1. Let $X$ be a single observation from the Bernoulli density $f(x|\theta) = \theta^x(1-\theta)^{1-x}$ where $0 < \theta < 1$. Let $T_1 = X$ and $T_2 = 1/2$.

(a) Are both $T_1$ and $T_2$ unbiased estimators for $\theta$? Is either? Justify your answer.

Answer: $E(T_1) = E(X) = \theta$ so $T_1$ is unbiased. $E(T_2) = E(1/2) = 1/2$ so in general, $T_2$ is not unbiased except when $\theta = 1/2$.

(b) Compare the mean square error of $T_1$ with that of $T_2$.

Answer: Since $T_1$ is unbiased, $MSE(T_1) = Var(T_1) = \theta(1-\theta)$. Since $T_2$ is constant then, $Var(T_2) = 0$ and $MSE(T_2) = (\theta - 1/2)^2$. Notice that both MSEs are quadratic functions on $(0,1)$. $MSE(T_1) < MSE(T_2)$ for $\theta$ near 0 or 1. $MSE(T_2) < MSE(T_1)$ when $\theta$ is near 1/2.

2. Let $X$ be a single observation with pdf $f(x|\theta) = (\theta/2)^{|x|}(1-\theta)^{1-|x|}$ where $x = -1, 0, 1$, $0 \leq \theta \leq 1$.

(a) What is a maximum likelihood estimator for $\theta$? Justify your answer.

Answer: We have two cases. If $|x| = 0$ then $f(x|\theta) = (1-\theta)$ which is a decreasing function on $\theta$. The maximum is reached at $\theta = 0$. If $|x| = 1$ then $f(x|\theta) = (\theta/2)$ which is an increasing function on $\theta$. The maximum is reached at $\theta = 1$. Either way, the MLE is $\hat{\theta} = |X|$.

Notice that taking the log of $f(x|\theta)$ and then its derivative with respect to $\theta$, is not correct because log $f(x|\theta)$ is not defined for $\theta = 0$ or $\theta = 1$. Still, if you follow this approach, you may obtain that the MLE is $\hat{\theta} = |X|$. Is one of this situations where you get the right answer with the wrong procedure!?

(b) Is the estimator in part (a) a uniformly minimum variance unbiased estimator (UMVUE)? Justify your answer.

Answer: We know that in this problem, $T = |X|$ is a sufficient and complete statistic for $\theta$ (exponential family of distribution result). Also,

$$E(|X|) = 2(\theta/2) + 0(1-\theta) = \theta$$

so $|X|$ is an unbiased estimator of $\theta$. By the Lehmann-Scheffé theorem, $|X|$ is an UMVUE.

Notice that we may use the Crámer-Rao lower bound (CRLB) approach if we impose the additional condition that $0 < \theta < 1$. Then,
\[
\frac{d^2 \log f(x|\theta)}{d\theta^2} = -\frac{|x|}{\theta^2} - \frac{(1 - |x|)}{(1 - \theta)^2}
\]

and

\[
E\left( -\frac{d^2 \log f(x|\theta)}{d\theta^2} \right) = \frac{1}{\theta (1 - \theta)}.
\]

The CRLB for the class of unbiased estimators of \( \theta \) is \( \theta (1 - \theta) \).

Finally, notice that \( E(|X|^2) = \theta \) and \( Var(|X|) = \theta (1 - \theta) \), so \( T = |X| \) is unbiased and reaches the lower bound.

(c) What is the form of the likelihood ratio test statistic for \( H_0 : \theta = 0.5 \) versus \( H_1 : \theta \neq 0.5 \).

\textit{Answer:} Under \( H_0 \), the supremum of \( f(x|\theta) \) is \((1/4)|x|(1/2)^{1-|x|}\). Without any restrictions the supremum is \((|x|/2)^{|x|}(1 - |x|)^{1-|x|}\). Then, the likelihood ratio statistic is:

\[
\lambda(x) = \left( \frac{1}{2|x|} \right)^{|x|} \left( \frac{1}{2(1 - |x|)} \right)^{1-|x|}
\]

We reject the null hypothesis if \( \lambda(x) < c \).

3. An experimenter observes \( X \), a random sample of size one, where \( X \) follows a \( N(\theta, 1) \). She wishes to test: \( H_0 : \theta = 0 \) versus \( H_1 : \theta = 1 \).

(a) Find the uniformly most powerful \( \alpha \) level test.

\textit{Answer:} This is a case of simple null hypothesis versus a simple alternative hypothesis. The UMP will be given by the Neyman-Pearson lemma. Here, \( f(x|\theta) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} (x - \theta)^2\right) \). Then

\[
\frac{f(x|\theta = 0)}{f(x|\theta = 1)} = \exp\left(-\frac{1}{2} [x^2 - (x - 1)^2]\right) = \exp\left(-\frac{1}{2} (2x + 1)\right).
\]

We reject the null hypothesis if and only if \( \exp(-0.5(2X + 1)) < c \), if and only if \( X > k \).

Under the null hypothesis \( X \) is a \( N(0, 1) \), then \( k \) must satisfy that:

\[
Pr[X > k|\theta = 0] = \alpha
\]

\( k = Z_{1-\alpha} \), the quantile \((1 - \alpha)\) under the normal CDF. (You may also write this quantile as \( Z_{\alpha} \).)
(b) For the test of part (a), what is the probability of the type II error? You may express this probability as a function of a quantile of the $N(0,1)$ distribution.

*Answer:* If $\beta$ is the probability of type II error:

$$\beta = Pr[X < Z_{1-a}|\theta = 1] = Pr[X - 1 < Z_{1-a}] = Pr[Z < Z_{1-a}] = \Phi(Z_{1-a} - 1)$$

where $\Phi(\cdot)$ is the CDF of the $N(0,1)$.

(c) Now suppose that she wishes to test $H_0 : \theta = 0$ versus $H_1 : \theta > 0$. Also, suppose that $H_0$ is rejected if and only if $X > 1.64$. Make a sketch of the power function. Justify your answer.

*Answer* The power function is:

$$\beta(\theta) = Pr[X > 1.64|\theta] = Pr[Z > 1.64 - \theta] = 1 - \Phi(1.64 - \theta)$$

The sketch must show that $\beta(\cdot)$ is an strictly increasing function of $\theta$ for values of $\theta$ greater (it is a function of $\Phi(1.64 - \theta)$). Also $\beta(\theta = 0) = .05$ so the probability of the type I error is $0.05$. 