ON THE RESOLVENT OF THE DYADIC PARAPRODUCT, AND
A NONLINEAR OPERATION ON \( RH_p \) WEIGTHS

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Abstract. The existence of a bounded inverse of \((I - \Pi_b)\) on \( L^p \) (\( \Pi_b \) is the dyadic paraproduct) does not imply the same for \((I - \lambda \Pi_b)\), \(-1 \leq \lambda < 1\) (we present a counterexample); but it guarantees the existence of \(1 < p_0\) such that there exist a bounded inverse in \( L^{p_0} \) for every \(-1 \leq \lambda \leq 1\). This is equivalent to showing that the \( RH^d_\lambda \) class of weights is not preserved under certain nonlinear operation involving \( \lambda \), but if \( \omega \in RH^d_\lambda \), then there exists \(1 < p_0\) such that the transformed weight \( \omega_\lambda \in RH^d_{p_0} \) for all \(-1 \leq \lambda \leq 1\).

1. Introduction

The necessary and sufficient conditions for the existence of a solution \( f \in L^p(\mathbb{R}) \) of the equation:

\[
(I - \Pi_b)f = g, \quad \|f\|_p \leq C\|g\|_p;
\]  

are known (see [8]). Here \( I \) is the identity operator, \( \Pi_b \) is the dyadic paraproduct (see (3) for the precise definition), associated to the function \( b \) of bounded mean oscillation (BMO), and \( g \) is a function in \( L^p(\mathbb{R}) \).

We are interested in studying this equation under a simple perturbation:

\[
(I - \lambda \Pi_b)f = g, \quad -1 \leq \lambda \leq 1.
\]  

It is known, by spectral theory, that as soon as (1) is solvable, then so is (2) in a neighborhood of \( \lambda = 1 \); clearly the same is true near \( \lambda = 0 \). What about \( 0 < \lambda < 1 \)? Does invertibility of \((I - \Pi_b)\) in \( L^p \) guarantees invertibility of \((I - \lambda \Pi_b)\) for \( 0 < \lambda < 1 \)?

The answer to this question is no. Counterexamples were constructed for \(-1 < \lambda < 0\) (see [8]) it was not clear then what the answer was for \( 0 < \lambda < 1 \).

The questions we are asking can be rephrased in terms of preserving reverse Hölder \( p \) \( (RH_p) \) weights (see [6]) under certain nonlinear operation involving \( \lambda \) (see §3 for definitions and precise statements).

Theorem 1.0.1. There exist a doubling weight \( \omega \in RH^d_p, \ 1 < p < \infty, \) and \( 0 < \lambda < 1 \), such that \( \omega_\lambda \notin RH^d_p \) (\( \omega_\lambda \) will be defined later).

The weight \( \omega_\lambda \) is given by an infinite product where \( \lambda \) appears in each factor (see (5)). Inserting \( \lambda = 1 \) we get back the weight \( \omega \) we started with. Unfolding the product, you get \( \omega_\lambda = 1 + \lambda b + ... \) where \( b \in BMO \). At first sight we are

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tempted to say that $\omega_\lambda$ plays a role much like the more traditional $\omega^\lambda = e^{\lambda b}$ does. Theorem 1.0.1 makes clear the substantial differences between them, since if $\omega \in RH^p$ then $\omega^\lambda \in RH^p$ for all $0 \leq \lambda \leq 1$ (this is just a trivial application of Hölder’s inequality!).

To the weight $\omega$ given by Theorem 1.0.1 we can associate a function $b \in BMO$, such that to $\omega^\lambda$ corresponds the function $\lambda b \in BMO$, and $(I - \Pi_\lambda b)$ is invertible in $L^p$, but $(I - \lambda \Pi_b) = (I - \Pi_\lambda b)$ is not.

Nevertheless we will show that:

**Theorem 1.0.2.** There exists $p_0 > 1$ such that if $(I - \Pi_b)$ is invertible in $L^p$, then $(I - \lambda \Pi_b)$ is invertible in $L^{p_0}$ for all $-1 \leq \lambda \leq 1$.

Notation and basic definitions are in §2. In §3 we recall a dyadic characterization of weights, the correspondence we mentioned between weights and functions in $BMO$ becomes clear. In §4 we recall the necessary and sufficient conditions for inverting $(I - \lambda \Pi_b)$ in $L^p$, and we prove Theorem 1.0.2. In the last section we construct the counterexample.

Throughout this paper $C$ will denote a constant that might change from line to line.

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2. Preliminaries

2.1. Dyadic intervals and Haar basis. We will work on $\mathbb{R}$ but everything holds in $\mathbb{R}^n$ (see [8]).

Let us denote by $\mathcal{D}$ the family of all dyadic intervals in $\mathbb{R}$, i.e. intervals of the form $(j2^{-k}, (j+1)2^{-k})$, $j, k \in \mathbb{Z}$. $\mathcal{D}_k$ denotes the $k$th generation of $\mathcal{D}$, consisting of those dyadic intervals of length $2^{-k}$. Given any interval $J$, $\mathcal{D}(J)$ denotes the family of dyadic subintervals of $J$; $\mathcal{D}_n(J) = \{I \in \mathcal{D}(J) : |I| = 2^{-n}|J|\}$, $|I|$ denotes the length of the interval $I$. Given an interval $J$ we will denote the right and left halves respectively by $J_r$ and $J_l$; they are the elements of $\mathcal{D}_1(J)$.

The **Haar function** associated to an interval $I$ is given by:

$$h_I = \frac{1}{|I|^{1/2}}(\chi_{L_I}(x) - \chi_{R_I}(x)),$$

here $\chi_I$ denotes the characteristic function of the interval $I$.

The Haar functions indexed on the dyadics, $\{h_I\}_{I \in \mathcal{D}}$, form a basis of $L^2(\mathbb{R})$.

2.2. Expectation and Difference operators. We define the expectation and difference operators for locally integrable functions by:

$$E_k f(x) = \frac{1}{|I|} \int_I f(t) dt = m_I f, \quad x \in I \in \mathcal{D},$$

$$\Delta_k f(x) = E_{k+1} f(x) - E_k f(x).$$

As operators defined on $L^p$, it is clear that $E_k = \sum_{j\leq k} \Delta_j$, and $\sum_j \Delta_j =$identity operator. It is not hard to check that:

$$\Delta_k f(x) = \sum_{I \in \mathcal{D}_k} \langle f, h_I \rangle h_I(x),$$

here $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $L^2$. This proves that the Haar system is complete.
2.3. Dyadic paraproducts and BMO. A locally integrable function \( b \) is in the
space of \textit{bounded mean oscillation}, \( BMO \), if
\[
\frac{1}{|J|} \int_J |b(x) - m_J b|^2 \leq C |J|,
\]
for every interval \( J \), recall that \( m_J b = \frac{1}{|J|} \int_J b \). This is equivalent to the \textit{Carleson condition}
\[
b \in BMO \iff \sum_{I \in \mathcal{D}(J)} b_I^2 \leq C |J| \quad \forall J,
\]
where here \( b_I = \langle b, h_I \rangle \). (It is just an application of Plancherel’s Theorem for
orthonormal basis.)

Define formally the \textit{dyadic paraproduct} associated to a function \( b \in BMO \) by:
\[
\Pi I f = \sum_k E_k f \Delta_k b.
\]

The dyadic paraproduct is a bilinear operator known to be bounded in \( L^p \), more
precisely,
\[
\|\Pi I f\|_p \leq C \|b\|_{BMO} \|f\|_p.
\]
See [7], or [2], for more about \( BMO \), paraproducts and related subjects.

2.4. Weights. A \textit{doubling weight} \( \omega \) is a positive locally integrable function such
that \( \omega(2I) \leq C \omega(I) \) for all intervals \( I \). (We are using the notations \( \omega(I) = \int_I \omega \),
and \( 2I \) is an interval concentric to \( I \) and with double length.)

A weight \( \omega \) is in \( A_\infty \) if given \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for any interval \( I \),
\( E \subset I \), such that \( |E| < \delta |I| \), then \( \omega(E) \leq \epsilon \omega(I) \). (Eg: \( \omega(x) = |x|^{\alpha} \) for \( -1 < \alpha \).)

Every \( A_\infty \) weight is doubling, but the converse is not true (see [4],[9]). There
are equivalent definitions of \( A_\infty \), for example:

A weight \( \omega \) is in \( A_\infty \) if for every interval \( I \)
\[
\frac{1}{|I|} \int_I \omega \leq C \exp \left( \frac{1}{|I|} \int_I \ln \omega \right).
\]
The smallest of such \( C \)’s is called the \( A_\infty \) constant of the weight \( \omega \).


A weight \( \omega \) is in \( RH_p \) (reverse H"older p) if for every interval \( I \)
\[
\left( \frac{1}{|I|} \int_I \omega^p \right)^{1/p} \leq C \frac{1}{|I|} \int_I \omega.
\]
The smallest of such \( C \)’s is called the \( RH_p \) constant of the weight \( \omega \).

(Eg: \( \omega(x) = |x|^\alpha \) for \( \alpha > -1/p \).)

The main properties of these classes of weights are the following:
(a) if \( \omega \in RH_p \), then \( \omega \in RH_{p+\epsilon} \) for some \( \epsilon > 0 \);
(b) if \( p < q \) then \( RH_p \subset RH_q \);
(c) \( A_\infty = \cup_{p>1} RH_p \).

Property (b) is a trivial consequence of Hölder’s inequality. Property (a) is a
classical result of Gehring (see [5],[6]). And property (c) can be found in [6], Thm.
2.11.

\textbf{Remark 1:} The \( A_\infty \) constant of a weight \( \omega \) forces a lower bound \( p_0 \), \( p_0 > 1 \),
on the range of \( p \)’s such that \( \omega \) is not in \( RH_p \). This can be seen in the proof of
property (c) (see [5],[6]).
There is a classical correspondence between $A_\infty$ weights and functions in $BMO$. Namely, if $\omega \in A_\infty$ then $b = \log \omega \in BMO$. Conversely if, $b \in BMO$ and has sufficiently small $BMO$ norm, then $e^b \in A_\infty$. We will consider a different correspondence in the next section, first introduced in [3].

3. Dyadic characterization of weights

In this section we will restrict our attention to functions defined on the unit interval $I_n = [0,1]$.

Let $\omega$ be a weight defined on $I_n$, such that $E_n \omega = m_{I_n} \omega = 1$. By the Lebesgue differentiation theorem, $\lim_{k \to \infty} E_k \omega(x) = \omega(x)$ for a.e. $x$; we can then write the telescoping product:

$$\omega(x) = \prod_{k=0}^{\infty} \frac{E_{k+1} \omega(x)}{E_k \omega(x)} = \prod_{k=0}^{\infty} \left(1 + \frac{\Delta_k \omega(x)}{E_k \omega(x)}\right).$$

Let us define the function $b$, at least formally, by

$$b = \sum_{k=0}^{\infty} \Delta_k b, \quad \Delta_k b = \frac{\Delta_k \omega}{E_k \omega}. \tag{4}$$

Still at the formal level, given a locally integrable function $b$, we can write:

$$\omega = \prod_{k=0}^{\infty} (1 + \Delta_k b). \tag{5}$$

The partial products are always defined and, in case of convergence, they correspond to the expectation of $\omega$ at level $k$, i.e.:

$$E_k \omega = \prod_{j=0}^{k-1} (1 + \Delta_j b). \tag{6}$$

**Definition 3.0.1.** A locally integrable function $b$ is of $RH_p^d$ type if

$$\sum_{I \in \mathcal{D}(J)} m_{I} \omega b_{I}^2 \leq C m_{I} \omega |J|, \quad \forall J \in \mathcal{D}(I_n); \tag{7}$$

here $m_{I} \omega = \prod_{I \subseteq I \in \mathcal{D}(I_n)} (1 + b_{I} h_{I}(x_I))$, where $x_I \in I$.

Under certain conditions, the formal equations (4) and (5) make sense. Properties of the weight $\omega$ can be read off properties on the corresponding function $b$, and viceversa. We have the following dictionary:

(i) if $|\Delta_k b| < 1$ then the partial products in (6) converge weakly to a positive measure.

(ii) $|\Delta_k b| < 1 - \epsilon$ for all $k \geq 0$ if and only if $\omega$ is a dyadic doubling weight.

(iii) $b \in BMOD$ if and only if $\omega \in A_\infty^d$. The $A_\infty$ constant of $\omega$ depends only on the $BMO$ norm of $b$.

(iv) $b$ is of $RH_p^d$ type if and only if $\omega \in RH_p^d$.

**Remark 2:** In this setting all conditions are dyadic, i.e., they hold on dyadic intervals, the superscript $d$ indicates that (e.g. $BMOD$, $A_\infty^d$, etc).

The results (i)-(iii) appeared first in [3]. This characterization of the dyadic $RH_p$ classes is due to S. Buckley (see [1]).

**Remark 3:** For the dyadic theory to resemble the classical theory (in particular if we want properties (a)-(c) to hold), we have to assume that the weight $\omega$ is dyadic
doubling (i.e. \( \omega(\bar{I}) \leq C\omega(I) \), where \( \bar{I} \) is the dyadic parent of \( I \)); or equivalently that the corresponding \( b \) satisfies \( |\Delta_k b| < 1 - \varepsilon \) (by (ii)).

4. Inverting \( (I - \lambda \Pi_b) \)

The following results are known (see [8]). We will state the theorems for functions defined on the unit interval \( I_0 = [0, 1] \), in that case we must assume that the functions have mean value zero on \( I_0 \). The results are true for functions in \( L^p(\mathbb{R}) \). Let \( L^p_b(I_0) = \{ f \in L^p(I_0) : \int_I f = E_0 f = 0 \} \).

When writing spaces of functions or classes of weights we will sometimes “forget” the domain of definition \( I_0 \), (e.g. \( L^p_b \), \( RH^d_b \), where it should read \( L^p_b(I_0) \), \( RH^d_b(I_0) \)).

**Theorem 4.0.1.** Given a locally integrable function \( b \) such that \( |\Delta_k b| < 1 - \varepsilon \) for all \( k \geq 0 \), the operator \( (I - \Pi_b) \) has a bounded inverse in \( L^p_b(I_0) \) if and only if \( b \) is of \( RH^d_b \)-type. Moreover, we have an explicit formula for the inverse operator

\[
(I - \Pi_b)^{-1} g(x) = \sum_{k=0}^{\infty} \Delta_k g(x) \prod_{j > k} (1 + \Delta_j b(x)).
\]

Let \( \omega = \prod_{j=0}^{\infty} (1 + \Delta_j b) \). We can write, by (6),

\[
\prod_{j > k} (1 + \Delta_j b(x)) = \frac{\omega(x)}{E_k \omega(x)(1 + \Delta_k b(x))}.
\]

Given a doubling weight \( \omega \), define formally the operator

\[
P_\omega g(x) = \sum_{k=0}^{\infty} \frac{\omega(x) \Delta_k g(x)}{E_k \omega(x)(1 + \Delta_k b(x))},
\]

(recall that \( \Delta_k b = \Delta_k \omega / E_k \omega \)).

**Theorem 4.0.2.** \( P_\omega \) is a well defined and bounded operator in \( L^p(I_0) \) if and only if \( \omega \in RH^d_b(I_0) \).

The operator \( P_\omega \) is an example of the *multiplier operator*:

\[
T f(x) = \sum_{k=0}^{\infty} \omega_k(x) \Delta_k f(x).
\]

Clearly if the multipliers are constant functions, \( \omega_k = a_k \), then \( T \) is bounded in \( L^p \) if and only if the sequence of \( a_k \)'s is bounded. The general conditions on the sequence of multipliers that will guarantee boundedness of \( T \) are not known. The necessary and sufficient conditions are known for a few particular cases (see [8]).

**Proof.** [Sketch of the proof of Theorem 4.0.1] Suppose that \( f \) is a solution of the equation \( f = g + \Pi_b f \). It is not hard to check that \( \Delta_k f = \Delta_k g + E_k f \Delta_k b \), using the properties of the expectation and difference operators, and the definition of the dyadic paraproduct. Recall that \( \Delta_k = E_{k+1} - E_k \), we obtain then the recurrence equation

\[
E_{k+1} f = \Delta_k g + (1 + \Delta_k b) E_k f.
\]

Solving the recurrence, and passing to the limit in \( L^p \) (using Theorem 4.0.2) we get that \( f = P_\omega g \). \( \square \)
The paraproduct is a bilinear operation, in particular \( \lambda \Pi_b = \Pi \lambda_b \). Therefore, by the previous theorems, questions about the invertibility of \((I - \lambda \Pi_b)\) are reduced to questions about the weight

\[
\omega_{\lambda} = \prod_{k=0}^{\infty} (1 + \lambda \Delta_k b),
\]

(8)
corresponding to the function \( \lambda b \).

Given a doubling weight \( \omega \in RH^d_p \), \( b \in BMO \) corresponding to \( \omega \) as described in the previous section, the operator \((I - \Pi_b)\) is invertible in \( L^p \). By spectral theory (the resolvent is an open set of \( \mathbb{C} \)), we know that \((I - \lambda \Pi_b)\) will be invertible in neighborhoods of \( \lambda = 1 \) and \( \lambda = 0 \). Is this last statement true for \(-1 < \lambda < 1\)?

This question can be translated into a question about weights,

Does multiplication by \(-1 < \lambda < 1\) on the \( b \) side preserves \( RH^d_p \) weights? Graphically:

\[
\begin{align*}
  b & \iff \omega \in RH^d_p \\
  \lambda b & \iff \omega_{\lambda} \in RH^d_p ?
\end{align*}
\]

The answer is negative. For \(-1 < \lambda < 0\) counterexamples were constructed in [8]. We will present a counterexample for \( 0 < \lambda < 1 \) in the last section.

If we replace \( RH^d_p \) by doubling \( A^d_{\infty} \), then the statement is true, more precisely:

**Lemma 4.0.3.** Given a doubling weight \( \omega \in A^d_{\infty} \), then \( \omega_{\lambda} \) is a doubling \( A^d_{\infty} \) weight for every \(-1 \leq \lambda \leq 1\). Moreover the \( A_{\infty} \) constants are uniformly bounded.

**Proof.** \( \omega \) is a doubling \( A^d_{\infty} \) weight \( \iff \lambda b \in BMO \) and \( |\Delta_k b| < 1 - \epsilon \) (properties (ii) and (iii)): \( \lambda b \in BMO \), and, since \(-1 \leq \lambda \leq 1\), certainly \( |\Delta_k \lambda b| = |\lambda \Delta_k b| < 1 - \epsilon \), and \( \|\lambda b\|_{BMO} \leq \|b\|_{BMO} \iff \omega_{\lambda} \) is a doubling \( A^d_{\infty} \) weight with \( A_{\infty} \) constant depending only on \( \|b\|_{BMO} \).

Nevertheless it is true that:

**Theorem 4.0.4.** Given a doubling weight \( \omega \in RH^d_p \), then there exists \( p_0 > 1 \) such that \( \omega_{\lambda} \in RH^d_{p_0} \) for all \(-1 \leq \lambda \leq 1\).

This implies that:

**Theorem 4.0.5.** Let \( b \) be a locally integrable function such that \( |\Delta_k b| < 1 - \epsilon \) for all \( k \). If \((I - \Pi_b)\) has a bounded inverse in \( L^p \) then there exists \( p_0 > 1 \) such that \( (I - \lambda \Pi_b) \) is invertible in \( L^{p_0} \) for all \(-1 \leq \lambda \leq 1\).

**Proof.** If \( |\Delta_k b| < 1 - \epsilon \) and \((I - \Pi_k)\) has a bounded inverse in \( L^p \) then, by Theorem 4.0.1, \( b \) is of \( RH^d_p \)-type \( \iff \) the corresponding \( \omega \) is doubling and in \( RH^d_p \) (by (ii) and (iv)) \( \Rightarrow \) there exists \( p_0 > 1 \) such that \( \omega_{\lambda} \in RH^d_{p_0} \) for all \(-1 \leq \lambda \leq 1\) (by Theorem 4.0.4), and \( \omega_{\lambda} \) is certainly doubling (by Lemma 4.0.3) \( \iff \lambda b \) is of \( RH^d_p \)-type and \( |\Delta_k \lambda b| < 1 - \epsilon_1 \), and using once more Theorem 4.0.1 we conclude that \( (I - \lambda \Pi_b) \) is invertible in \( L^{p_0} \) for all \(-1 \leq \lambda \leq 1\).

**Proof.** [proof of Theorem 4.0.4] Given a doubling weight \( \omega \in RH^d_p \) then \( \omega \in A^d_{\infty} \) (by property (b)). By Lemma 4.0.3 it is also true that for \(-1 \leq \lambda \leq 1\), \( \omega_{\lambda} \) are doubling \( A^d_{\infty} \) weights, with \( A_{\infty} \) constants uniformly bounded. That implies (see Remark 1 in page 3) the existence of \( p_0 > 1 \) so that \( \omega_{\lambda} \in RH^d_{p_0} \) for all \( p > p_0 \) and for all \(-1 \leq \lambda \leq 1\).
5. Counterexample

**Theorem 5.0.6.** There exist a doubling dyadic weight \( \omega \) on \([0,1]\), \(1 < p < \infty\), and \(0 < \lambda < 1\), such that \( \omega \in RH_p^d\) but \( \omega_\lambda \notin RH_p^d\).

The proof of this theorem will follow easily from the next two lemmas.

**Lemma 5.0.7.** There exists a one-parameter family of weights \( \omega^t \), \(1/2 \leq t \leq 1\), with the following properties:

1) \( \int_0^1 \omega^t = 1 \) for all \(1/2 \leq t \leq 1\),

2) \( \omega^t \) is doubling for all \(1/2 \leq t \leq 1\),

3) \( \omega^t \in RH_p^d([0,1])\) if and only if \(1 < p < p_t\), where \(p_t = \ln 8/\ln[8t^2(t - 1)]\) for \(1/2 \leq t < (1 + \sqrt{5})/4\), and \(p_t = \infty\) for \((1 + \sqrt{5})/4 \leq t \leq 1\).

The weights \( \omega^t \) constructed in the previous lemma have the property that their structure is preserved under the operation \((\omega^t)_\lambda\), in particular we can keep track of the \(RH_p^d\) classes that they belong to. More precisely:

**Lemma 5.0.8.** \((\omega^t)_\lambda = \omega^{t \lambda}\), where \(t_\lambda = 1/2 + \lambda(t - 1/2)\), for all \(1/2 \leq t \leq 1\).

**Proof.** [Proof of Theorem 5.0.6] Choose \((1 + \sqrt{5})/4 \leq t_o \leq 1\) (eg. \(t_o = 7/8\)). Set \(\omega = \omega^{t_o}\). By Lemma 5.0.7, \(\omega\) is doubling and \(\omega \in RH_p^d\) for all \(1 < p < \infty\). In particular, \(\omega \in RH_p^{d^*}\) for \(p_o = p(t_o) > 1\), where \((t_o)\lambda = 1/2 + \lambda(t_o - 1/2) > 1/2\), and we choose \(\lambda > 0\) such that 

\[ (t_o)\lambda < (1 + \sqrt{5})/4 \]

( for \(t_o = 7/8\), and \((t_o)\lambda = 2/3\) we get \(\lambda = 4/9 < 1\)).

Combining lemmas 5.0.8 and 5.0.7, we get that \(\omega_\lambda = (\omega^{t_o})_\lambda = \omega^{t_o \lambda}\) is doubling and belongs to \(RH_p^d\) only for \(1 < p < p(t_o)\); in particular it does not belong to \(RH_p^{d^*}\) for \(p_o = p(t_o)\) (in our example \(p_o = p_{2/3} = \ln 8/\ln(32/27) > 1\)).

**Proof.** [Proof of Lemma 5.0.7]

Fix \(1/2 \leq t \leq 1\).

Let \(I_k = (2^{-k}, 2^{-k+1}]\) for \(k = 1, 2, \ldots\). Clearly \(I_o = (0,1] = \cup_{k=1}^{\infty} I_k\). Define the step function

\[ \omega^t(x) = \sum_{k=1}^{\infty} c_k(t) \chi_{I_k}(x), \]

where if we let \(s = 1 - t\), then

\[ c_k(t) = \begin{cases} 2^k(t^2s)^{n}s & k = 3n + 1 \\ 2^k(t^2s)^{n+1}ts & k = 3n + 2 \\ 2^k(t^2s)^{n+1}t^3 & k = 3n + 3 \end{cases} \]

**Remark:** The numbers \(t\) and \(s\) represent the proportion of the mass of \(\omega\) on a given interval \(I_k^*\) that we have distributed among its children \(I_{k+1}, I_{k+1}^*\); where \(I_k^* = \cup_{j > k} I_j\) (the sibling of \(I_k\)).

1) Just computing and since \(s + t = 1\), we get

\[ \int_0^1 \omega^t = \sum_{k=1}^{\infty} c_k(t) |I_k| = (s + ts + t^3) \sum_{n=0}^{\infty} (t^2s)^n = \frac{s + ts + t^3}{1 - t^2s} = 1 \]
2) We want to show that $\omega^d$ is a dyadic doubling weight. We must check that
the mass in any dyadic interval $I$ is comparable with that of its parent $\bar{I},$
\text{i.e.} $\omega^d(\bar{I}) \leq C\omega^d(I^*)$, for all $I \in \mathcal{D}(I_0)$. It is enough to consider only those
intervals $\bar{I}_0 = I_0 \cup I_0^*$. Because of the scale invariance it is enough to consider
only the case $k = 1$, $\bar{I}_1 = I_0 = I_1 \cup I_1^*$, $I_1^* = \cup_{j \geq 2} I_j$. By 1) $\omega^d(\bar{I}_1) = 1$, by
definition $\omega^d(I_1) = s = 1 - t \leq 1/2$. It follows that $\omega^d(I_1^*) = t \geq 1/2$. Clearly
$\omega^d(\bar{I}) \leq s^{-1}\omega(\bar{I})$.

3) We want to show that $\omega^d \in RH^d_p$ for $1 < p < p_t$. Scaling once more, it is
easy to check that for $p < p_t$
$$
\int_0^1 (\omega^d)^p \leq C \left( \int_0^1 \omega^d \right)^p = C.
$$
Now $(\omega^d)^p(x) = \sum_{k+1}^\infty c_k^p \chi_k^p (x)$, hence
$$
\int_0^1 (\omega^d)^p(x) dx = \sum_{k+1}^\infty c_k^{p^{2-k}}
= \left( (2s)^p + (2^2 ts)^p + (2^3 t^2)^p \right) \sum_{n=0}^\infty \left( \frac{2^n t^{2^n}}{2^n} \right)^n.
$$
This series converges if and only if
$$
\frac{(2^3 t^2)^p}{2^3} < 1.
$$
Recall that $s = 1 - t$. Set $f(t) = 8t^2(1-t)$. It is a straightforward
calculation to check that for $(1 + \sqrt{5})/4 \leq t \leq 1$ then $f(t) \leq 1$. Hence in this
range of $t$'s (9) holds for every $1 < p$, i.e. $p_t = \infty$. For $1/2 \leq t < (1 + \sqrt{5})/4$ we have that $f(t) > 1$. Therefore (9) holds only if and only if $p < p_t = \ln 8/\ln(8^2(1-t))$.

This finishes the proof of the lemma.

Proof. [Proof of Lemma 5.0.8] We can use the results in §3 to write:
$$
\omega^d = \prod_{k=0}^\infty (1 + \Delta_k b^d),
$$
$$
\omega^d_\lambda = \prod_{k=0}^\infty (1 + \lambda \Delta_k b^d).
$$
We can find explicitly $b^d$. Recall that
$$
\Delta_k b^d = \frac{\Delta_k \omega^d}{E_k \omega^d} = \frac{E_{k+1} \omega^d - E_k \omega^d}{E_k \omega^d}.
$$
By the definition of $\omega^d$, it is clear that $\Delta_k b^d(x)$ is not 0 only for those $x \in I_k^* = \cup_{j \geq k} I_j$, and in that case, $|\Delta_k b^d(x)| = t - s$ (by the remark in the proof
of Lemma 5.0.7). This implies that for $0 < \lambda < 1$
$$
|\lambda \Delta_k b^d(x)| = \begin{cases}
\lambda(t-s) & x \in I_k^* \\
0 & \text{otherwise}
\end{cases}
$$
We can write $\lambda(t-s) = t_\lambda - s_\lambda$ where $t_\lambda = 1/2 + \lambda(t-1/2)$ and $s_\lambda = 1/2 - \lambda(t-1/2)$; clearly $t_\lambda + s_\lambda = 1$. The structure of the weight $\omega^d_\lambda$ is exactly the same as the structure of the initial weight $\omega^d$, except that we now replace $t$ by $t_\lambda$. 

\[\square\]
More refined versions of this counterexample will appear elsewhere. A slightly more delicate construction will allow us to construct examples where the index $p_0$ in Theorem 5.0.6 can be made as close to one as we want (in our example the worst $p_0 = p_2/3$).

References


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