Parseval’s Formula

We will now address the situation of when the value of a function $f$ can be recovered from its Fourier series: in other words, when is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)?$$

Recall that if $f$ is a continuous function on $[-\pi, \pi]$ then

$$|f| = \sqrt{\frac{1}{\pi} \int_{-\pi}^{\pi} (f(x))^2}.$$ 

1. Suppose $V = \mathbb{R}^3$ with the usual inner product.
   a. Show that $|v| = \sqrt{\langle v, v \rangle}$ is the usual Euclidean notion of size, namely the distance from $v$ to $\vec{0}$.
   b. Which vector $w$, in the subspace generated by $\{e_1 = (1,0,0), e_2 = (0,1,0)\}$ is closest to $v$?
   c. How can you express $w$ from problem b in terms of the inner product on $\mathbb{R}^3$?
   d. How does this relate to Problem 6 on the Inner Product problem set?
2. Let

\[ v_k = f_k(x) = \frac{a_0}{2} + \sum_{n=1}^{k} a_n \cos (nx) + b_n \sin (nx) \]

so \( v_k \) is the truncated Fourier expansion of \( f(x) \). Show that \( v_k \) is the vector, in the vector space of functions generated by \( \{ \cos (0x), \ldots, \cos (kx), \sin (x), \ldots, \sin (kx) \} \), which is closest to \( f \), that is which minimizes \( |f - v_k| \).
3. Show that the best trigonometric approximation, relative to the distance $| \cdot |$, for $f(x)$ is the Fourier Series

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx).$$

Is the distance from $f$ to its Fourier series necessarily zero? When the distance is zero does this mean that $f(x) = F(x)$ for all values of $x$? Suppose for the moment that $|f(x) - F(x)| = 0$. Give a plausible argument why

$$|f|^2 = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

(1)

Is your argument rigorous? What convergence issues need to be ironed out? Similarly, if we have a second function $g(x)$ with convergent Fourier series

$$g(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(nx) + d_n \sin(nx)$$

then

$$\langle f(x), g(x) \rangle = \frac{a_0 c_0}{2} + \sum_{n=1}^{\infty} a_n c_n + b_n d_n.$$  

(2)

The equations (1) and (2) are referred to as Parseval’s Formula.
4. It would be nice to have a large collection of functions for which one knows that

\[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx). \]

According to previous work, two reasonable hypotheses in order to guarantee this would be:

a. \( f(-\pi) = f(\pi) \),
b. \( f \) is twice differentiable with continuous second derivative. Show that the Fourier series for \( f \) does indeed converge under hypotheses a and b. In a situation where one knows that

\[ \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx). \]

converges (and indeed converges uniformly) it follows that it is a continuous function. In this situation, Parsevals’ formula says that

\[ \left| f - \left( \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \right) \right| = 0. \]

If \( f \) is continuous, show that you can then conclude that \( f = F \).
As you can see, the convergence issues which arise when studying Fourier series are quite subtle. Suppose \( f \) is a function which is nice enough so that its Fourier coefficients \( a_n \) and \( b_n \) are well defined and let \( F \) denote the corresponding Fourier series. Parseval’s formula says that \( |f - F| = 0 \) which means that there is no area underneath the graph of \( |f - F|^2 \). This does not mean that \( f = F \) but it certainly means that it cannot be non-zero over an entire interval. One would like to somehow identify all functions which differ from one another only off of a “negligeable” set and then one could truly say that any \( f = F \) in this new universe. In order to develop these ideas, one needs to study a little bit of measure theory and then introduce the space \( L^2 \) on the interval \([-\pi, \pi]\). The essential point to be gathered from this quick survey of Fourier analysis is that the functions \( \sin(nx) \), \( \cos(mx) \) are very special periodic functions, special because they form a basis for all (sufficiently nice) periodic functions. Just as a vector in \( \mathbb{R}^n \) can be prescribed by giving its coordinates with respect to the usual orthonormal basis, so a periodic function can be expressed as a sum of sine and cosine functions.