Answer all questions.

1. Let \( f_n \) be a sequence of continuous maps \([0, 1] \rightarrow \mathbb{R}\) such that
\[
\int_0^1 (f_n(x) - f_m(x))^p \, dx \to 0
\]
as \( n, m \to \infty \), where \( p > 1 \).
Let \( K : [0, 1] \times [0, 1] \rightarrow \mathbb{R} \) be continuous.
Define \( g_n : [0, 1] \rightarrow \mathbb{R} \) by \( g_n(x) = \int_0^1 K(x, y) f_n(y) \, dy \). Prove that the sequence \( g_n \) converges uniformly to some function \( g \).

2. Let \( \Omega \subset \mathbb{R}^2 \) be a bounded region whose boundary \( \partial \Omega \) is a smooth simple closed curve and let \( u \) and \( v \) be real valued \( C^2 \) functions defined on a neighborhood of \( \bar{\Omega} = \Omega \cup \partial \Omega \).
Suppose \( \nabla^2 u = 0 = \nabla^2 v \) in \( \Omega \) (\( \nabla^2 \) = Laplacian). Show
\[
\int_{\partial \Omega} u(\nabla v \cdot n) = \int_{\partial \Omega} v(\nabla u \cdot n)
\]

3. Let \( f(x, y, z) = (2x^6 + y + z, \sin(\pi x) + yz - y) \). Show that near \((-1,1,1)\) the equation
\( f(x, y, z) = 0 \) defines \( x \) and \( y \) as functions of \( z \). Find \( \frac{dx}{dz}(1) \) and \( \frac{dy}{dz}(1) \).

4. Let \( f : [a, b] \rightarrow \mathbb{R} \) be a continuous map. Let \( \varepsilon > 0 \). Show there is a polynomial \( p \) satisfying
\[
p(a) = f(a)
\]
\[
p'(a) = 0
\]
\[
|f(x) - p(x)| < \varepsilon \quad \forall \ x \in [a, b]
\]
5. Recall that a set of real numbers is called a $G_δ$ set if it is a countable intersection of open sets. Prove that the set of points where a function $f : \mathbb{R} \to \mathbb{R}$ is continuous is a $G_δ$ set.

6. Let $0 \leq \lambda < 1$. Give a construction of a closed set $A_λ \subset [0, 1]$ such that:
   (i) $A_λ$ has Lebesgue measure $λ$
   (ii) the complement of $A_λ$ is dense in $[0, 1]$.

7. Recall that a function $f : [a, b] \to \mathbb{R}$ is called Lipschitz continuous if there exists some constant $M$ such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in [a, b]$. Prove that if $f : [a, b] \to \mathbb{R}$ is Lipschitz continuous, then $f$ is absolutely continuous. Give an example of an absolutely continuous function $f : [a, b] \to \mathbb{R}$ that is not Lipschitz continuous. (Prove that your example is not Lipschitz continuous).

8. Let $\{f_n\}_{n=1}^\infty$ be a sequence of functions in $L^p(\mathbb{R})$, $1 \leq p < \infty$, which converges pointwise almost everywhere to a function $f \in L^p$. Show that $\{f_n\}$ converges to $f$ in the $L^p$ norm if and only if $\lim_{n \to \infty} \|f_n\|_{L^p} = \|f\|_{L^p}$. 