The Sasaki Cone
versus
the Kähler Cone

CHARLES BOYER
University of New Mexico
Contact Manifold (compact)

A contact 1-form $\eta$ such that

$$\eta \wedge (d\eta)^n \neq 0.$$

defines a contact structure

$$\eta' \sim \eta \iff \eta' = f\eta$$

for some $f \neq 0$, take $f > 0$. or equivalently a codimension 1 sub-bundle $\mathcal{D} = \ker \eta$ of $TM$. $(\mathcal{D}, d\eta)$ symplectic vector bundle
Unique vector field $\xi$, called the Reeb vector field, satisfying

$$\xi \lrcorner \eta = 1, \quad \xi \lrcorner d\eta = 0.$$ 

The characteristic foliation $\mathcal{F}_\xi$ each leaf of $\mathcal{F}_\xi$ passes through any nbd $U$ at most $k$ times $\iff$ quasi-regular, $k = 1 \leftrightarrow$ regular, otherwise irregular

Quasi-regularity is strong, most contact 1-forms are irregular.
Contact bundle $\mathbb{D} \to$ choose almost complex structure $J$ extend to $\Phi$ with $\Phi \xi = 0$

Get a compatible metric

$$g = d\eta \circ (\Phi \otimes 1) + \eta \otimes \eta$$

Quadruple $S = (\xi, \eta, \Phi, g)$ called contact metric structure

The pair $(\mathbb{D}, J)$ is a strictly pseudo-convex almost CR structure.
**Definition:** The structure $S = (\xi, \eta, \Phi, g)$ is **K-contact** if $\mathcal{L}_\xi g = 0$ (or $\mathcal{L}_\xi \Phi = 0$). It is **Sasakian** if in addition $(\mathcal{D}, J)$ is integrable.

**Note:** Quasi-regular $\Rightarrow$ K-contact.

- **Manifold K-contact** $\iff$ transverse structure almost Kähler
- **Manifold Sasakian** $\iff$ transverse structure Kähler

$(\xi, \eta, \Phi, g) \iff (d\eta, J, g_T)$
Sasaki cone $\kappa(\mathbb{D}, J) = \{\xi\}$ s.t. $\mathcal{S} \in (\mathbb{D}, J)$ is Sasakian $\subset t_k$ with $1 \leq k \leq n + 1$ $\Rightarrow$ finite dim’l moduli of Sasakian structures with underlying CR structure $(\mathbb{D}, J)$

Alternative Deformation
Fix foliation: $\mathcal{F}_\xi \Rightarrow$ basic cohomology groups $H^p_B(\mathcal{F}_\xi)$

Transverse holomorphic $\Rightarrow$ basic $(p,q)$-forms $\Omega^{p,q}_B \Rightarrow$ basic cohomology groups $H^{p,q}_B(\mathcal{F}_\xi)$. 
Riemannian foliation $\Rightarrow$ **transverse Hodge theory** holds.

$\mathcal{S} = (\xi, \eta, \Phi, g)$ Sasakian str.
Space $\mathcal{S}(\xi)$ of Sasakian structures
$\mathcal{S}' = (\xi, \eta', \Phi', g')$ Reeb v.f. $\xi$

$\eta \mapsto \eta' = \eta + d\zeta$, $\zeta$ is basic. $[d\eta]_B \in H_{B,1}^{1,1}(\mathcal{H}_\xi)$ fixed

Subspace $\mathcal{S}(\xi, \bar{J})$ where $\Phi$ projects to the complex structure $\bar{J}$ on normal bundle $\nu(\xi) = TM/L_\xi$. 
The pair \((\xi, \bar{J})\) defines a transverse holomorphic structure on \(M\), and gives infinite dim’l space \(S(\xi, \bar{J})\) of Sasakian structures.

**Extremal Sasakian Metrics**
(B, Galicki, Simanca)

\[ E(g) = \int_M s_g^2 d\mu_g, \]

Varying in \(S(\xi, \bar{J})\) gives critical point of \(E(g) \iff \partial_g^\# s_g\) is transversely holomorphcic.
Moduli of extremal Sasakian structures $\mathcal{E}(\mathcal{D}, J) \subset \kappa(\mathcal{D}, J)$

Scalar curvature $s_g$ constant $\Rightarrow$ extremal.

Thm (B, Galicki, Simanca) $\mathcal{E}(\mathcal{D}, J)$ is open in $\kappa(\mathcal{D}, J)$.

$\mathfrak{h}(\xi, \overline{J})$ Lie algebra of transversally holomorphic vector fields

Sasaki-Futaki invariant

$$\mathcal{F}_\xi(X) = \int_{M} X(\psi_g) d\mu_g$$
where \( X \in \mathfrak{h}(\xi, \bar{J}) \) and \( \psi_g \) is the harmonic rep in \( H^{1,1}_B(\mathcal{F}_\xi) \) of the transverse Ricci form \( \rho^T_g \).

If \( S \) is extremal \( \tilde{\zeta}_\xi(\cdot) = 0 \iff s_g \) is constant.

Assume Sasakian, \( X, Y \) sections of \( \mathcal{D} \Rightarrow \)

\[
\text{Ric}_g(X, Y) = \text{Ric}^T_g(X, Y) - 2g(X, Y)
\]

\[
\rho_g(X, Y) = \rho^T_g(X, Y) - 2d\eta(X, Y)
\]

\( s_g = s_T - 2n \) (scalar curvatures)

\( s_g \) constant \iff \( s_T \) constant.

\( \rho^T_g \) represents \( c_1(\mathcal{F}_\xi) \) basic 1st Chern class \( \rightarrow c_1(\mathcal{D}) \)
orbifold Boothby-Wang: Manifold $M$ compact with $(\xi, \eta, \Phi, g)$ quasi-regular $\Rightarrow$ quotient $\mathcal{Z} = M/F_{\xi}$ almost Kähler orbifold

Converse: $\mathcal{Z} = M/F_{\xi}$ almost Kähler orbifold. $\omega$ Kähler form with $[\omega] \in H^2_{\text{orb}}(\mathcal{Z}, \mathbb{Z})$. Total space $M$ of $S^1$ orbibundle over $\mathcal{Z}$ has K-contact structure. $(\mathcal{Z}, \omega)$ is projective algebraic orbifold $\iff (\xi, \eta, \Phi, g)$ is Sasakian.
Holomorphy Potentials

\[ \mathcal{H}_g^B = \text{Ker}(\bar{\partial}\partial\#)^*\bar{\partial}\partial\# \]

A map \( \partial_g\# : \mathcal{H}_g^B \rightarrow \mathfrak{h}^T(\xi, \bar{J})/L_\xi \) gives transversely holomorphic vector fields that are \((1, 0)\)-gradients.

\( S \) is extremal \( \iff \) \( s_g \in \mathcal{H}_g^B \).

If Sasaki cone \( \kappa(\mathcal{D}, J) \) has dim \( k \)
\( \Rightarrow \) \( \text{Aut}_0(S) = T^k \) where \( S \) is generic.

moment map \( \mu : M \rightarrow t_k^* \) whose image \( \subset \mathcal{H}_g^B \). Interested in when \( s_g \in \text{Im}\mu \).
Extremal: Special Cases

- Sasaki-$\eta$-Einstein ($S\eta E$)

$$c_1(D) = 0$$

$Ric_g = ag + b\eta \otimes \eta$, $a, b$ constants.

$s_g$ constant. $M$ compact,

$$c_1(F_\xi) \leq 0 \Rightarrow \dim \kappa(D, J) = 1.$$dim 1 $\leftrightarrow$ Transverse homothety

- $\dim \kappa(D, J) > 1. \Rightarrow c_1(F_\xi) > 0$ or indefinite.

- $c_1(F_\xi) > 0 \label{S\eta E} \Rightarrow SE$, many examples including exotic spheres, $k(S^2 \times S^3)$ etc ($\dim \kappa(D, J) = 1.$) (B, Galicki, Nakamaye, Kollár).
• $c_1(F_\xi) > 0$ and $c_1(D) = 0$ Toric $(\dim \kappa(D, J) = n + 1.) \Rightarrow$ SE (Futaki, Ono, Wang, others)

• $c_1(F_\xi)$ indefinite $\Rightarrow c_1(D) \neq 0 \Rightarrow \nexists S_\eta E$. Toric $(\dim \kappa(D, J) = n + 1.)$ generally not known whether extremal (B, Galicki, Ornea)

**Question:** When is $\epsilon(D, J) = \kappa(D, J)$?

Only two known cases:

(1) standard CR structure on $S^{2n+1}$

Toric $(\dim \kappa(D, J) = n + 1.)$

All $S \in \kappa(D, J)$ have extremal reps,
but only transverse homothety of the round sphere has constant scalar curvature, and only the round sphere is SE. (B, Galicki, Simanca)

(2) The Heisenberg group $\mathfrak{h}^{2n+1}$ with standard CR structure (non-compact), $\dim \kappa(\mathcal{D}, J) = n$. All $S \in \kappa(\mathcal{D}, J)$ have extremal reps, but there is only one with constant scalar curvature, $S_{\eta E}$ with $\Phi$-holomorphic curvature $= -2$
Obstructions to extremality
Lichnerowicz: Lowest eigenvalue of Laplacian
(Gauntlett, Martelli, Sparks, Yau)