Preface

This book is an undergraduate introduction to pure mathematics. It can serve as a course bridging the gap between “procedural” mathematics (that emphasizes calculation) and “conceptual” mathematics (that emphasizes ideas); it can also serve as a rigorous introduction to the basic concepts of logic and mathematics. This being said, the present book seems to drastically differ from most available textbooks, as we shall explain below.

Indeed most books introducing logic and mathematics start by assuming elementary mathematics as largely known in its “procedural” form; in particular they assume a certain familiarity with “handling” numbers, elementary geometric figures, and some elementary functions. This leads to a certain circularity in the presentation: mathematics is being used as an illustration (and even as a tool) in the construction of the logical apparatus while, on the other hand, logic is being used to construct mathematical proofs. This circularity is generally presented as acceptable, although, for excellent reasons, students are often uncomfortable accepting it. Moreover, most books on the subject implicitly subscribe to a correspondence theory of truth; according to such a theory mathematical sentences have a meaning, they are supposed to refer to an “infinite universe of mathematical reality,” and they are supposed to be either true or false depending on their meaning and on the “state of affairs” in the mathematical universe. This leads to an implicit acceptance of a “maximal ontology,” such as that of Cantor set theory, where “actual infinities” are being treated as “reality” and statements about them are being treated as referring to “empirical facts.”

The present book will adopt, however, a radically different position: we will declare that the “infinite universe of mathematical reality” is a fiction; and that mathematics does not need this “fairy tale universe” in order to exist. All that mathematics needs are the mathematical words themselves which should be regulated by a “pre-mathematical logic” of language. For instance we will agree that empirical “infinity” is not available but that we can use the word “infinity” as a mere word, devoid of any reference, provided that strict rules are observed in its use. Consequently, in contrast with most available books, we will begin here by asking the students to “forget,” for a while, about all the mathematical objects/concepts that they were ever exposed to; one will have to act as if one does not know the “meaning” of the words implies, contradiction, etc., or the “meaning” of the symbols $1, 2, 3, +, \times, \mathbb{Z}, =, \sin, \cos,$ etc. Then the course will introduce each of these words/symbols in a non-circular manner. In this way we will build, from scratch, first a pre-mathematical logic, then mathematics itself, and, finally, a mathematical logic. Pre-mathematical logic will deal exclusively with language and will be independent of mathematics; on the other hand mathematical logic will appear as a chapter of mathematics. Distinguishing between these two logics will help break
the circularity mentioned earlier. On the other hand we will replace the “maximal ontology” of the “mathematical universe” with a “minimal ontology” according to which the only things “admitted into existence” will be (written) symbols and their combinations referred to as texts. Mathematical sentences will then be regarded as having no meaning and no reference; and everywhere in our proofs semantics will be replaced by syntax. A direct consequence of this will be that the concepts of truth and falsehood will become irrelevant in (and will be actually banned from) all of our discussion of mathematics.

The attitude of this book generally fits into what is called the formalist approach to mathematics, although the brand of formalism which we chose to adopt here is rather extreme. Nevertheless the author feels that extreme formalism can be easily grasped by students, who generally seem to enjoy playing the game of reconstructing mathematics from language itself. This is, of course, more than a game. Indeed one can argue that the only tenable alternatives to formalism seem to involve reduction of mathematics to either psychology (of the kind involved in intuitionism) or metaphysics (of the kind involved in platonism). Both these reductions seem to postulate a somewhat mysterious dimension beyond mathematics which this course would like to reject. There is also the pragmatic view that identifies mathematics with a toolbox for physical sciences. A moment’s reflection shows that this approach is not as straightforward as it seems but, rather, runs into subtle epistemological problems; and attempts to solve these problems inevitably lead back to psychology or metaphysics. Finally there are other possible ways to introduce mathematics by viewing it not as discourse but as a creative process. One aspect of this, which, again, involves a mysterious psychological component, is the discovery of new mathematics by individual mathematicians. Another aspect is the evolution of mathematical ideas throughout history. By the way, there are quite a few excellent books written along historical lines; we kept historical comments to a minimum here because the historical order of things is often quite different from the order of things required by a non-circular mathematical discourse.

It should be clear by now why the present introduction to mathematics is called minimal. It is so called because it seeks to avoid any reference to non-mathematical disciplines such as metaphysics, physics, psychology, or history. What is left, after eliminating such references, is the mathematical discourse itself; and reaching this minimal point, where discourse is left to function by itself, is an intellectual adventure that should probably be part of any serious attempt to understand mathematics. Once this austere landscape of discourse has been revealed, students can start repopulating it, at will, with elements belonging to the various disciplines that we tried to avoid here.

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