$$u = U + \sum_{j=1}^{l} \left\{ \varepsilon^{2j-2}u_{j} + \varepsilon^{2j}u_{j} \right\} + u',
$$

$$p = P + \tilde{P}(t) + \sum_{j=1}^{l} \left\{ \varepsilon^{2j-2}p_{j} + \varepsilon^{2j}(P_{j} + \tilde{P}_{j}(t)) \right\} + p',
$$

where $u', p'$ denote the remainder terms. The functions $u_{j}, p_{j}$ are of boundary layer type and

$$u_{j} = O(\varepsilon^{2}), \quad v_{j} = O(\varepsilon^{4}), \quad p_{j} = O(1) \quad \text{at} \quad x = 0.
$$

3.2 Estimate of the Remainder and Existence of a Solution

We write the solution $u, p$ in the form

$$u = u^{as} + u', \quad p = p^{as} + p',
$$

where $u^{as}, p^{as}$ denote the finite sum asymptotic expansion constructed above. For the remainder terms one obtains equations

$$u' + (u^{as} \cdot \nabla)u' + (u' \cdot \nabla)u^{as} + (u' \cdot \nabla)u' + \nabla p' = \nu \Delta u' + \varepsilon^{2l}{\tilde{\mathbf{H}}},
$$

$$\varepsilon^{2} \left\{ p' + (u^{as} \cdot \nabla)p' + (u' \cdot \nabla)p^{as} + (u' \cdot \nabla)p' \right\} + \Delta \cdot u' = \varepsilon^{2l}{\tilde{\mathbf{G}}}
$$

$$u' = 0, p' = 0 \quad \text{at} \quad t = 0,
$$

$$u' = \varepsilon^{2l}{\tilde{\mathbf{G}}} \quad \text{at} \quad x = 0, x = 1,
$$

$$p' = \varepsilon^{2l}{\tilde{\mathbf{p}}}^{b} \quad \text{at} \quad x = 0.
$$

The coefficients and data of the above system are smooth functions of $y, t$. The $x$-derivatives satisfy

$$\frac{\partial^{j}u^{as}}{\partial x^{j}} = O(1 + \varepsilon^{2-2j}e^{-\alpha'x/\varepsilon^{2}}),$$

$$\frac{\partial^{j}p^{as}}{\partial x^{j}}, \frac{\partial^{j}{\tilde{\mathbf{H}}}}{\partial x^{j}}, \frac{\partial^{j}{\tilde{\mathbf{G}}}}{\partial x^{j}} \quad \text{are} \quad O(1 + \varepsilon^{-2j}e^{-\alpha'x/\varepsilon^{2}}),$$

where $\alpha(y, t) \geq \alpha_{0} > \alpha' > 0$.

The $y, t$-derivatives of the $x$-derivatives satisfy the same estimates. Also, by our start-up assumption, the data $\tilde{\mathbf{H}}, \tilde{\mathbf{G}}, \tilde{\mathbf{p}}^{b}$ vanish in $0 \leq t \leq \delta$. 
We can make the boundary data homogeneous by changing variables
\[ u' \rightarrow \tilde{u} = u' - \varepsilon^{2l}(x \tilde{G}(1, y, t) + (1 - x)\tilde{G}(0, y, t)), \]
\[ p' \rightarrow \tilde{p} = p' - \varepsilon^{2l}p^b(y, t). \]
This changes the data \( \tilde{H}, \tilde{g} \) to \( \tilde{H}, \tilde{g} \), but the same estimates as above are retained. We introduce new variables
\[ \tilde{u} = \varepsilon^{2l-1}\tilde{u}, \quad \tilde{p} = \varepsilon^{2l-2}\tilde{p} \]
and obtain, omitting the \( \tilde{\cdot} \) sign,
\[ u_t + (u^a \cdot \nabla)u + (u \cdot \nabla)u^a + \varepsilon^{2l-1}(u \cdot \nabla)u + \frac{1}{\varepsilon}\nabla p = \nu \Delta u + \varepsilon H, \]
\[ p_t + (u^a \cdot \nabla)p + \varepsilon(u \cdot \nabla)p^a + \varepsilon^{2l-1}(u \cdot \nabla)p + \frac{1}{\varepsilon}\nabla \cdot u = g, \]
(3.11)
with homogeneous initial and boundary data
\[ u = 0, p = 0 \quad \text{at} \quad t = 0, \]
\[ u = 0 \quad \text{at} \quad x = 0, x = 1, \quad (3.12) \]
\[ p = 0 \quad \text{at} \quad x = 0. \]
Let us first treat the linear problem where the terms multiplied by \( \varepsilon^{2l-1} \) are neglected. We start with an energy estimate. The main technical difficulty is that the coefficient \( \varepsilon p^a_x \) appearing in (3.11) is large at \( x = 0 \). We utilize its layer behaviour.

**Lemma 3.1.** Suppose \( u, p \) solve
\[ u_t + (u^a \cdot \nabla)u + (u \cdot \nabla)u^a + \frac{1}{\varepsilon}\nabla p = \nu \Delta u + \varepsilon H, \quad (3.13a) \]
\[ p_t + (u^a \cdot \nabla)p + \varepsilon(u \cdot \nabla)p^a + \frac{1}{\varepsilon}\nabla \cdot u = g \quad (3.13b) \]
and satisfy the homogeneous conditions (3.12). Here we recall
\[ p^a_x = O\left(1 + \frac{1}{\varepsilon^2}e^{-\alpha' x/\varepsilon^2}\right), \quad \alpha' > 0. \]
For any fixed time interval \( 0 \leq t \leq T \) there is a constant \( c = c(T, \nu) \) with
\[ \|u(\cdot, t)\|^2 + \|p(\cdot, t)\|^2 + \nu \int_0^T \|\nabla u\|^2 \, dt \leq c \int_0^T (e^{-2\|H\|^2 + \|g\|}) \, dt, \quad (3.14) \]
in \( 0 \leq t \leq T \). (Here \( \|u\|^2 = \|u_x\|^2 + \|u_y\|^2, \|\nabla u\|^2 = \|u_x\|^2 + \|u_y\|^2 + \|v_x\|^2 + \|v_y\|^2 \).) The constant \( c \) is independent of \( \varepsilon \) and the data \( H \) and \( g \).
Proof. As usual, we consider
\[ \frac{1}{2} \frac{d}{dt} \{ \|u\|^2 + \|p\|^2 \} = (u, u_t) + (v, v_t) + (p, p_t), \]
use Eqs. (3.13), and apply integration by parts. From
\[ (u, u^{as}u_x) = -(u_xu^{as}, u) - (uu_x^{as}, u) \]
we obtain
\[ -(u, u^{as}u_x) = \frac{1}{2} (u, u_x^{as}u) \leq c_1 \|u\|^2. \]
Also, using the assumption that \( u^{as} \geq 0 \) at \( x = 1 \), we obtain
\[ -(p, u^{as}p_x) \leq c_1 \|p\|^2. \]
Finally,
\[ -\varepsilon (p, up_x^{as}) \leq c_2 \left( \varepsilon \|p\| \|u\| + \frac{1}{\varepsilon} I \right), \]
with
\[ I = \int_0^1 e^{-\alpha x/\varepsilon^2} \left( \int_0^1 \left| pu \right| dy \right) dx \]
\[ \leq \int_0^1 e^{-\alpha x/\varepsilon^2} \left( \int_0^1 p^2 dy \int_0^1 u^2 dy \right)^{1/2} dx \]
\[ \leq \left( \max_x \int_0^1 u^2 dy \right)^{1/2} \left( \int_0^1 e^{-2\alpha x/\varepsilon^2} \right)^{1/2} \|p\| \]
\[ \leq c_3 \varepsilon \left( \max_x \int_0^1 u^2 dy \right)^{1/2} \|p\|. \]
By Fourier expansion in \( y \) and a Sobolev inequality in one space dimension,
\[ \max_x \int_0^1 u^2 dy \leq 2\|u\|^2 + \|u_x\|^2. \]
Therefore,
\[ -\varepsilon (p, up_x^{as}) \leq \frac{\nu}{2} \|u_x\|^2 + c_4 \{\|u\|^2 + \|p\|^2\}. \]
All other terms are treated similarly, and one obtains
\[ \frac{d}{dt} \{ \|u\|^2 + \|p\|^2 \} \leq -\nu \|\nabla u\|^2 + c_5 \{\|u\|^2 + \|p\|^2\} + \varepsilon^2 \|H\|^2 + \|g\|^2. \]
This proves the lemma.
Now we want to estimate derivatives. Let $D$ denote $\partial / \partial y$ or $\partial / \partial t$. The functions $Du$, $Dp$ satisfy (3.12) and the equations

\[(Du)_t + (u^{as} \cdot \nabla)(Du) + (Du \cdot \nabla)u^{as} + \frac{1}{\varepsilon} \nabla Dp = \nu \Delta Du + \varepsilon DH + R,\]

\[(Dp)_t + (u^{as} \cdot \nabla)(Dp) + \varepsilon ((Du) \cdot \nabla)p^{as} + \frac{1}{\varepsilon} \nabla \cdot (Du) = Dg + r,\]

where

\[-R = (Du^{as} \cdot \nabla)u + (u \cdot \nabla) Du^{as},\]
\[-r = (Du^{as} \cdot \nabla)p + \varepsilon (u \cdot \nabla) Dp^{as}.\]

Since $Du^{as}, Du^{as}_x, Du^{as}_y$ are $O(1)$ in maximum norm, we obtain from (3.14) that

\[\int_0^T \|R\|^2 \; dt \leq c \int_0^T \{\varepsilon^2 \|H\|^2 + \|g\|^2\} \; dt.\]

The function $r$ contains the term

\[(Du^{as})_x p_x.\]

We have made the assumption $U \geq \gamma > 0$, and therefore

\[u^{as}(x, y, t) \geq \gamma' > 0.\]  \hspace{1cm} (3.15)

Thus we can use Eq. (3.13b) to express $p_x$ by $p_t$, $p_y$, $(1/\varepsilon) \nabla \cdot u$, etc. Therefore,

\[-r = ap_y + bp_t + \bar{r},\]

with

\[a = Du^{as} - \frac{v^{as} Du^{as}}{u^{as}}, \quad b = -\frac{Du^{as}}{u^{as}}.\]

The function $\bar{r}$ can be estimated in terms of the data,

\[\int_0^T \|\bar{r}\|^2 \; dt \leq \frac{c}{\varepsilon^2} \left\{ \int_0^T \varepsilon^2 \|H\|^2 + \|g\|^2 \right\} \; dt.\]

From the combined system for $u_x$, $p_t$, $u_y$, $p_y$, we obtain an estimate for these derivatives. This process can be continued, and we can estimate any
number of \( y, t \)-derivatives. (The estimates depend on negative powers of \( \varepsilon \).) To estimate \( x \)-derivatives, we use the original differential equations (3.13). For example,

\[
-\nu u_{xx} + u^{as}u_x + \frac{1}{\varepsilon} p_x = \phi^{(1)},
\]

\[
-\nu v_{xx} + u^{as}v_x = \phi^{(2)},
\]

\[
u u^{as}p_x + \frac{1}{\varepsilon} u_x = \phi^{(3)},
\]

where the \( \phi^{(j)} \) are already estimated. Using the boundary conditions \( u = v = p = 0 \) at \( x = 0 \), \( u = v = 0 \) at \( x = 1 \), we obtain estimates for \( u_{xx}, v_{xx}, p_x \) by o.d.e. arguments. Estimates for higher \( x \)-derivatives and mixed derivatives follow from the differentiated system.

Thus, given any \( k = 0, 1, \ldots \) there is \( n = n(k) \) with

\[
\max_{0 \leq t \leq T} \left\{ \|u(\cdot, t)\|_{H^k}^2 + \|p(\cdot, t)\|_{H^k}^2 \right\}
\]

\[
\leq \frac{c}{\varepsilon^n} \int_0^T \left\{ \varepsilon^2 \|H(\cdot, t)\|_{H^k}^2 + \|g(\cdot, t)\|_{H^k}^2 \right\} dt.
\]

The coefficients \( u^{as}, p^{as} \) depend on \( l \), i.e., on the number of terms in the asymptotic expansion. However, the above estimate is independent of \( l \), since the used bounds of the coefficients are uniform in \( l \). Therefore, by making \( l \) sufficiently large and \( \varepsilon \) sufficiently small, we can treat the nonlinear system (3.11) as a perturbation of the linear system (3.13) and obtain finite time existence and estimates. See, e.g., [5] for the standard arguments. This proves

**Theorem 3.2.** Consider the compressible problem (1.1a), (1.1b), (3.1a)–(3.1c) and the incompressible problem (1.2a), (1.2b), (3.3a)–(3.3c) under the assumptions described above. Then, given any \( T > 0 \), the compressible problem has a unique smooth solution \( u, p \) in \( 0 < t < T \) for \( 0 < \varepsilon \leq \varepsilon_0(T) \). To leading order,

\[
u = U + u_1 + O(\varepsilon^2),
\]

\[
p = P + \tilde{P}(t) + p_1 + O(\varepsilon^2).
\]

The boundary layer functions \( u_1, p_1 \) and the adjusted constant \( \tilde{P}(t) \) are described above. The remainder terms are \( O(\varepsilon^2) \) in maximum norm.
4. SUPPRESSION OF THE BOUNDARY LAYER

For numerical calculations one usually wants to choose boundary conditions in such a way that a boundary layer does not occur. For the problem (1.1a), (1.1b) this can be achieved by taking at inflow

\[ u + \varepsilon^2 a p = G^{(1)}(x = 0) > 0, \quad v = G^{(2)}(x = 0), \]
\[ \nabla \cdot u = g \quad \text{at } x = 0. \quad (4.1) \]

Here \( a \) is a parameter with
\[ a > \frac{1}{2} u; \]
this guarantees that the linearized equations satisfy an energy estimate. (We retain the outflow condition
\[ u = G^{(1)}(x = 1) > 0, \quad v = G^{(2)}(x = 1) \quad \text{at } x = 1. \]

For \( \varepsilon \to 0 \), (4.1) goes (formally) over into a condition that can be imposed on the incompressible problem (1.2a), (1.2b). This was not true for (3.1a), (3.1b) at inflow.

Also, (4.1) can be written in the form
\[ u + \varepsilon^2 a p = G^{(1)}, \quad v = G^{(2)}, \quad u_x - g - G_y^{(2)}, \]
showing that (4.1) is of standard type for mixed hyperbolic–parabolic systems if \( \varepsilon > 0 \).

We want to sketch the derivation of an asymptotic expansion. In the first step we solve the incompressible problem

\[ U_t + (U \cdot \nabla) U + \nabla P = \nu \Delta U + F, \]
\[ \nabla \cdot U = g + \varepsilon^2 \tilde{g} \]
with boundary conditions
\[ U + \varepsilon^2 a P = G^{(1)}(x = 0), \quad V = G^{(2)}(x = 0) \quad \text{at } x = 0, \]
\[ U = G^{(1)}(x = 1), \quad V = G^{(2)}(x = 1) \quad \text{at } x = 1. \]

The function \( \tilde{g} \) will be chosen below. The difference
\[ u' = u - U, \quad p' = p - P - \tilde{P} \]
satisfies Eqs. (2.1), where, however, the right-hand side \( \varepsilon^2 g_1 \) is replaced by
\[ \varepsilon^2 g_1 - \varepsilon^2 \tilde{g} + \varepsilon^4 g_2. \]
The boundary conditions read
\[ u' + \varepsilon^2 u p' = 0, \quad v' = 0, \quad \nabla \cdot u' = -\varepsilon^2 \tilde{g} \quad \text{at } x = 0, \quad (4.2) \]
\[ u' = v' = 0 \quad \text{at } x = 1. \]

Now we choose \( \tilde{g} = g_1. \)

Then the largest inhomogeneous term appears in the boundary condition. To remove the term, we consider again the boundary layer equations (3.6). A boundary layer solution \( p_1, u_1, v_1 \) is again given by (3.7), (3.8); the boundary conditions (4.2) are fulfilled up to terms of order \( O(\varepsilon^4) \) if we choose \( p_1^b \) so that \( u_{1x} = -\varepsilon^2 \tilde{g}; \) i.e.,
\[ p_1^b = -\frac{\varepsilon^2 \tilde{g}(x = 0)}{aU} = O(\varepsilon^2). \]

Therefore,
\[ p_1 = O(\varepsilon^2), \quad u_1 = O(\varepsilon^4), \quad v_1 = O(\varepsilon^6). \]

Now we can proceed as in Section 3.1 and build up the asymptotic expansion. The boundary layer part of the solution is uniformly \( O(\varepsilon^2) \); the first derivatives are uniformly bounded.

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