1. **Exercise 8.10**: Given a gamma($\alpha, \beta$) prior, from exercise 7.24, we know that the posterior distribution follows a gamma($\alpha', \beta'$) where

$$\alpha' = \alpha + \sum_{i=1}^{n} X_i; \beta' = \beta/(n\beta + 1)$$

(a) Expressions for the posterior probabilities of $H_0$ and $H_1$ are:

$$P(H_0|\bar{x}) = \int_{0}^{\lambda_0} \text{Gamma}(\alpha', \beta') d\lambda; P(H_1|\bar{x}) = \int_{\lambda_0}^{\infty} \text{Gamma}(\alpha', \beta') d\lambda$$

where $\text{Gamma}(\alpha', \beta')$ represents a gamma pdf with parameters $\alpha'$ and $\beta'$.

(b) With this prior, the parameters for the posterior distribution for $\lambda$ are:

$$\alpha' = (5 + 2 \sum_{i=1}^{n} X_i)/2; \beta' = 2/(2n + 1)$$

Define $Y = (2n + 1)\lambda$. Via “change of variable”, $Y$ has a gamma pdf with parameters $\alpha'$ and $\beta = 2$, i.e. a chi-square distribution with $p = 5 + 2 \sum_{i=1}^{n} X_i$ degrees of freedom. Then,

$$P(H_0|\bar{x}) = P(\lambda \leq \lambda_0|\bar{x}) = P(Y \leq (2n + 1)\lambda_0)$$

Since $Y$ is a chi-square random variable, we only need find the cumulative probability up to the value $(2n + 1)\lambda_0$ under the chi-square distribution table with $p$ degrees of freedom.

2. **Exercise 8.14**: We reject the null hypothesis if $\sum_{i=1}^{n} X_i > c$. Under $H_0$, we wish to have:

$$Pr[\sum_{i=1}^{n} X_i > c | \theta = 0.49] = 0.01$$

With the CLT approximation, $Z = (\sum X_i - n(0.49))/\sqrt{n(0.49)(0.51)}$ is approximately a $N(0,1)$. Then,

$$c = n(0.49) + (2.33)(0.4999)\sqrt{n} \quad (1)$$

Under $H_1$, we wish to have:

$$Pr[\sum_{i=1}^{n} X_i > c | \theta = 0.51] = 0.99$$

$$c = n(0.51) - (2.33)(0.4999)\sqrt{n} \quad (2)$$
If we find $n$ that satisfies (1) and (2), we obtain that:

$$n = 13566.82$$

If we want to detect such a small difference with so small errors, we will need a pretty large sample size!

3. Exercise 8.16:

(a) Since we always reject $H_0$, $Pr[\text{Reject } H_0 | \theta] = 1$ for any value of $\theta$. Then, the type I error is always 1 for $\theta \in \Theta_0$ and the type II is always 0 for $\theta \in \Theta_1$.

(b) Since we always accept $H_0$, $Pr[\text{Reject } H_0 | \theta] = 0$ for any value of $\theta$. Then, the type I error is always 0 and the type II is always 1.

4. Exercise 8.18:

(a) The power is the probability of rejecting $H_0$ given any value of the parameter. Given the rejection region for this problem, the power function is:

$$\beta(\theta) = Pr[\bar{X} > \theta_0 + c(\sigma/\sqrt{n}) \text{ or } \bar{X} < \theta_0 - c(\sigma/\sqrt{n}) | \theta]$$

This is equivalent to:

$$\beta(\theta) = Pr[Z > c + \sqrt{n}(\theta_0 - \theta)/\sigma \text{ or } Z < -c + \sqrt{n}(\theta_0 - \theta)/\sigma]$$

where $Z = \frac{\bar{X} - \theta}{\sigma/\sqrt{n}}$ follows a $N(0,1)$ distribution. Then,

$$\beta(\theta) = 1 - \Phi\left(c + \frac{\sqrt{n}(\theta_0 - \theta)}{\sigma}\right) + \Phi\left(-c + \frac{\sqrt{n}(\theta_0 - \theta)}{\sigma}\right)$$

where $\Phi(\cdot)$ is the CDF of a $N(0,1)$.

(b) If a type I error probability of 0.05 is desired, $c$ is the 0.975 quantile of $N(0,1)$ distribution, i.e. $c = 1.96$. If we want a type II error probability of at most 0.25 at $\theta_0 + \sigma$, this implies that

$$\Phi(1.96 - \sqrt{n}) - \Phi(-1.96 - \sqrt{n}) \leq 0.25$$

Notice that for $n \geq 1$, $\Phi(-1.96 - \sqrt{n}) \approx 0$. All we need is to choose $n$ such that

$$\Phi(1.96 - \sqrt{n}) \leq 0.25$$

This implies that $1.96 - \sqrt{n} \leq -0.67$ or $n \geq 6.92$. With $n = 7$ and $c = 1.96$, we have the required conditions.

5. Exercise 8.21: All there is to this exercise is to re-write the proof in page 389. Particularly, you need to change the integral sign for a summation sign on equation (8.3.3). The rest of the details are also applicable to the discrete random variable case.

6. Exercise 8.22:
(a) Based on the Neyman-Pearson Lemma, you may show that the UMP is given by the condition: Reject $H_0$ if and only if $\sum_{i=1}^{n} X_i \leq c$. Under $H_0$, $\sum_{i=1}^{n} X_i$ follows a Binomial $(n, 0.5)$. Since $\alpha = 0.0547$, using the Binomial CDF, $c = 2$. To obtain the power of this test, we need to compute the following Binomial probability with $n = 10$ and $p = 0.25$.

$$Pr\left[\sum_{i=1}^{n} X_i \leq 2 | p = 0.25\right] = 0.5256$$

(b) For different values of $p$, we need to compute:

$$\beta(p) = Pr\left[\sum_{i=1}^{10} X_i \geq 6 | p\right]$$

for a Binomial(10, $p$)

Here are some values that you may use to sketch this function: $\beta(0.1) = 0.0001$, $\beta(0.2) = 0.0064$, $\beta(0.25) = 0.197$, $\beta(0.5) = 0.377$, $\beta(0.6) = 0.6331$, $\beta(0.8) = 0.9672$.

Certainly, the size of the test is $\alpha = \beta(0.5) = 0.377$.

(c) It will only be for the cumulative probabilities obtained for a Binomial(10, 0.5). These probabilities are: 0.0010, 0.0107, 0.0547, 0.1719, 0.3770, 0.6230, 0.8281, 0.9453, 0.9893, 0.9990, 1.0.