Note: All the questions must be answered.

PART 1: ODEs

1. Consider $x' = A(t)x$, where $A(t)$ is a continuous $2 \times 2$ matrix with $A(t + 1) = A(t)$. Let $\Phi(t)$ be the fundamental matrix solution, and assume that $\text{trace} [\Phi(1)] = 2\alpha$, where $\alpha$ is a real constant. Show that if $\int_0^1 \text{trace} [A(t)] \, dt = -\ln 2$ and if $|\alpha| < 3/4$, then all solutions approach $x = 0$ as $t \to \infty$.

2. Consider the ODE

\[
\begin{align*}
x' &= \lambda x - y - x(4x^2 + 7y^2) \\
y' &= x + \lambda y - y(4x^2 + 7y^2).
\end{align*}
\]

Show that if $\lambda > 0$, then there is at least one nonconstant periodic solution. (Hint: Write the ODE in polar coordinates)

3. Consider the linear ODE

\[
x' = Ax + B(t)x, \quad x(0) = x_0.
\]

Suppose that all of the eigenvalues of the $n \times n$ matrix $A$ have negative real part, and that

\[
\int_0^\infty \|B(t)\| \, dt < \infty.
\]

Show that

\[
\lim_{t \to \infty} |x(t)| = 0.
\]

(Hint: Derive an inequality for $|x(t)|$)

4. Show that $x = 0$ is an asymptotically stable equilibrium point for the system

\[
\begin{align*}
x'_1 &= -x_2 - x_1x_2 - x_1^3 \\
x'_2 &= x_1 - x_1^2x_2 - x_2^3.
\end{align*}
\]

(Hint: The function $V(x_1, x_2) = x_1^2 + x_2^2$ may prove to be useful)
PART 2: PDEs

1 Show that the solution of the nonlinear equation

\[ u_x + u_y = u^2 \]  \hspace{1cm} (1)

satisfying \( u = t \) along the curve \( x = t, \ y = -t \), becomes infinite along the hyperbola \( x^2 - y^2 = 4 \).

2 Solve

\[ u_{tt} - c^2 u_{xx} = 0 \]

in \( x > 0, \ t > 0 \), with initial conditions

\[ u(x, 0) = f(x), \ u_t(x, 0) = g(x) \]

and boundary condition \( u_t(0, t) = \alpha u_x(0, t) \), where \( \alpha \) is a constant.

(b) Show that in general, no solution exists when \( \alpha = -c \).

3 Prove that the solution of Poisson equation,

\[ \Delta u = f, \ x \in \Omega \]  \hspace{1cm} (2)

with boundary condition

\[ \frac{\partial u}{\partial n} + \alpha u = g, \ x \in \partial \Omega \]

is unique when \( \alpha > 0 \). (Here \( \frac{\partial u}{\partial n} \) refers to the normal derivative).

(b) For \( \alpha = 0 \) show that \( \int_{\Omega} f dx = \int_{\partial \Omega} g dS \) is a necessary condition for the solution of the Neumann problem to exist.