Each problem is worth 10 points for full credit.

Problem 1: Consider the initial–value problem for Burgers’ equation

\[ u_t + \frac{1}{2}(u^2)_x = 0, \quad u(x, 0) = f(x), \]

where

\[ f(x) = \begin{cases} 
1 & \text{for } |x| < 1 \\
0 & \text{for } |x| > 1
\end{cases} \]

Determine a physically reasonable weak solution with one shock wave and one rarefaction wave.

Problem 2: (fundamental solution of \(-\Delta\) in \(\mathbb{R}^2\)) For \(x = (x_1, x_2)\) set

\[ \Phi(x) = \frac{1}{2\pi} \ln \left( \frac{1}{|x|} \right), \quad x \neq 0. \]

Let \(f \in C_0^2(\mathbb{R}^2)\) and set

\[ u(x) = \int_{\mathbb{R}^2} \Phi(y) f(x - y) dy. \]

Show that \(u \in C^2(\mathbb{R}^2)\) and \(-\Delta u = f\).

Problem 3: Let

\[ \Phi(x) = \frac{1}{4\pi |x|}, \quad x \in \mathbb{R}^3, \quad x \neq 0. \]

Let \(f \in C_0(\mathbb{R}^3)\) and let \(u = \Phi * f\) be the Newtonian potential of \(f\). Assume that \(f(x) > 0\) for \(|x| < 1\) and \(f(x) = 0\) for \(|x| \geq 1\). Show that there is a constant \(c_0 > 0\) so that

\[ u(x) \geq \frac{c_0}{1 + |x|} \quad \text{for all } x \in \mathbb{R}^3. \]

Here \(c_0\) depends on \(f\), but is independent of \(x\). This shows that the decay rate \(|x|^{-1}\) for \(u\) cannot be improved, in general.

Problem 4: Let \(g \in C_0^1(\mathbb{R}^3)\). Let \(f\) be a first derivative of \(g\); for example, let \(f = D_1 g\). As before, let \(u = \Phi * f\). Show the improved decay estimate

\[ |u(x)| \leq \frac{C}{1 + |x|^2} \quad \text{for all } x \in \mathbb{R}^3. \]

Here the constant \(C\) depends on \(f\), but is independent of \(x\).