There are a total of 100 points, plus ten possible bonus points. No books, notes or calculators are allowed. Good luck!

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**Blackboard formulas and results promised**

1. **Definitions of trigonometric, hyperbolic, logarithmic and power functions:**
   - \( \cos z = \frac{e^{iz} + e^{-iz}}{2} \), \( \sin z = \frac{e^{iz} - e^{-iz}}{2i} \).
   - \( \cosh z = \frac{e^z + e^{-z}}{2} \), \( \sinh z = \frac{e^z - e^{-z}}{2} \).
   - For \( z \neq 0 \), \( \log z = \ln |z| + i \text{Arg} z \) (single-valued).
   - For \( z \neq 0 \), \( \log z = \ln |z| + i \text{arg} z \) (multiple-valued).
   - For \( z \neq 0, a \in \mathbb{C} \), \( z^a = e^{a \log z} \) (multiple-valued when \( a \) is not an integer).

2. **Lemma:** If \( \phi : S \subset \mathbb{R}^2 \to \mathbb{R}, S \) is a domain (an open and connected set), \( \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial y} = 0 \) on \( S \) then \( \phi \) is a constant function.

3. **Cauchy-Riemann Equations in polar coordinates:** Given \( f(z) = u(r, \theta) + iv(r, \theta) \) for \( z \neq 0 \), then \( u \) and \( v \) satisfy the CR equations if and only if
   \[ \frac{\partial v}{\partial \theta} = r \frac{\partial u}{\partial r}, \quad \frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r}. \]

4. **Limits at infinity and equal to infinity**
   - We say \( \lim_{z \to z_0} f(z) = \infty \) if and only if \( \lim_{z \to z_0} \frac{1}{f(z)} = 0 \).
   - We say \( \lim_{z \to \infty} f(z) = L \) if and only if \( \lim_{z \to 0} f(1/z) = L \).
   - We say \( \lim_{z \to \infty} f(z) = \infty \) if and only if \( \lim_{z \to 0} \frac{1}{f(1/z)} = 0 \).
1. (50 points + 10 bonus points) Please decide whether the following statements are true or false. If TRUE give a very short explanation as to why. If FALSE present a a counterexample or a corrected formula/statement. (There are 12 questions: 10 correct give you full credit, everything additional is bonus).

(a) If $z^5 = 2e^{i\pi}$ then $z = 2^{1/5}e^{i\pi/5}$. □ TRUE □ FALSE

(b) $|−i^3| = i$ □ TRUE □ FALSE

(c) $f(z) = e^z$ is a one-to-one function on $\mathbb{C}$. □ TRUE □ FALSE

(d) $\lim_{z \to (-3i)} z^2 e^z = -9 \cos 3 + i \cdot 9 \sin 3$. □ TRUE □ FALSE

(e) Arg($z$) is a continuous function on $\mathbb{C} \setminus \{0\}$. □ TRUE □ FALSE

(f) Log($z^4$) = 4Log($z$) for all $z \neq 0$. □ TRUE □ FALSE
(g) $|\cos z| \leq 1$ for all $z \in \mathbb{C}$. □ TRUE □ FALSE

(h) If $f$ is an entire function then $h(z) = e^z f(1/z^2)$ is analytic on $\mathbb{C} \setminus \{0\}$, and its derivative $h'(z) = h(z) + e^z f'(1/z^2)(-2/z^3)$. □ TRUE □ FALSE

(i) The function $u(x, y) = 2y - 3$ is harmonic in $\mathbb{C}$. □ TRUE □ FALSE

(j) If $u$ and $v$ are harmonic functions on $\mathbb{C}$ then $f(z) = u(x, y) + iv(x, y)$ is analytic on $\mathbb{C}$. □ TRUE □ FALSE

(k) If $f(z)$ is an entire function with $u(x, y)$ its real part and $v(x, y)$ its imaginary part. Then $u$ is the harmonic conjugate of $v$. □ TRUE □ FALSE

(m) If $f$ is real-valued on $\mathbb{C}$ and analytic then $f$ must be a constant function. □ TRUE □ FALSE
2. (10 points) Let $z = 1 - i$. Write $z$ in polar coordinates.

Write (a) $z^{1/3}$, (b) $\log z$ and (c) $\frac{i}{z}$ in the form $a + ib$ with $a, b \in \mathbb{R}$ (note: some of these are multiple-valued).

3. (10 pts) Find the limit if it exists otherwise justify why it does not exist. Here $y \in \mathbb{R}$ and $z \in \mathbb{C}$.

(a) $\lim_{z \to 0} \frac{\overline{z}}{z}$

(b) $\lim_{z \to i\pi} \frac{e^z + 1}{z - i\pi}$
4. (10 pts) Given the function \( f(z) = (x + \frac{y^3}{3}) + i(y - x + \frac{x^3}{3}) \) defined on the complex plane \( \mathbb{C} \).

(a) Determine all points of continuity of \( f \).

(b) Determine all points at which the function \( f \) is differentiable.

(c) Determine all points at which the function is analytic.
5. (10 pts) Can you find an entire function \( f(z) \) whose real part is \( u(x, y) = e^x \sin y \)? If yes, find such function \( f(z) \) and write it as a function of \( z \) only.

6. (10 pts) Find a function \( \phi(r, \theta) \) that is harmonic in the domain \( \{ z \in \mathbb{C} : 1 < |z - 3i| < 5 \} \) and that has boundary values \( \phi = 2 \) when \( |z - 3i| = 1 \) and \( \phi = 12 \) when \( |z - 3i| = 5 \).