Practice Problems for Exam # 1 - MATH 401/501 - Spring 2016

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1. Show by induction that the statement

\[ P(n) : \ 1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x} \]

is true for all natural numbers \( n \), and any fixed rational number \( |x| \neq 1 \).

2. Given functions \( f : X \rightarrow Y \), \( g : Y \rightarrow Z \), show that if \( g \circ f \) is surjective (onto) then \( g \) must be surjective. Is it true that \( f \) must also be surjective? If true prove it, if false present a counterexample.

3. Given a set \( X \) and subsets \( A \) and \( B \) of \( X \). Show that \( X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B) \).

4. Let \( a \) and \( b \) be integer numbers. Show that if \( ab = 0 \) then \( a = 0 \) or \( b = 0 \). (You can use the fact that this is true for natural numbers).

5. Let \( r \) and \( q \) be rational numbers. Show that if \( rq = 0 \) then \( r = 0 \) or \( q = 0 \). (You can use the fact that this is true for integer numbers).

6. Given a rational number \( r \), and natural numbers \( n \) and \( m \). We define \( r^0 := 1 \) and given the rational number \( r^n \) then we define the rational number \( r^{n+1} := r^n \times r \).

   (a) Show that \( (r^n)^m = r^{n \times m} \).

   Hint: fix one of the natural numbers and induct on the other.

   (b) Assume now that \( r \neq 0 \), \( p \) and \( q \) are integers, and show that \( (r^p)^q = r^{p \times q} \). Where we define for a negative integer \( p = -n \), \( n \in \mathbb{N} \), \( r^p = r^{-n} := (r^n)^{-1} \). Useful auxiliary lemma is to show that: \( (r^n)^{-1} = (r^{-1})^n \).

7. Let \( \epsilon > 0 \) be a positive rational number (a “step” or “unit”). Show that given a positive rational number \( x \geq 0 \), there exists a natural number \( n \) (depending both on the step \( \epsilon \) and on \( x \)) such that \( |x| < n \epsilon \). In words: given any positive step size we can overcome any fixed rational number with a finite number of steps.

8. Given a rational number \( x \), show directly from the definition of absolute value, that \( |-x| = |x| \).

9. In this problem, use any property of absolute value you wish, make sure you state properly the properties you are using.

   (a) Show that if \( x, y, z \in \mathbb{Q} \), \( |x - y| < 1/2 \), then \( |xz - yz| < |z|/2 \).

   (b) Show that if \( w, x, y, z \in \mathbb{Q} \), \( |w - x| \leq 1/2 \) and \( |y - z| \leq 1/2 \) then \( |(w + y) - (x + z)| \leq 1 \). Can you find rational numbers \( x, w, x, y, z \) such that the hypothesis are satisfied and \( |(w + y) - (x + z)| = 1 \)?

10. Show that the “reverse triangle inequality” holds for \( x, y \in \mathbb{Q} \): \( |x| - |y| \leq |x - y| \).
The following are additional problems in case you want more. Do not worry about them for the purpose of the exam on Thursday.

- Suppose \( f : X \to Y \), and suppose that \( A, B \) are subsets of \( X \) and \( C, D \) are subsets of \( Y \). The \textit{direct image of} \( A \) \textit{under} \( f \) is the subset of \( Y \) defined by

\[
f(A) := \{ y \in Y : y = f(x), x \in A \}.
\]

The \textit{inverse image of} \( C \) \textit{under} \( f \) is the subset of \( X \) defined by

\[
f^{-1}(C) := \{ x \in X : f(x) \in C \}.
\]

Determine which inclusion relationship must hold for the following pairs of sets:

(a) \( f(A \cap B) \) and \( f(A) \cap f(B) \),

(b) \( f^{-1}(C \cap D) \) and \( f^{-1}(C) \cap f^{-1}(D) \).

- Given sets \( A \) and \( B \), the power set \( A^B \) is the collection of all functions \( f : B \to A \). Show that given any sets (finite or not) \( A, B, C \) then

\[
\#((A^B)^C) = \#(A^{B \times C}).
\]